A Lorentz Transition Distribution Model for High Frequency Crude Oil Futures

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Abstract: With the deepening of the financialization of the oil market, the importance of the oil futures market is highlighted. The highly volatile crude oil future market inevitably has major influence on financial markets, national economy and even national security. Therefore, modelling accurately the volatility of crude oil futures prices has important theoretical and practical significance for investors and for preventing energy market risks. In this paper, we study the distribution of high-frequency crude oil futures price changes. Many empirical studies have shown that the distribution of price changes in financial market has fatter tail than lognormal distribution. Thus, we build a model using both the transition distribution and the Lorentz stable distribution to describe the characteristics of oil future market. Employing the Fokker-Planck equation, we find the explicit formalism of the distribution of oil future price changes. Using empirical data from China's future market, we have proved the consistency of our theoretical model with the real market.

1 INTRODUCTION

In the energy structure of all countries, crude oil constitutes one of the most crucial components. It also plays an important part in the economic and social development of various countries. In recent years, the financialization of the oil market has deepened. The oil futures market has an impact on the price discovery of crude oil market. Therefore, the high volatility of crude oil futures prices will inevitably have a significant impact on financial markets, national economy and even national security. Therefore, modelling accurately and estimating the volatility of crude oil futures prices has important theoretical and practical significance for investors, and for preventing and mitigating the energy market risks. In this paper, we study the distribution of high-frequency crude oil futures price variations.

Traditionally, the solution to geometric Brownian motion by the Black-Scholes model (Black and Scholes, 1973; Merton, 1973) gives the common lognormal probability distribution for changes in financial asset prices. However, many empirical studies have shown that the distribution of financial assets' price changes has fatter tail than lognormal distribution (Bouchaud and Potters, 2003; Wilmott, 1998).Many progress have been made academically to improve the probability density distribution function of financial assets. Some scholars argue that the volatilities of financial assets are driven by mean-reverting stochastic processes (Engle and Patton, 2001; Blanc et al, 2014). Some scholars believe that the volatility of financial products should be a random variable rather than a constant number as in Black-Scholes model (Hull and White, 1987; Fouque et al., 2000). Autoregressive Conditional Heteroskedasticity models (Engle, 1982; Dumas et al., 1998) use a function of the actual size of the error term in the preceding time period to describe the variance of the current error term. When the error variance is assumed to follow the autoregressive moving average, the model becomes a generalized autoregressive conditional heteroskedasticity (GARCH) model (Bollerslev, 1986; Francq and Zakoian, 2010; Chicheportiche and Bouchaud, 2014; Blanc et al., 2014). In financial market, GARCH models are often applied to describe time series with volatility clustering and time-varying volatility.

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In particular, there has been extensive research on the empirically observed power-law tails or the scaling behaviour of financial assets (Mandelbrot, 1963; Bouchaud, 2000). Mandelbrot (1963) was the first to notice the scaling properties of financial assets and found that the distribution of financial assets' price variation follows a power law. Since then, many scholars have analyzed the power-law distribution of financial data price tails from the perspective of econphysics (Ballocchi et al., 1999; Mantegna and Stanley, 1995; Ghashghaie et al., 1996; Stanley and Plerou, 2001;Voit, 2001). Yet, most of these studies focus on the stock and foreign exchange markets. Many scholars have studied the price fluctuation of crude oil market (Wei et al., 2010; Wen et al., 2016; Gong et al., 2017; Wei et al., 2017; Zhang et al., 2019a; Zhang et al., 2019b; Li et al., 2022). But there is still a lack of research on the distribution of high-frequency crude oil futures prices.

In this paper, we attempt to study the distribution of high-frequency crude oil futures price changes. We construct a two-stage distribution of a stochastic time series for crude oil futures markets using transition probability distribution and Lorentz stable distribution. The Lorentz stable distribution is used to describe the stochastic price changes of high-frequency time series of crude oil futures. And we use the transition distribution to model the price transition from $F(t)$to $F(t+\Delta t)$ in high frequency crude oil future market. Using Fokker-Planck equation, we obtain the explicit expression of our theoretical model. Using empirical data from China's future market, we have proved the consistency of our theoretical model with the real market.

The paper is organized as follows. In Section 2, we build our theoretical model. We describe the possible abnormal stochastic process of high-frequency crude oil futures prices and present the Lorentz stable distribution of high-frequency crude oil futures prices. Then, we build the two-stage model for the stochastic high frequency crude oil futures market and give the explicit formalism of our theoretical model. In Section 3, we calibrate our theoretical model using high-frequency crude oil future SC2209 in China's future market. The results show our theoretical model can describe the real market well, with the R^2 =0.9849. The final section gives the conclusion.

2 MODEL

2.1 Anomalous Geometric Brownian Motion and Lorentz Stable Distribution

It is generally assumed that financial asset follows the geometric Brownian motion

$$dS = S(\sigma dB + \mu dt). \tag{1}$$

In this paper, we analyse the price of future market. We use *F(t)* to represent the price of the future product at time *t*. Therefore, we have

$$dF = F(\sigma dB + \mu dt). \tag{2}$$

Take the first order differential of time, we can have

$$\frac{dF(t)}{dt} = \mu F + F\sigma\eta(t), \tag{3}$$

where $\eta(t)$ represents the noise. We use a functional probability distribution $[dP(\eta)]$ to describe the noise. Thus, the probability distribution of a Gaussian white noise can be described as

$$[dP(\eta)] = [d\eta]e^{-\frac{1}{2\Omega(F)}\int \eta^2(t)dt}. \tag{4}$$

in which $\Omega(F)$ depicts the width of noise distribution.

For the Gaussian white noise, the 1-point and 2-point correlations are characterized as

$$E[\eta(t)] = 0, \tag{5}$$
$$E[\eta(t)\eta(t')] = \Omega(F)\delta(t-t').$$

Given that for initio time t_0 , the value of $F(t_0) = F_0$, the log-return is of the form

$$r(t) = lnF(t)/F(0). \tag{6}$$

Take the first order differential of time, we can have

$$\frac{dr(t)}{dt} = \mu - \frac{\sigma^2}{2} + \sigma\eta(t). \tag{7}$$

We define the relative log-return $z(t)$ as

$$z(t) = r(t) - \mu t. \tag{8}$$

The Langevin equation of the relative log-return $z(t)$ can be written as

$$\frac{dz(t)}{dt} = -\frac{\sigma^2}{2} + \sigma\eta(t). \tag{9}$$

For the stochastic variable $z(t)$, the probability distribution $P(z,t)$ is of the form

$$P(z,t) = E[\delta(z(t) - z)]. \tag{10}$$

Differentiating the probability distribution $P(z,t)$ and using equation (3) (Hohenberg and Halperin, 1977), we can obtain

$$\frac{\partial P(z,t)}{\partial t} = E\left[\left(-\frac{\sigma^2}{2} + \sigma\eta(t)\right)\frac{\partial}{\partial z(t)}\delta(z(t)-z)\right]. \tag{11}$$

The probability distribution $P(z,t)$ satisfies Fokker-Planck equation, therefore we have

$$\frac{\partial P(z,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial z}\left[\sigma\Omega(z)\frac{\partial P(z,t)}{\partial z} + \sigma^2 P(z,t)\right]. \quad (12)$$

As for the stationary solution $P_0(z)$, it also satisfies the stationary Fokker-Planck equation. Therefore we can obtain

$$\frac{\partial}{\partial z}\left[\sigma\Omega(z)\frac{\partial P_0(z)}{\partial z} + \sigma^2 P_0(z)\right] = 0. \quad (13)$$

Integrate the above equation, we can have

$$\sigma\Omega(z)\frac{\partial P_0(z)}{\partial z} + \sigma^2 P_0(z) = C, \quad (14)$$

where C is an integral constant.

As $z_{min} = -\infty, z_{max} = \infty$, the integral constant which represents the probability current equals zero. Therefore, this equation can be reduced as

$$\Omega(z)\frac{\partial P_0(z)}{\partial z} + \sigma P_0(z) = 0. \quad (15)$$

Solving out the stationary Fokker-Planck equation exactly, we have

$$P_0(z) = \frac{1}{N}\exp\left(-\int \frac{\sigma}{\Omega(z)}dz\right), \quad (16)$$

Here N represents normalization constant.

The width of diffusion is set to be

$$\Omega(z) = \frac{\sigma}{2}\frac{\gamma^2 + z^2}{z}. \quad (17)$$

Thus, we can obtain the Lorentz distribution as

$$P_0(z) = \frac{\gamma}{\pi}\frac{1}{\gamma^2 + z^2}. \quad (18)$$

If z is a sum of two Lorentzian random variables z_1 and z_2, the probability density distribution of $z = z_1 + z_2$ under the assumption of independence of z_1 andz_2 is

$$P_2(z) = P_0(z_1) \otimes P_0(z_2) \quad (19)$$
$$= \int_{-\infty}^{\infty} P_0(z_1) P_0(z - z_1)\, dz_1.$$

To calculate the probability density distribution $P_2(z)$, we define the characteristic function of the Lorentzian stochastic process

$$\phi(q) \equiv \int_{-\infty}^{\infty} P_0(z_1) e^{iqz}\, dz. \quad (20)$$

It is not difficult to get the characteristic function of the Lorentzian stochastic process

$$\phi_0(q) = e^{-\gamma|q|}. \quad (21)$$

The convolution theorem of Fourier transform implies that the characteristic function of the stochastic variable z is given by

$$\phi_2(q) = \left(\phi_0(q)\right)^2 = e^{-2\gamma|q|}. \quad (22)$$

By making use of the inverse Fourier transform, we can obtain the probability density function for the stochastic variable $z = z_1 + z_2$,

$$P_2(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_2(q)\, e^{-iqz} dq \quad (23)$$
$$= \frac{2\gamma}{\pi}\frac{1}{4\gamma^2 + z^2}.$$

In the general case, the probability density function for the stochastic variable $z = z_1 + z_2 + \cdots + z_n$ is of the form

$$P_n(z) = P_0(z_1) \otimes P_0(z_2) \otimes \cdots \otimes P_0(z_n) \quad (24)$$
$$= \int_{-\infty}^{\infty} P_0(z_1) P_0(z_2) \cdots P_0(z_{n-1}) P_0(z - z_1$$
$$- z_2 - \cdots - z_{n-1})\, dz_1 dz_2 \cdots dz_{n-1}.$$

The convolution theorem of Fourier transform guarantees that the characteristic function of the stochastic variable z is as

$$\phi_n(q) = \left(\phi_0(q)\right)^n = e^{-n\gamma|q|}. \quad (25)$$

Using the inverse Fourier transform, we can have the probability density function for the stochastic variable $z = z_1 + z_2 + \cdots + z_n$,

$$P_n(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_n(q)\, e^{-iqz} dq \quad (26)$$
$$= \frac{n\gamma}{\pi}\frac{1}{n^2\gamma^2 + z^2}.$$

Thus, the Lorentzian distribution is stable.

In Figure 1 and Figure 2, we exhibit the comparison of Lorentz distribution with some other distributions. We compare the Lorentz distribution with the Gaussian distribution in figure 1. It can be seen that in comparison with Gaussian distribution, the Lorentz distribution is a better fit for the fat-tail distribution observed in real financial market. In figure 2, we present a comparison of truncated Lévy flight with Lorentz distribution. When the stochastic variable z is relatively large, the Lorentz stable distribution approaches $1 / z^2$ while the truncated Lévy flight approaches $1 / z^{1.5}$.

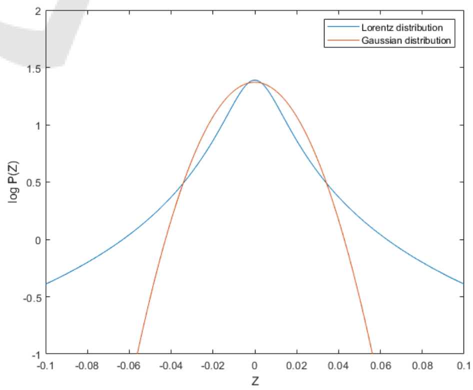

Figure 1: A comparison of the Gaussian distribution with Lorentz distribution. It can be seen that in comparison with Gaussian distribution, the Lorentz distribution is a better fit for the fat-tail distribution observed in real financial market.

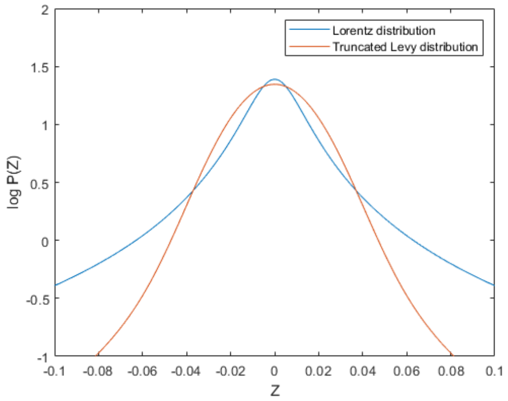

Figure 2: A comparison of Truncated Lévy flight with Lorentz distribution. When the stochastic variable z is relatively large, the Lorentz stable distribution approaches $1\,/\,z^2$ while the truncated Levy flight approaches $1\,/\,z^{1.5}$.

2.2 The Transition Probability Distribution

In the previous part, we have already obtained the Fokker-Planck equation,

$$\frac{\partial P(z,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial z}\left[\sigma\Omega(z)\frac{\partial P(z,t)}{\partial z}+\sigma^2 P(z,t)\right]. \quad (27)$$

We can rewrite the above Fokker-Planck equation as

$$\frac{\partial P(z,t)}{\partial t}=L_{FP}P(z,t) \quad (28)$$

$$L_{FP}\equiv\frac{1}{2}\frac{\partial}{\partial z}\left[\sigma\Omega(z)\frac{\partial}{\partial z}+\sigma^2\right].$$

The price changes of oil future can be defined as

$$Z_{\Delta t}=lnF(t+\Delta t)-lnF(t). \quad (29)$$

As for the probability density of z at time $t+\Delta t$ under the condition that it has the value (t) , we define the conditional probability density as

$$P(z(t+\Delta t),(t+\Delta t)|z(t),t) \quad (30)$$
$$=\langle\delta\big(z(t)-z(t+\Delta t)\big)\rangle.$$

It can be deduced that for initial condition $P(z,t)=P(z(t+\Delta t),(t+\Delta t)|z(t),t)$ should also follow the Fokker-Planck equation (28), namely

$$\frac{\partial P(z(t+\Delta t),(t+\Delta t)|z(t),t)}{\partial t} \quad (31)$$
$$=L_{FP}P(z(t+\Delta t),(t+\Delta t)|z(t),t).$$

We can find one formal solution of the above equation as

$$P(z(t+\Delta t),(t+\Delta t)|z(t),t) \quad (32)$$
$$=e^{L_{FP}\Delta t}\delta\big(z(t+\Delta t)-z(t)\big).$$

Making use of iteration (Dyson, 1949), we can obtain the following equation

$$P(z(t+\Delta t),(t+\Delta t)|z(t),t) \quad (33)$$
$$=\delta\big(z(t+\Delta t)-z(t)\big)[1+\Pi]$$

$$\Pi=\sum_{n=1}^{\infty}\int_t^{t+\Delta t}dt_1\int_t^{t_1}dt_2\cdots\int_t^{t_{n-1}}dt_n\,L_{FP}(z,t_1)\cdots L_{FP}(z,t_n).$$

When the time interval Δt is relatively small, the solution reads

$$P(z(t+\Delta t),(t+\Delta t)|z(t),t) \quad (34)$$
$$=\delta\big(z(t+\Delta t)-z(t)\big)[1+$$
$$L_{FP}(z,t)\Delta t+O(\Delta t)^2].$$

By using the integral presentation of the δ function, we can have

$$\delta(z-z')=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iu(z-z')}du. \quad (35)$$

Thus,

$$P(z(t+\Delta t),(t+\Delta t)|z(t),t)= \quad (36)$$
$$\left[1+\frac{1}{2}\frac{\partial^2}{\partial z(t+\Delta t)^2}\sigma\Omega(z)-\frac{1}{2}\frac{\partial}{\partial z(t+\Delta t)}\left(\sigma\frac{d\Omega(z)}{dz}-\sigma^2\right)\right]$$
$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iu(z(t+\Delta t)-z(t))}du.$$

Replacing $z(t+\Delta t)$ by $z(t)$ in the drift coefficient and diffusion coefficient (Risken, 1984; Wissel, 1979), the above equation can be rewritten as

$$P(z(t+\Delta t),(t+\Delta t)|z(t),t) \quad (37)$$
$$=\frac{1}{\sqrt{2\pi\sigma\Omega(z)\Delta t}}\exp\left(-\frac{\left[Z_{\Delta t}-\frac{1}{2}\left(\sigma\frac{d\Omega(z)}{dz}-\sigma^2\right)\Delta t\right]^2}{2\sigma\Omega(z)\Delta t}\right).$$

Therefore, the probability density distribution of oil future price changes $Z_{\Delta t}$ can be expressed as

$$P(Z_{\Delta t}) \quad (38)$$
$$=\int\frac{1}{\sqrt{2\pi\sigma\Omega(z)\Delta t}}\exp\left(-\frac{\left[Z_{\Delta t}-\frac{1}{2}\left(\sigma\frac{d\Omega(z)}{dz}-\sigma^2\right)\Delta t\right]^2}{2\sigma\Omega(z)\Delta t}\right)\frac{\gamma}{\pi}\frac{1}{\gamma^2+z^2}dz$$

Here $\Omega(z)=\frac{1}{\sigma}\frac{\gamma^2+z^2}{2z}$.

In figure 3, we compare the model we build with Gaussian distribution and the truncated Lévy flight. As can be seen in figure 3, in comparison with Gaussian distribution, the Lorentz transition distribution and truncated Lévy flight can describe the leptokurtic feature of financial assets' price variations better. When the price variations are small, Lorentz transition distribution and truncated Lévy flight perform similarly. When the price variations are relatively large, Lorentz transition distribution has fatter tail than truncated Lévy flight.

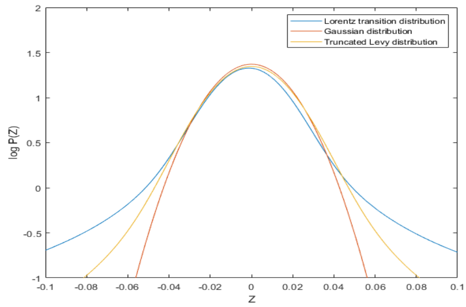

Figure 3: A comparison of distribution models. In general, the Lorentz transition distribution and truncated Lévy flight can describe the leptokurtic feature of financial assets' price variations better than Gaussian distribution. When the price variations are small, Lorentz transition distribution and truncated Lévy flight perform similarly. When the price variations are relatively large, Lorentz transition distribution has fatter tail than the truncated Lévy flight.

We calculate the correlation between the probability of no price change and different time intervals Δt for different parameters γ and σ and present the result in figure 4.

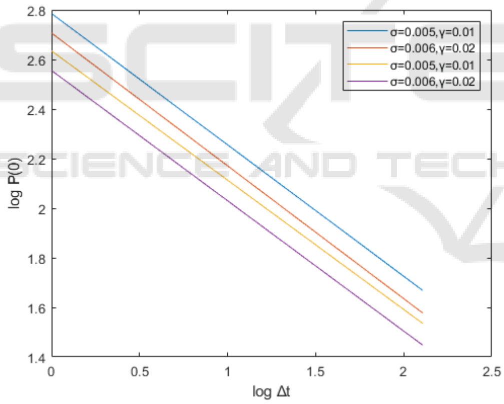

Figure 4: The correlation between the probabilities of no price change $P(Z_{\Delta t} = 0)$ with the time interval Δt.

In figure 5, we plot the Lorentz transition probability distribution of parameters γ =0.012, and σ =0.005463 for different time intervals (Δt =1, 10, 30, 60, 80 and 100 minutes) (in logarithmic form).

As can be seen in figure 5, our newly built Lorentz transition distribution model has leptokurtic distribution and is also mostly symmetric with finite variance. When the time interval Δt increases, the Lorentz transition distribution is likely to spread.

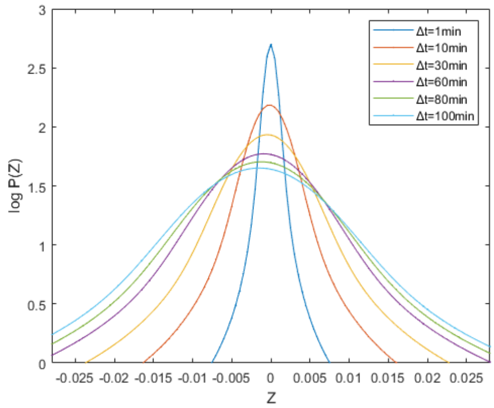

Figure 5: The Lorentz transition probability distribution of parameters γ =0.012, and σ =0.005463 for different time intervals (Δt =1, 10, 30, 60, 80 and 100 minutes). Lorentz transition distribution model has leptokurtic distribution and is also mostly symmetric with finite variance. When the time interval Δt increases, the Lorentz transition distribution is likely to spread.

3 CALIBRATION OF THE MODEL IN CRUDE OIL FUTURE MARKET

Now, we try to analyse statistically the features of the high frequency crude oil future in China's future market by using the new distribution model that we developed at last sections. We obtain the 1-minute high frequency data of crude oil future SC2209 from the Wind database. Our data period ranges from May 11, 2022, to August 4, 2022. We denote the price of crude oil future SC2209 as $F(t)$, and the successive variation of the crude oil future price is denoted as $Z_{\Delta t}$.

The price changes of the crude oil future SC2209 is measured as follows:

$$Z_{\Delta t} = lnF(t + \Delta t) - lnF(t). \quad (39)$$

We calculate the probability distribution $P(Z_{\Delta t})$ of crude oil future price variations for different time values (Δt =1, 10, 30, 60, 80 and 100 minutes). Figure 6 and 7 are semilogarithmic plots of $P(Z_{\Delta t})$ of different time interval Δt. As can be seen in figure 6 and figure 7, the distribution of high frequency crude oil futures have fatter tail than log-normal distribution. The crude oil future price variations cannot be depicted well by a random walk. The distributions are leptokurtic and tend to spread as the time interval Δt increases.

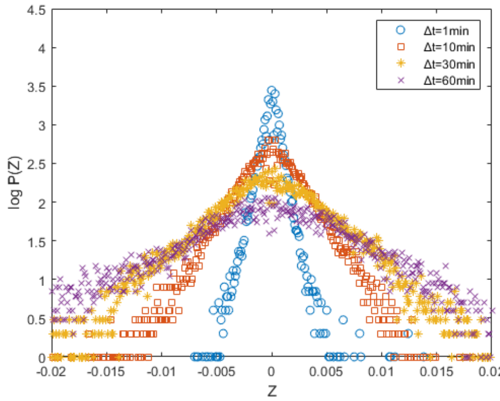

Figure 6: Probability density distributions $P(Z_{\Delta t})$ of crude oil future price variation measured at different time intervals (Δt) 1, 10, 30 and 60 minutes for high-frequency data in China's future market during the period from May 11, 2022, to August 4, 2022.

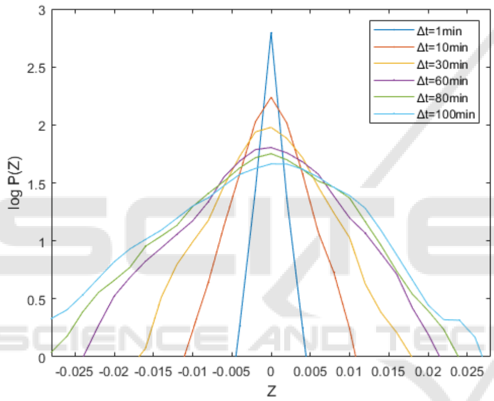

Figure 7: Probability density distributions $P(Z_{\Delta t})$ of crude oil future price variation at different time intervals (Δt) 1, 10, 30, 60, 80 and 100 minutes for high-frequency data in China's future market during the period from May 11, 2022, to August 4, 2022.

In figure 8, we exhibit the correlation between the probability of no price change $P(Z_{\Delta t} = 0)$ and the time interval Δt. The regression of $\log P(Z_{\Delta t} = 0)$ on $\log \Delta t$ gives a coefficient of -0.5019 with 95% confidence bounds.

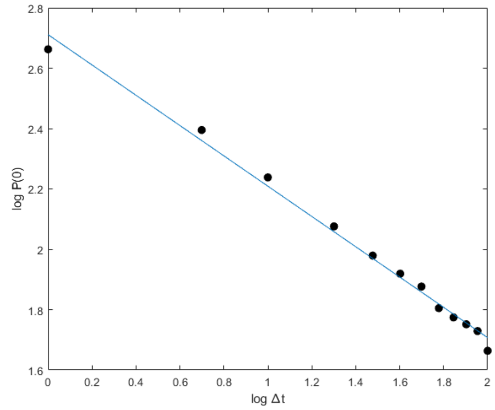

Figure 8: The correlation between the probability of no price change $P(Z_{\Delta t} = 0)$ for the crude oil future SC2209 and different time intervals Δt. The slope of best-fit straight line is -0.5019.

Now, it's time for us to give a best-fit of the high frequency crude oil future SC2209 by using the Lorentz transition probability distribution. We use the Matlab fitting toolbox to find the best-fit parameters for our theoretical model. The best-fit of the Lorentz transition probability distribution has the parameters γ = 0.012, and σ=0.005463 with the high frequency crude oil future SC2209 during the period from May 11, 2022, to August 4, 2022.The R^2 is 0.9849.The fitting results are shown in figure 9. The high R^2 of the fitting result indicate that our theoretical model can describe the characteristics of crude oil future distribution well. As accurate modelling and predicting the volatility of crude oil futures prices has important theoretical and practical significance for investors, and for preventing and mitigating the energy market risks. The Lorentz transition model with its accuracy can be applied empirically.

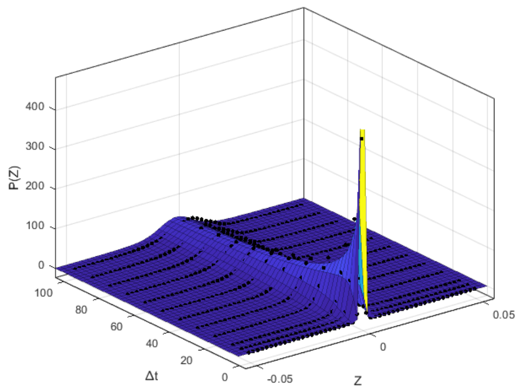

Figure 9: The best-fit of Lorentz transition probability distribution of parameters γ = 0.012, and σ=0.005463 with the crude oil future SC2209 time series during the period from May 11, 2022, to August 4, 2022.The R^2 is 0.9849.

To show the calibration results clearer, in figure 10, we exhibit the best-fit of Lorentz transition distribution with the high frequency crude oil future SC2209 during the period from May 11, 2022, to August 4, 2022 with different time interval Δt=1, 10, 30, 60, 80, 100 minutes, respectively. As can be seen in figure 10, the Lorentz transition probability distribution describes well the price variation distribution of crude oil future SC2209.

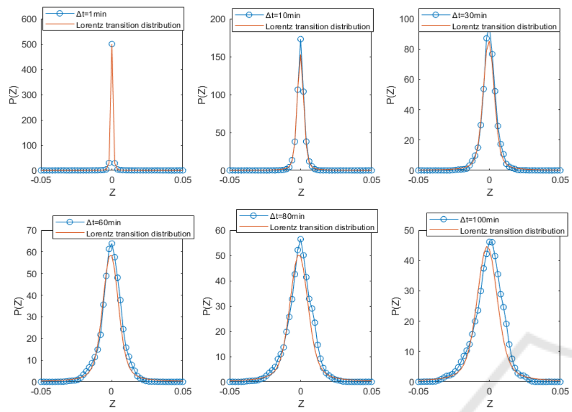

Figure 10: The best-fit of the probability density distributions $P(Z_{\Delta t})$ of price variation for the crude oil future SC2209 with time interval Δt=1, 10, 30, 60, 80, 100 minutes with $\gamma = 0.012$, and σ=0.005463.

4 CONCLUDING REMARKS

With the deepening of the financialization of the oil market, the importance of the oil futures market is highlighted. The high volatility of crude oil futures prices inevitably has a major influence on global financial markets and the healthy development of world economy. Therefore, modelling accurately and estimating the volatility of crude oil futures prices has important theoretical and practical significance for investors and for preventing energy market risks. In this paper, we study the distribution of high- frequency crude oil futures price changes.

Many empirical studies have shown that the distribution of financial assets' price changes has fatter tail than lognormal distribution. Various efforts have been made to improve the modelling of financial assets' price variations. In particular, many scholars have paid special attention to the power-law tail and scaling property of the price variation distributions. In this paper, we try to model the leptokurtic distribution of high frequency crude oil futures using a combination of transition probability distribution and Lorentz stable distribution. The newly built model has fatter tail than log-normal distributions. Using high frequency data of crude oil future in China's future market, we calibrate our theoretical model and have proved the consistency of our theoretical model with the real market.

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REFERENCES

Ballocchi, G., Dacorogna, M. M., Gencay, R., and Piccinato, B. (1999). Intraday statistical properties of Eurofutures. *Derivatives Quarterly,* 6: 28-44.

Black, F. and Scholes, M.,(1973). The pricing of options and corporate liabilities. *Journal of Political Economy,* 81: 637-659.

Blanc, P. Chicheportiche, R., and Bouchaud, J. P. (2014). The fine structure of volatility feedback II: Overnight and intra-day effects. *Physica A: Statistical Mechanics and its Applications*, 402: 58-75.

Bollerslev,T.(1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31(3): 307-327.

Bouchaud J. P. and Potters, M. (2003). *Theory of Financial Risk and Derivative Pricing: from Statistical Physics to Risk Management* . Cambridge University Press, Cambridge, 2nd edition.

Bouchaud, J. P. (2000). Power-laws in economy and finance: some ideas from physics, https://arxiv.org/pdf/cond-mat/0008103v1.

Chicheportiche R. and Bouchaud, J. P. (2014). The fine-structure of volatility feedback I: Multi-scale self-reflexivity. *Physica A: Statistical Mechanics and its Applications*, 410: 174-195.

Dumas, B. Fleming, J., and Whaley, R. (1998). Implied volatility functions: empirical tests. *Journal of Finance*, 53(6) : 2059-2106.

Dyson, F. J. (1949). The radiation theories of Tomonaga, Schwinger, and Feynman. *Phys. Rev.* 75: 486-502.

Engle R. F. and Patton, A.J. (2001). What good is a Volatility Model? *Quantitative Finance*, (2) :237-245.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom Inflation. *Econometrica,* 50(4): 987-1007.

Fouque, J. P. Papanicolaou G., and Sircar, R.(2000). Mean-Reverting Stochastic Volatility. *International Journal of Theoretical and Applied Finance,* 3(1): 101-142.

Francq C. and Zakoian, J. (2010). *GARCH Models, in Structure, statistical inference and financial applications.* Wiley, UK.

Ghashghaie, S., Breymann, W., Peinke, J., Talkner, P., and Dodge, Y. (1996). Turbulent cascades in foreign exchange markets. *Nature*, 381: 767-770.

Gong,X., Wen, F., Xia, X.H., Huang, J., and Pan, B. (2017). Investigating the risk-return trade-off for crude oil futures using high-frequency data. *Applied Energy*, 196: 152-161.

Hohenberg P. C. and Halperin, B. I. (1977). Theory of dynamic critical phenomena. *Reviews of Modem Physics,* 49: 435-478.

Hull J. and White, A. (1987). The pricing of options with stochastic volatilities. *Journal of Finance*, 42: 281-300.

Li, X. Wei, Y. Chen, X. Ma, F. Liang, C. and Chen, W. (2022). Which uncertainty is powerful to forecast crude oil market volatility? New evidence. *Journal of International Money and Finance,* 27(4): 4279-4297.

Mandelbrot, B. B. (1963). The Variation of Certain Speculative Prices. *Journal of Business,* 36: 394-419.

Mantegna R. N. and Stanley, H. E. (1995). Scaling Behaviour in the Dynamics of an Economic Index. *Nature,* 376: 46-49.

Merton, R. (1973). Theory of rational option pricing. *Bell Journal of Economics*, 4:141-183.

Risken, H. (1984). *The Fokker-Planck Equation*. Springer-Verlag, Berlin.

Stanley H. E. and Plerou, V. (2001). Scaling and universality in economics: empirical results and theoretical interpretation. *Quantitative Finance,* 1:563-567.

Voit, J. (2001).*The Statistical Mechanics of Financial Markets.* Springer, Berlin.

Wei, Y., Liu, J., Lai, X., and Hu, Y. (2017). Which determinant is the most informative in forecasting crude oil market volatility: fundamental, speculation, or uncertainty? *Energy Economics,* 68:141-150.

Wei, Y., Wang, Y., and Huang D. (2010). Forecasting crude oil market volatility: further evidence using GARCH-class models. *Energy Economics*, 32:1477-1484.

Wen, F., Gong, X., and Cai, S. (2016). Forecasting the volatility of crude oil futures using HAR-type models with structural breaks. *Energy Economics*, 59:400-413.

Wilmott, P.(1998). *Derivatives.* John Wiley & Sons, New York.

Wissel, C. (1979). Manifolds of Equivalent Path Integral Solutions of the Fokker-Planck Equation. *Z. Physik B*, 35:185-191.

Zhang, Y. Ma, F. and Wang, Y. (2019a). Forecasting crude oil prices with a large set of predictors: can LASSO select powerful predictors? *Journal of Empirical Finance*, 54:97-117.

Zhang, Y., Ma, F., and Wei, Y. (2019b). Out-of-sample prediction of the oil futures market volatility: a comparison of new and traditional combination approaches. *Energy Economics*, 81:1109-1120.