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Review

The stochastic finite element method: Past, present and future

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ABSTRACT

A powerful tool in computational stochastic mechanics is the stochastic finite element method (SFEM). SFEM is an extension of the classical deterministic FE approach to the stochastic framework i.e. to the solution of static and dynamic problems with stochastic mechanical, geometric and/or loading properties. The considerable attention that SFEM received over the last decade can be mainly attributed to the spectacular growth of computing power rendering possible the efficient treatment of large-scale problems. This article aims at providing a state-of-the-art review of past and recent developments in the SFEM area and indicating future directions as well as some open issues to be examined by the computational mechanics community in the future.

Contents

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1. Introduction

It is today widely recognized that computational methods permit the analysis and design of large-scale engineering systems. The considerable influence of inherent uncertainties on system behavior has also led the scientific community to recognize the importance of a stochastic approach to engineering problems. Issues related to uncertainty quantification and its influence on the reliability of the computational models, are continuously gaining in significance. While analytical procedures were most useful for exploring and developing the field of stochastic mechanics, it is now for the computational methods with the help of powerful computing resources and technology to expand and generalize these procedures and hence make them applicable to complex realistic engineering systems. For this reason, the procedures of computational stochastic mechanics are receiving lately considerable attention [125].

Engineering experience has shown that uncertainties are involved not only in the assessment of loading but also in the material and geometric properties of engineering systems. The rational treatment of these uncertainties, achieved by means of probability theory and statistics, cannot be addressed rigorously when following the traditional deterministic approach. This approach, which is almost exclusively used in engineering practice even today, is based on the extreme (minimum, maximum) and mean values of system parameters. In this framework, it is implicitly assumed that the results obtained from a deterministic analysis are representative of all possible scenarios of system loading and strength. This is not true in most cases. It is however, sure that the deterministic approach cannot lead to an "optimum" system design. Stochastic methods do provide this possibility at the expense of increasing the complexity of the system model and, consequently, of the required computational effort for the solution of the problem. The exploitation of the available computational resources (hardware and system software) and the development of enhanced solution algorithms (application software) are therefore of paramount importance in the application of stochastic methods to real-world problems and to their further dissemination to the engineering community.

A powerful tool in computational stochastic mechanics is the stochastic finite element method (SFEM). SFEM is an extension of the classical deterministic FE approach to the stochastic framework i.e. to the solution of stochastic (static and dynamic) problems involving finite elements whose properties are random. From a mathematical point of view, SFEM can be seen as a powerful tool for the solution of stochastic partial differential equations (PDEs) and it is treated as such in numerous studies where convergence and error estimation issues are examined in detail. In fact, these two aspects of SFEM are complementary and inter-dependent. The considerable attention that SFEM received over the last decade can be mainly attributed to the spectacular growth of computational power rendering possible the efficient treatment of large-scale problems. This article aims at providing a state-of-the-art review of past and recent developments in the SFEM area. It also aims at indicating future directions as well as some open issues to be examined in the future.

A fundamental issue in SFEM is the modeling of the uncertainty characterizing the system parameters (input). This uncertainty is quantified by using the theory of stochastic functions (processes/fields). The first half of this article (Section 2) is thus devoted to methods existing in the literature for the simulation (generation of sample functions) of stochastic processes and fields. In the second half (Section 3), a thorough description of the available variants of SFEM is provided. The most important techniques used for the discretization of stochastic fields are first introduced and a discussion on their performance is made based on results reported in the literature. The issue of using two different meshes for the discretization of the system and the stochastic field, respectively, is also examined. The formulation of the stochastic finite element matrix, which is the key-point of the method, is then presented. For the calculation of system response statistics, two methods are discussed in detail: Monte Carlo simulation (MCS) along with the perturbation approach (based on a Taylor series expansion of the response vector) and their recent variants. Section 3 closes with the spectral stochastic finite element method (SSFEM), which is a specific formulation of SFEM based on the expansion of the response vector in polynomial chaos series. Some recent and promising developments concerning this formulation are pointed out. The advantages and drawbacks of the SSFEM are critically reviewed and summarized from a variety of applications existing in the literature. Finally, issues related to specialized software development are discussed.

The author hopes that this article will serve as a useful source of information to scientists and engineers interested by SFEM and will help to further disseminating the method for the solution of real-world problems which are inherently influenced by a number of uncertain parameters during their life time.

2. Uncertainty modeling: representation of stochastic processes and fields

The first step in the analysis of uncertain systems (in the framework of SFEM) is the representation of the input of the system. This input usually consists of the mechanical and geometric properties as well as of the loading of the system (left and right hand side of the equilibrium equation, respectively). Characteristic examples are the Young modulus, Poisson ratio, yield stress, cross section geometry of physical systems, material and geometric imperfections of shells, earthquake loading, wind loads, waves etc. A convenient way for describing these uncertain quantities in time and/or space has always been the implementation of stochastic processes and fields, the probability distribution and correlation structure of which can be defined through experimental measurements. However, in most cases, due to the lack of relevant experimental data, assumptions are made regarding these probabilistic characteristics. Two main categories of stochastic processes and fields can be defined based on their probability distribution: Gaussian and non-Gaussian. A detailed review of the existing techniques for the simulation of Gaussian and non-Gaussian stochastic processes and fields along with their respective applications in computational stochastic mechanics, is presented in the next two sub-sections. For the sake of brevity, the presentation is made for stochastic fields (variable in space). The same expressions hold for stochastic processes but with time t as the independent variable.

2.1. Simulation methods for Gaussian stochastic processes and fields

Despite the fact that most of the uncertain quantities appearing in engineering systems are non-Gaussian in nature (e.g. material, geometric properties, wind, seismic loads), the Gaussian assumption is often used due to its simplicity and the lack of relevant experimental data. Furthermore, Gaussian random fields occur naturally in applications as a result of the central limit theorem and are the model of maximum entropy when only information on the second-order moments is available [172]. From the wide variety of methods developed for the simulation of Gaussian stochastic processes and fields, two are most often used in applications: the spectral representation method [167,168] and the

Karhunen–Loève (K–L) expansion [67]. A unified approach for generating Gaussian random field simulation methods (including spectral representation and K–L expansion) has been proposed in [144].

2.1.1. The spectral representation method

In the general case, the spectral representation method expands the stochastic field $f(x)$ as a sum of trigonometric functions with random phase angles and amplitudes. The version having only random phase angles is adopted in most applications because it leads to sample functions that are ergodic in the mean value and autocorrelation [78]. The amplitudes are then deterministic and depend only on the prescribed power spectrum of the stochastic field:

$$\hat{f}^{(i)}(x) = \sum_{n=0}^{N-1} A_n \cos(\kappa_n x + \phi_n^{(i)}), \tag{1}$$

where $A_n = \sqrt{2S_{ff}(\kappa_n)\Delta\kappa}$, $\kappa_n = n\Delta\kappa$, $\Delta\kappa = \kappa_u/N$ and $n = 0,1,2,\ldots, N-1$. It also holds that $A_0 = 0$ or $S_{ff}(\kappa_0 = 0) = 0$. The parameter κ_u is a cut-off wave number defining the "active region" of the power spectral density function (SDF) $S_{ff}(\kappa)$ of the stochastic field. Since κ_u has a specific value, the wave number step $\Delta\kappa \to 0$ as $N \to \infty$. In addition, for a given number of terms N, $\Delta\kappa$ is constant, while $\phi_0^{(i)}, \phi_1^{(i)}, \ldots, \phi_{N-1}^{(i)}$ are independent random phase angles uniformly distributed in the range $[0, 2\pi]$ and are produced by a random number generator.

Each sample function given by Eq. (1) has the following properties [167]:

1. It is asymptotically a Gaussian stochastic field as $N \to \infty$ due to the central limit theorem.
2. Its mean value and autocorrelation function are identical to the corresponding targets as $N \to \infty$.
3. Under the condition $A_0 = 0$ or $S_{ff}(\kappa_0 = 0) = 0$, it can be shown that $\hat{f}^{(i)}(x)$ is periodic with period $T_0 = 2\pi/\Delta\kappa$.

Weakly ergodic sample functions are produced when both the phase angles and amplitudes are random [78]. This is the main reason for which spectral representation with random phase angles and deterministic amplitudes is used in most applications.

Spectral representation algorithms are nowadays available covering various kinds of Gaussian stochastic fields: multi-dimensional, multi-variate (vector), non-homogeneous e.g. [33,78,93,105,167–169]. The simulation of the non-homogeneous fields is based on the notion of the evolutionary power spectrum [105,171]. The computational cost of digital generation of homogeneous Gaussian sample functions can be drastically reduced by using the fast Fourier transform technique (FFT). Spectral representation is even useful for the simulation of non-Gaussian fields. There exist a wide variety of methods related to the translation concept (memory-less non-linear transformation of a Gaussian field to a non-Gaussian one) which are using this technique for the generation of sample functions of the underlying Gaussian field e.g. [16,21,34,80,100,147] (see Section 2.2). Spectral representation has also been successfully implemented in the framework of Monte Carlo simulation (MCS) for the solution of realistic problems with the stochastic finite element approach e.g. [4,20,99,133,135–138,145,146,173].

2.1.2. The Karhunen–Loève (K–L) expansion

The K–L expansion can be seen as a special case of the orthogonal series expansion where the orthogonal functions are chosen as the eigenfunctions of a Fredholm integral equation of the second kind with the autocovariance function as kernel (covariance decomposition) [67,195]:

$$\hat{f}(x) = \bar{f}(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}\xi_n\phi_n(x), \tag{2}$$

$$\int_D C_{ff}(x_1, x_2)\phi_n(x_1)dx_1 = \lambda_n\phi_n(x_2), \tag{3}$$

where $\bar{f}(x)$ is the mean of the field (usually considered as equal to zero), λ_n and $\phi_n(x)$ are the eigenvalues and eigenfunctions of the autocovariance function $C_{ff}(x_1,x_2)$, respectively, ξ_n is a set of uncorrelated random variables and N is the number of K–L terms. In the case of zero-mean, homogeneous Gaussian stochastic fields, the autocovariance function depends only on the distance $\xi = x_2 - x_1$ between two points and coincides with the autocorrelation function R_{ff} i.e. $C_{ff}(x_1,x_2) = C_{ff}(\xi) = R_{ff}(\xi)$.

The K–L expansion offers a unified framework for the simulation of homogeneous and non-homogeneous stochastic fields, although some problems have been identified regarding the homogeneity of the generated sample functions [55,83,174,176]. It is particularly suitable for the representation of strongly correlated stochastic fields where only a few terms, corresponding to the N larger eigenvalues, are required in order to capture most of the random fluctuation of the field (see below comments on the solution of the Fredholm integral equation and Fig. 1). This approach is usually combined in the literature with the polynomial chaos (PC) approximation for the calculation of the response variability of uncertain finite element systems e.g. [1–3,19,22,28,29,41,66,67,94,108,113,187,189]. The combination is called the spectral stochastic finite element method (SSFEM). In this case, the uncertain (Gaussian) input parameters are modeled via the K–L expansion while the probabilistic characteristics of system response are determined using the PC decomposition (see Section 3 for a detailed presentation of the SSFEM). There also exist some cases where the K–L expansion has been implemented in the framework of MCS e.g. [112,157–159]. It should be noted that for homogeneous random fields defined over an infinite domain, the K–L expansion reduces, theoretically, to the spectral representation method [65,88].

Despite its theoretical importance, the implementation of K–L expansion is often hindered by the difficulty encountered for solving the Fredholm integral equation. As analytic solutions of this integral equation are only known for simple geometries and special forms of the autocovariance function, special numerical treatment is required in the case of realistic problems involving complex domains. These numerical methods (e.g. Galerkin) usually lead to dense matrices that are very costly to compute and solve the corresponding equations. It is important to note that the accuracy in the computation of the eigenpairs of the autocovariance function strongly influences the efficiency of K–L series

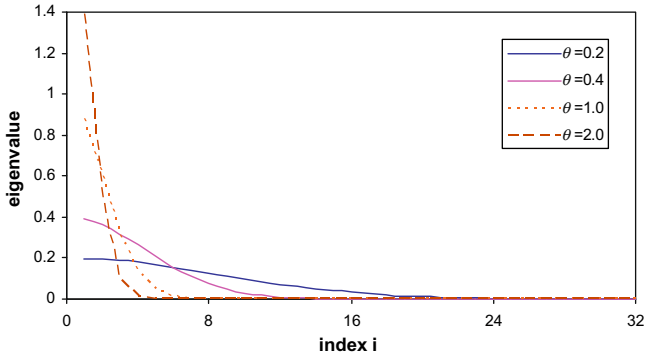

Fig. 1. Eigenvalue decay in K–L expansion for scale of fluctuation θ=0.2, 0.4, 1.0, 2.0: case of square exponential autocovariance. Reprinted from [174], Copyright © 2007, with permission from Elsevier.

[88,140,164,174,176]. Enhanced methods for the solution of the Fredholm integral equation have been proposed in [140,58,164]. The first method is a mixed wavelet-Galerkin scheme replacing the conventional bases (polynomial, trigonometric, etc.) by wavelets that exhibit a number of desired properties which improve the performance of the Galerkin method for the required solution of integral equations. The second approach is a generalized fast multi-pole accelerated Krylov eigen-solver applicable to general, piecewise analytic correlation kernels and leading to significant speed up in some specific cases [164].

A comprehensive comparison between the spectral representation and K–L expansion methods can be found in [83,88,174] where it is shown that strongly correlated stochastic fields with smooth autocovariance function may be easier to simulate with the K–L expansion when using a small number of terms $N(\leqslant 20)$ in Eqs. (1) and (2), (Fig. 2). However, the performance of spectral representation improves by increasing the number of retained terms. In most cases, an absolute minimum of 128 terms must be used for N in the spectral representation method in order to ensure some level of convergence to Gaussianity through the central limit theorem. Finally, the homogeneity and ergodicity of sample functions generated by the K–L series are questionable and its computational performance less satisfactory than that of spectral representation.

2.1.3. Other series expansion methods

In addition to the spectral representation and K–L expansion, there also exist some other methods for the simulation of Gaussian stochastic processes and fields such as the turning bands method (TBM), the autoregressive moving average (ARMA)–autoregressive (AR) models, the optimal linear estimation (OLE) and the expansion optimal linear estimation (EOLE) methods. The TBM involves the simulation of random fields in two or higher dimensions by using a sequence of one-dimensional fields along lines crossing the domain (Fig. 3). The formulation of TBM depends on knowledge of the 1D autocorrelation function $R_1(\xi)$. If this function is known, the line fields can be produced using some efficient 1D algorithm (e.g. FFT). The autocorrelation function $R_1(\xi)$ is chosen such that the multi-dimensional correlation structure $R_n(\boldsymbol{\xi})$ is reflected over the ensemble. Mantoglou and Wilson [109] suggested the computation of $R_1(\xi)$ through an integral equation and supply explicit solutions, for either the equivalent 1D autocorrelation function or for the 1D SDF, for a variety of multi-dimensional correlation structures. The TBM produces accurate results only when a large number of lines are used at the expense of decreased computational efficiency [52]. The ARMA–AR models permit the simulation of both stationary and non-stationary stochastic processes using recursive expressions for the calculation of some coefficients relating a Gaussian white noise process with the process to be simulated e.g. [35]. The degree of success of such time-series generation is usually measured in terms of the closeness of the prescribed target autocorrelation function and the corresponding sample autocorrelation function computed from the generated sample functions (Fig. 4).

The OLE method was introduced in [102] and is sometimes referred to as the Kriging method. It is a special case of the method of regression on linear functionals. In the context of OLE, the approximated field $\hat{f}(\mathbf{x})$ is defined by a linear function of nodal values $\mathbf{f} = \{f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n)\}$ as follows:

$$\hat{f}(\mathbf{x}) = \alpha(\mathbf{x}) + \sum_{n=1}^{N} b_n(\mathbf{x}) f_n = \alpha(\mathbf{x}) + \mathbf{b}^T(\mathbf{x}) \cdot \mathbf{f}. \tag{4}$$

The functions $\alpha(\mathbf{x})$ and $b_n(\mathbf{x})$ are determined by minimizing in each point $\mathbf{x}$ the variance of the error $\mathrm{Var}[f(\mathbf{x}) - \hat{f}(\mathbf{x})]$ under the condition that $\hat{f}(\mathbf{x})$ is an unbiased estimator of $f(\mathbf{x})$ in the mean i.e. $E[f(\mathbf{x}) - \hat{f}(\mathbf{x})] = 0$. The EOLE method is an extension of OLE

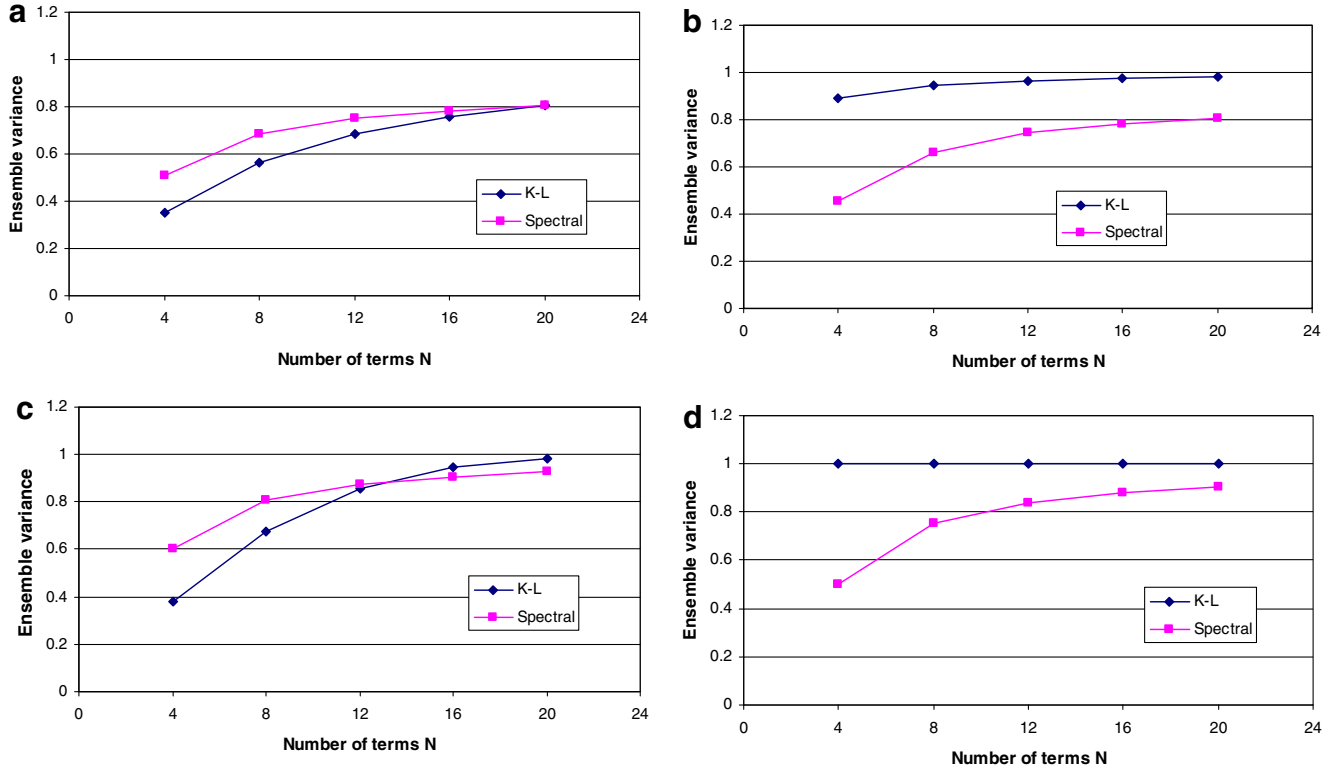

Fig. 2. Convergence of spectral representation and K–L expansion to the target variance as a function of the number of retained terms N in Eqs. (1) and (2): (a) exponential autocovariance, θ=0.2; (b) θ=2.0; (c) square exponential autocovariance, θ=0.2; (d) θ=2.0. Reprinted from [174], Copyright © 2007, with permission from Elsevier.

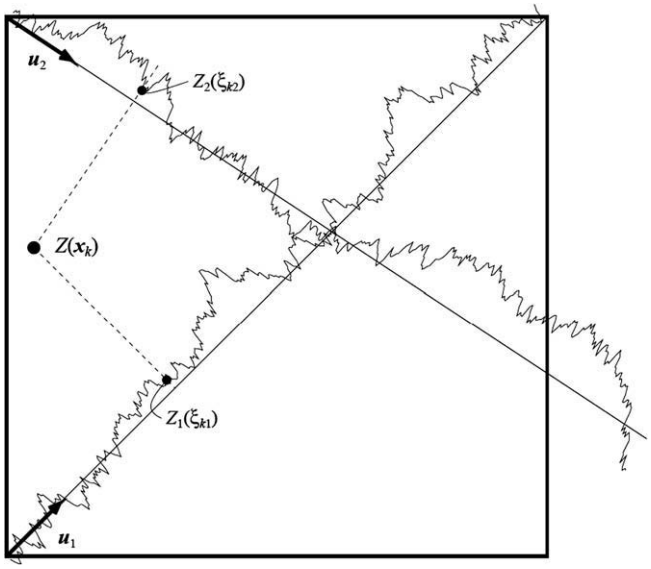

Fig. 3. Illustration of the TBM concept: contributions from the line process $Z_i(\xi_i)$ at the closest point are summed into the field $Z(\boldsymbol{x})$ at $\boldsymbol{x}_k$. Reprinted from [52] with permission from ASCE.

using a spectral representation of the vector of nodal variables **f**. As in the K–L expansion, the series can be truncated after N terms and the eigenvalues λ_n are sorted in descending order. A comprehensive comparison between the K–L expansion and the EOLE method can be found in [102,176]. In these papers, variance error estimators are provided, which allow checking the accuracy of the random field discretization for different correlation structure, scale and order of expansion. It appears that even in the case of the exact K–L expansion (i.e. when an exponential covariance kernel is used) the K–L maximal error is not always smaller than the EOLE error for a given number of retained terms (Fig. 5). In addition, the K–L point-wise variance error estimator for a given order of expansion is smaller than the EOLE error in the interior of the discretization domain but larger at the boundaries. However, the K–L approach provides the lowest mean error over the domain (Fig. 6).

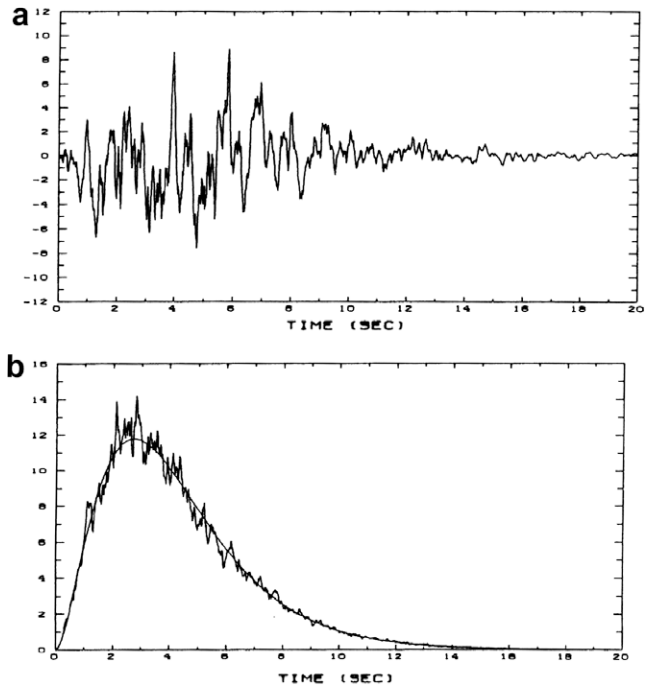

Fig. 4. (a) Sample function of a non-stationary process generated using an ARMA model; (b) target (continuous line) and sample (oscillating line) autocorrelation functions computed from the generated sample function of Fig. 4a. Reprinted from [35] with permission from ASCE.

2.2. Simulation methods for non-Gaussian stochastic processes and fields

The problem of simulating non-Gaussian stochastic processes and fields has received considerable attention recently in the field of stochastic mechanics. This is due to the fact that several quantities arising in practical engineering problems (e.g. material, geometric properties, soil properties, wind, wave, earthquake loads) exhibit non-Gaussian probabilistic characteristics. Non-Gaussian fields are also useful for the determination of spectral-distribution-free upper bounds of the response variability of stochastic systems [134]. In particular, the simulation of highly skewed narrow-banded stochastic processes and fields is well recognized today as a testbed that reveals the limitations of the existing simulation methods [34]. Since all the joint multi-dimensional density functions are needed to fully characterize a non-Gaussian stochastic field, a number of studies have been focused on producing a more realistic definition of a non-Gaussian sample function from a simple transformation of some underlying Gaussian field with known second-order statistics e.g. [16,21,34,80,100,110,141,142,147].

Simulation methods for non-Gaussian stochastic processes and fields can be grouped into two main categories. Those which seek to produce sample functions matching the prescribed power spectral density function (SDF) and lower-order statistics (mean,

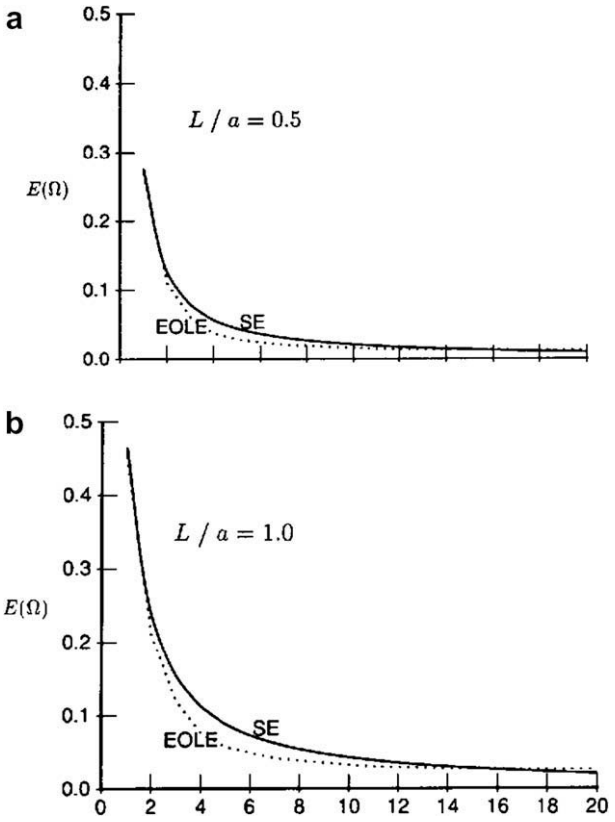

Fig. 5. Comparison of errors for K–L (denoted as SE) and EOLE methods with exponential autocorrelation. Reprinted from [102] with permission from ASCE.

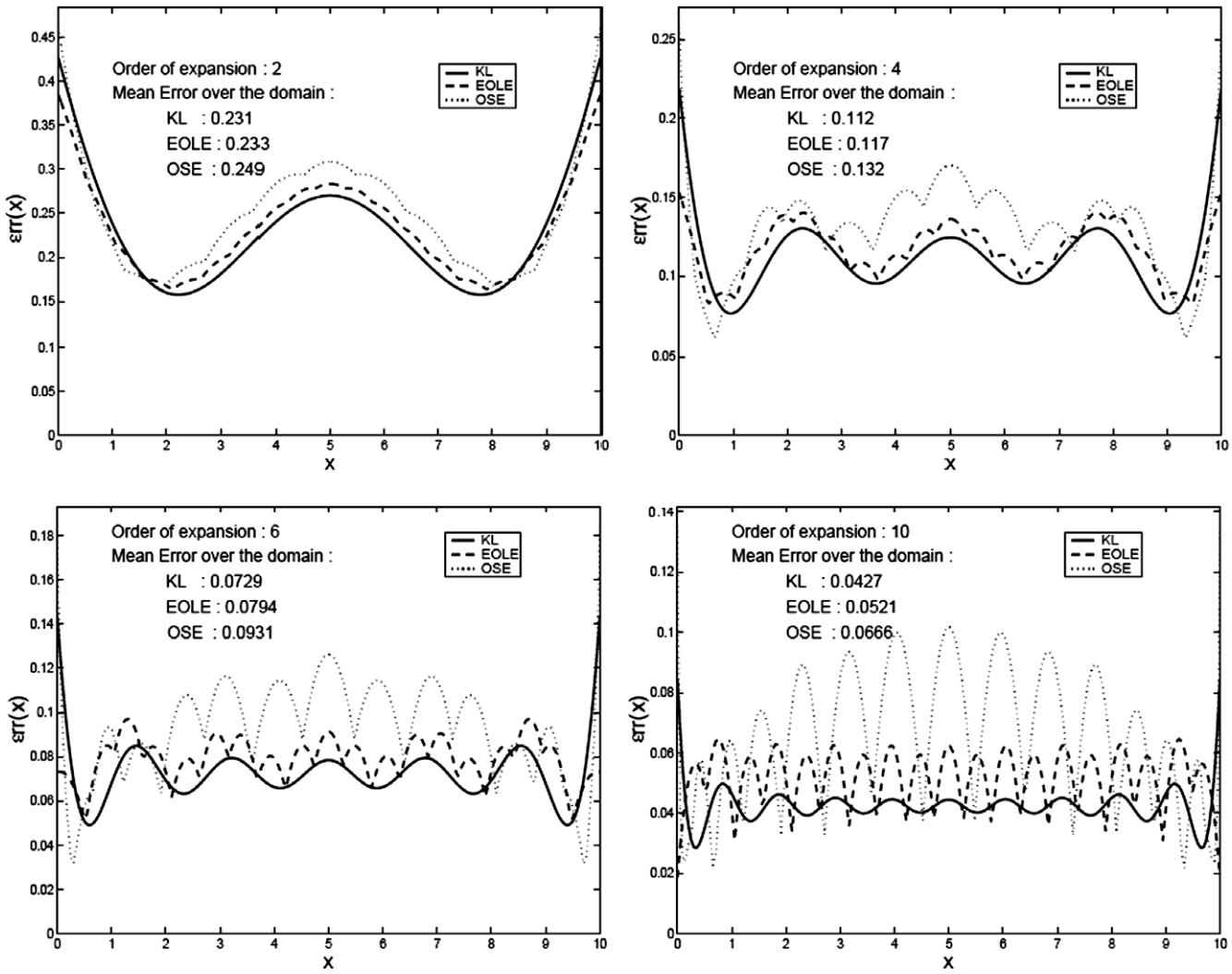

Fig. 6. Point-wise estimator for variance error, represented for different discretization schemes and orders of expansion (exponential autocorrelation function) [176].

variance, skewness and kurtosis) of a target stochastic field e.g. [85,86] and those seeking to generate sample functions compatible to complete probabilistic information. The first type of methods are suitable for the simulation of wind and wave loads, for which generation of non-Gaussian sample functions according to prescribed lower-order moments will provide accurate results for the stochastic response [86]. However, sample functions having only the prescribed lower moments are not sufficient for the successful solution of problems where the accurate characterization of the tails of the distributions is of importance (e.g. soil liquefaction [145,146]). This is due to the potential non-uniqueness of the marginal probability distribution of realizations of a non-Gaussian field that is defined only by its lower-order moments. Research studies have shown that the occurrence of soil liquefaction is significantly influenced by the tails of the marginal probability distributions of the random soil properties used in the analysis. Different marginal probability distributions with similar lower-order moments, but dissimilar tails, will lead to widely varying amounts of observed soil liquefaction as described in detail in [146]. When dealing with such types of problems, the use of methods belonging to the second category is required.

2.2.1. Correlation distortion methods

2.2.1.1. Translation fields. The methods falling into the second category are more challenging in the sense that they seek to generate sample functions compatible to complete probabilistic information, namely the marginal probability distribution and the SDF of the stochastic field. The correlation distortion methods [21,34,80,100,147] are the main representatives of this group. In all these approaches, the generation of a zero-mean homogeneous non-Gaussian field with SDF $S_{ff}^{T}(\boldsymbol{\kappa})$ is based on the translation field concept [77] i.e. on a nonlinear memory-less transformation of an underlying zero-mean homogeneous Gaussian field with SDF $S_{gg}(\boldsymbol{\kappa})$:

$$f(\mathbf{x}) = F^{-1} \cdot \Phi[g(\mathbf{x})], \tag{5}$$

where Φ is the standard Gaussian cumulative distribution function and F is the non-Gaussian marginal cumulative distribution function of $f(\mathbf{x})$. Methods [21,34,100,147] are iterative because their objective is to match the prescribed probabilistic characteristics at the individual sample level through spatial averaging, whereas the technique described in [80] requires only one step because its objective is to meet the same goal through ensemble averaging.

An important issue arising in the context of translation fields is that the choice of the marginal distribution of $f(\mathbf{x})$ imposes constrains to its correlation structure [79–81]. In other words, F and $S_{ff}^{T}(\boldsymbol{\kappa})$ (or $R_{ff}^{T}(\boldsymbol{\xi})$) have to satisfy a specific compatibility condition derived directly from the definition of the autocorrelation function of the translation field:

$$R_{ff}^{T}(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{-1}[\Phi(g_1)]F^{-1}[\Phi(g_2)]\cdot\phi[g_1,g_2;R_{gg}(\xi)]dg_1dg_2, \quad (6)$$

where $g_1 = g(\mathbf{x})$, $g_2 = g(\mathbf{x}+\xi)$, $\phi[g_1,g_2;R_{gg}(\xi)]$ denotes the joint density of $\{g_1,g_2\}$ and ξ is the space lag. If F and $S_{ff}^{T}(\kappa)$ are proven to be incompatible through Eq. (6), i.e. if $R_{ff}^{T}(\xi)$ has certain values lying outside a range of admissible values and/or the solution $R_{gg}(\xi)$ is not positive definite and therefore not admissible as an autocorrelation function, there is no translation field with the prescribed characteristics. In this case, one has to resort to translation fields that match the target marginal distribution and/or the SDF approximately [80]. It must be noted that translation fields have a number of useful properties such as the analytical calculation of crossing rates and extreme value distributions [77,80]. This class of random fields can be used to adequately represent various non-Gaussian phenomena e.g. the peak dynamic response distribution of nonlinear beams, the loads encountered during the atmospheric re-entry of a spacecraft or the spatial variability of the crystallographic orientation in random polycrystalline microstructures [56,6,84].

2.2.1.2. Methods extending the translation field concept. The aforementioned issue arising in the context of translation fields is amended by using (i) an iterative procedure involving the repeated updates of the SDF of the underlying Gaussian stochastic field $g(\mathbf{x})$ and, (ii) an extended empirical non-Gaussian to non-Gaussian mapping leading to the generation of a non-Gaussian field $f(\mathbf{x})$ with the prescribed F and $S_{ff}^{T}(\kappa)$ [34,100]. Yamazaki and Shinozuka [192] defined the iterative procedure in such a way that when the final realization of $g(\mathbf{x})$ is generated according to the updated $S_{gg}(\kappa)$ and then mapped to $f(\mathbf{x})$ via Eq. (5), the resulting non-Gaussian sample function will have both the prescribed marginal probability distribution and SDF. The formula used to update $S_{gg}(\kappa)$ is the following:

$$S_{gg}^{(j+1)}(\kappa) = \frac{S_{gg}^{(j)}(\kappa)}{S_{ff}^{(j)}(\kappa)} S_{ff}^{T}(\kappa). \quad (7)$$

This algorithm provides fairly good results for slightly non-Gaussian fields with broad-banded SDFs. However, as observed by Deodatis and Micaletti [34], there is a limitation with regard to the simulation of highly skewed non-Gaussian stochastic fields. In this case, the resulting non-Gaussian sample functions have the prescribed SDF but their marginal probability density function (PDF) differs significantly from the target one (Fig. 7). This limitation is due to the specific form of the updating formula of Eq. (7). The major problem is that, after the first iteration, the underlying Gaussian field is no more Gaussian and homogeneous for reasons thoroughly explained in [34].

Deodatis and Micaletti [34] proposed an algorithm having the same structure as that of Yamazaki and Shinozuka but with several improvements:

(i) Improved updating scheme of $S_{gg}(\kappa)$

$$S_{gg}^{(j+1)}(\kappa) = \left[\frac{S_{ff}^{T}(\kappa)}{S_{ff}^{(j)}(\kappa)}\right]^{\alpha} S_{gg}^{(j)}(\kappa). \quad (8)$$

From extensive numerical experimentation, the authors concluded that a value of α equal to 0.3 gives the better results in terms of convergence.

(ii) Extended empirical non-Gaussian to non-Gaussian mapping

$$f(\mathbf{x}) = F^{-1}\cdot F^{*}[g(\mathbf{x})], \quad (9)$$

F^{*} is the empirical marginal probability distribution of $g(\mathbf{x})$ updated in each step.

(iii) Use of frequency shifting in order to circumvent some convergence issues arising around $\kappa = 0$.

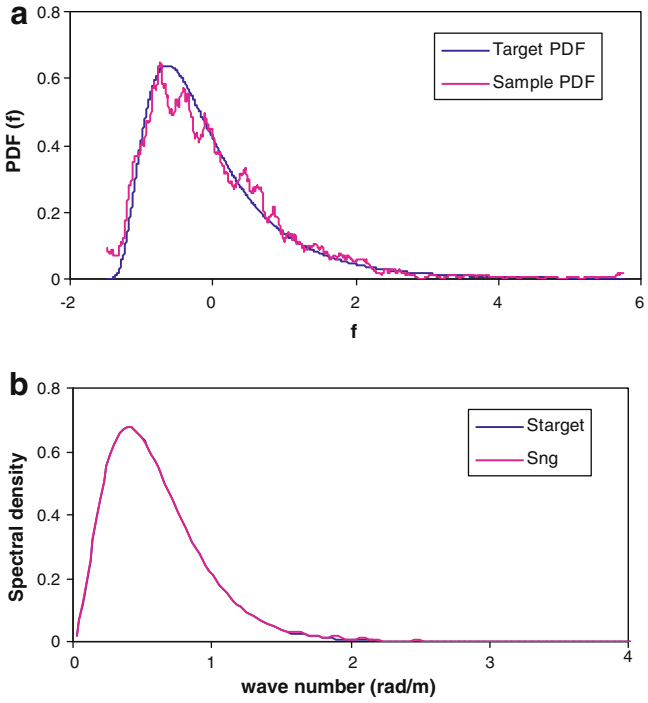

Fig. 7. Comparison of target and sample PDF and SDF of a moderately skewed (skewness = 1.838) lognormal stochastic field produced using the Yamazaki–Shinozuka algorithm (correlation length parameter $b = 5$).

As a result of the mapping of Eq. (9), the generated non-Gaussian fields are not translation fields in a strict sense but match the prescribed characteristics (PDF and SDF) with remarkable accuracy (Fig. 8). The algorithm becomes computationally demanding in the case of non-Gaussian fields with large skewness and narrow-banded spectra (see the numerical examples of [100]).

Phoon et al. [141] used the K–L expansion for the simulation of non-Gaussian fields together with an iterative mapping scheme to fit the target marginal distribution function. The method offers a unified framework for the simulation of homogeneous and non-homogeneous stochastic fields and has been further improved in order to cover the case of highly skewed distributions [142].

Efficient variations of the aforementioned procedures have been proposed in [100,110]. Masters and Gurley [110] presented a general non-Gaussian cumulative distribution function (CDF) mapping technique in which the generated sample functions converge to both the target PDF and SDF through iterative corrections to both probability and spectral content. As in [141], the distortion to the PDF is used as a criterion to determine the need for further iteration. This technique makes use of the spectral representation method for the generation of the underlying Gaussian sample functions and achieves a good matching of the target non-Gaussian PDF in many cases. However, it should be mentioned that this approach is accurate and efficient for stochastic fields having distributions close to the Gaussian. An interesting variant of [34] has been introduced in [100]. This computationally efficient technique is again translation-based and uses an iterative procedure similar to that of [34] for the generation of homogeneous non-Gaussian fields with the prescribed characteristics. However, the function fitting ability of neural networks (NN) is exploited for the approximation of the SDF of the underlying Gaussian field in a very small number of iterations and the algorithm is remarkably efficient even in the limiting case of narrow-banded fields with very large skewness (Fig. 9 – Table 1). The convergence criterion used in this method is the following:

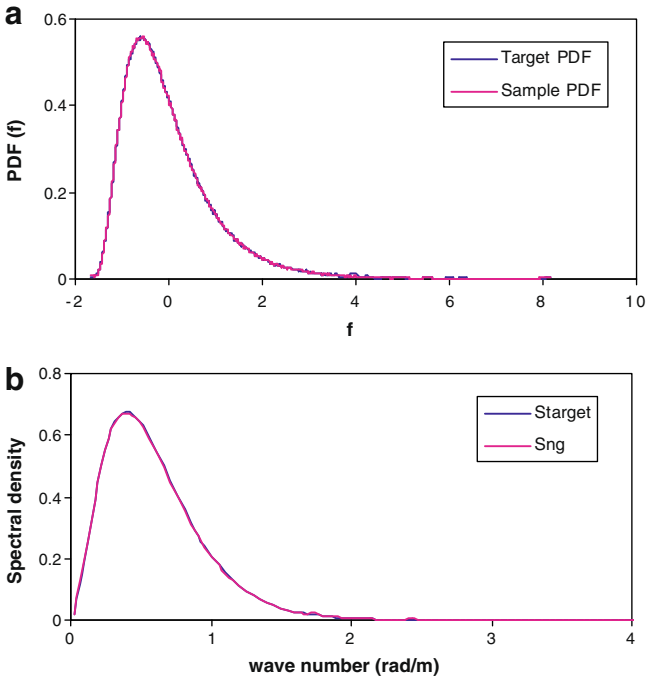

Fig. 8. Comparison of target and sample PDF and SDF of the moderately skewed lognormal stochastic field of Fig. 7 produced using the Deodatis–Micaletti algorithm. Reprinted from [100], Copyright © 2005, with permission from Elsevier.

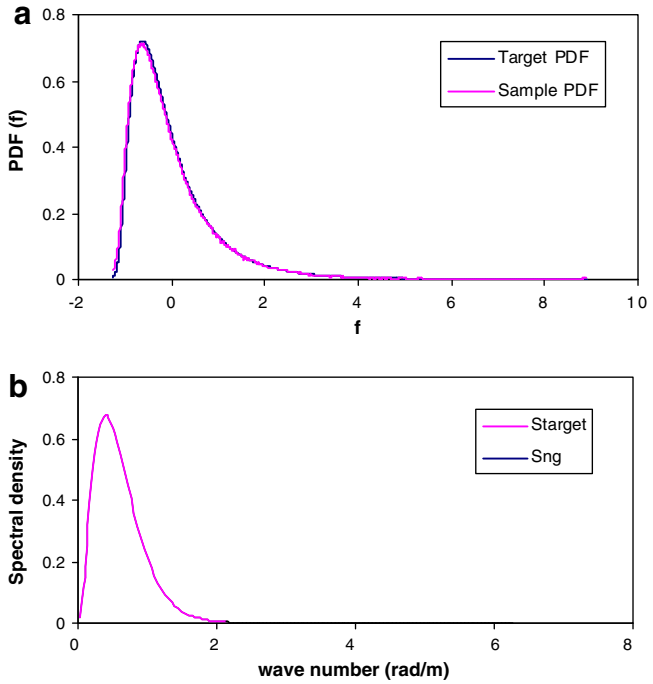

Fig. 9. Comparison of target and sample PDF and SDF of a highly skewed (skewness = 2.763) lognormal stochastic field produced using the NN-based enhanced hybrid method (EHM) (correlation length parameter $b = 5$). Reprinted from [100], Copyright © 2005, with permission from Elsevier.

$$E(w) = \frac{1}{2}\sum_{j=1}^{N}[S_{ff}(\kappa_j) - S_{ff}^{T}(\kappa_j)]^2, \tag{10}$$

and the neural network weights w are adjusted at every iteration.

Recently, another algorithm for the simulation of strongly non-Gaussian stochastic fields has been proposed in [16]. It involves an iterative scheme generating sample functions that match a prescribed non-Gaussian marginal distribution and a prescribed SDF. The simulated field possesses all the properties of translation fields. The method also determines the SDF of an underlying Gaussian field according to translation field theory. Several numerical examples demonstrate the capabilities of the methodology and determine the limits of its applicability. This is the latest development in a class of simulation algorithms based on the translation field concept.

2.2.1.3. Binary random fields – simulation of random media. As mentioned before, the simulation of highly skewed narrow-banded stochastic processes and fields is a highly computationally demanding task. Limiting case of this class of random fields are the binary fields, which are often used in modeling two-phase random media. A translation model for non-stationary, non-Gaussian random processes has been developed in [53] and successfully applied to the simulation of a 1D binary process representing a two-phase functionally graded composite material. However, translation models are in many cases inadequate to accurately describe the micro-structural features of random media as the requirement of positive definiteness is not often met [76]. An alternative methodology for the simulation of binary random fields according to their prescribed autocorrelation function has been lately introduced by Koutsourelakis and Deodatis [97]. It essentially contains two parts. In the first part, an algorithm is introduced to obtain samples of a binary field from a nonlinear transformation with memory of a Gaussian field. In the second step, an iterative algorithm is implemented allowing the determination of the probabilistic characteristics of the underlying Gaussian field, so that the resulting binary field has a prescribed autocorrelation function (Fig. 10). The method has a wide range of applicability and its computational cost is relatively small. The accurate modeling of constituent properties and microstructure of random heterogeneous materials (e.g. concrete, geomaterials, composites) using non-Gaussian stochastic fields has been addressed by a large number of researchers e.g. [6,38,39,54,60,73,75,76,84,90,97,114,117,129,152,180,193]. As it is stated in a recent state-of-the-art article [75], a joint experimental-stochastic mechanics research is imperative in this area in order to validate the stochastic models and improve the safety and reliability of engineering material systems.

Table 1
Computational performance of D–M and EHM algorithms for the highly skewed non-Gaussian field of Fig. 9. Reprinted from [100], Copyright © 2005, with permission from Elsevier.

Method	Iterations	Time (s)
Deodatis–Micaletti	29,279	146
EHM–SD	82	2.0
EHM–CG (Fletcher and Reeves)	25	0.6
EHM–Quickprop	45	1.2
EHM–Rprop	32	0.8

2.2.2. Methods based on polynomial chaos (PC) expansion

Sakamoto and Ghanem [156] proposed an alternative way to generate sample functions of non-Gaussian non-stationary stochastic processes according to their prescribed (non-stationary) marginal PDF and correlation function with the expansion of the non-Gaussian process at discrete points using classical polynomial chaos (PC) decomposition:

$$u(x) = \sum_{j=0}^{P} u_j(x)\Psi_j, \tag{11}$$

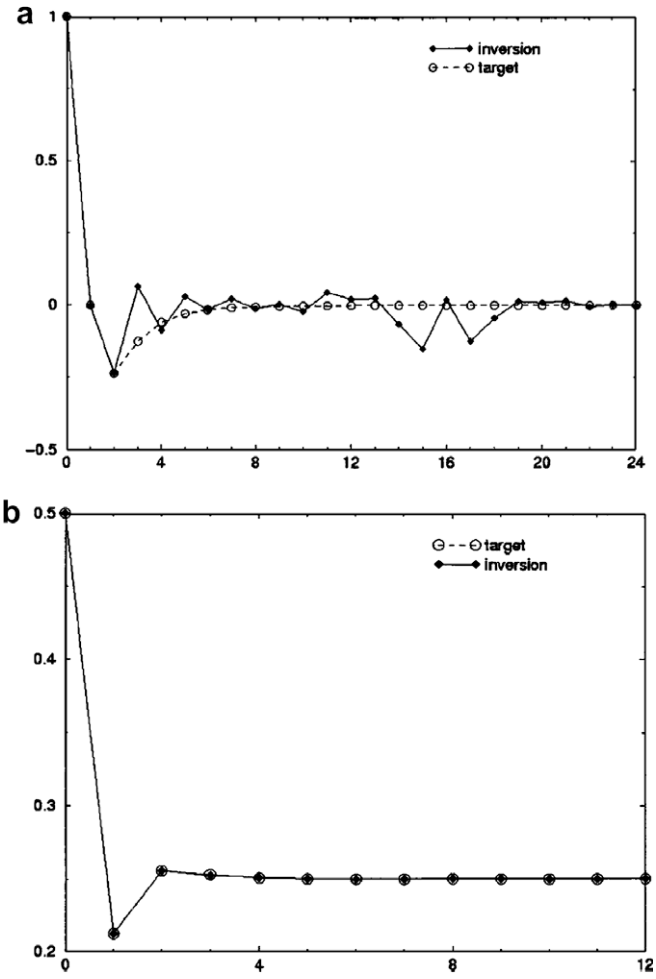

Fig. 10. (a) Comparison of target Gaussian autocorrelation with Gaussian autocorrelation calculated using iterative inversion algorithm; (b) comparison of target binary autocorrelation with binary autocorrelation calculated using inversion algorithm. Reprinted from [97] with permission from ASCE.

where $\{\Psi_j\}_{j=0}^{P} = P_j(\{\xi_n\}_{n=1}^{N})$ denotes the set of Hermite polynomials defined in some underlying Gaussian set $\{\xi_n\}_{n=1}^{N}$, P is the number of PC expansion terms and N is the number of terms retained in the K–L expansion. It is noted that K–L expansion (Eq. (2)) is used for the simulation of the underlying Gaussian process. The method leads in most cases to good approximations of the target non-Gaussian distribution. Puig et al. [149,150] examined the convergence behaviour of PC expansion and proposed an optimization technique for the determination of the underlying Gaussian autocorrelation function. Some limitations of PC approximations have been recently pointed out by Field and Grigoriu [55,83]. Specifically, it has been demonstrated that the accuracy of the PC approximation is not always improved as additional terms are retained and the PC approximation of certain processes may become computationally demanding because of the large number of coefficients u_j that need to be calculated. This is usually the case of problems involving sharp non-linearity and abrupt slope changes or bifurcations [1,55] (Fig. 11).

Xiu and Karniadakis [190] suggested an optimal description of different distribution types by using a more general PC framework called Askey chaos. Precisely, they presented a new method for solving stochastic differential equations based on Galerkin projections and extensions of Wiener's polynomial chaos. In this framework, the stochastic processes are represented using an optimum trial basis from the Askey family of orthogonal polynomials

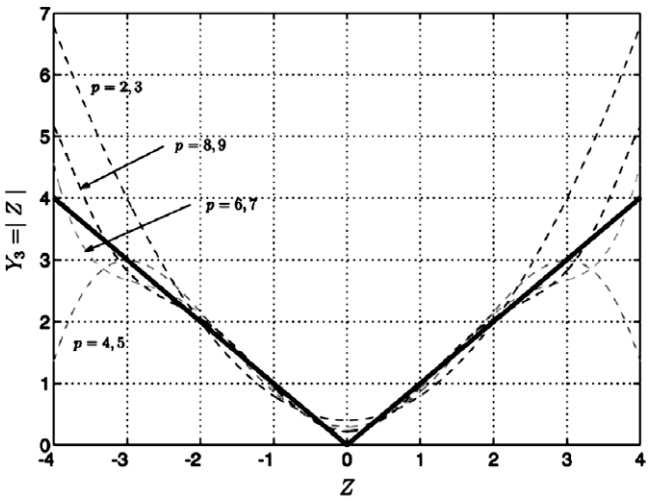

Fig. 11. Exact map $Y_3 = g_3(Z)$ (solid line) vs. PC approximate map $Y_3^{(p)} = g_3^{(p)}(Z)$ (dashed lines) for different orders of PC (random variable $Z \sim N(0,1)$). Reprinted from [55], Copyright © 2003, with permission from Elsevier.

$\{\Phi_j\}_{j=0}^{P} = P_j(\{\zeta_n\}_{n=1}^{N})$ that reduces the dimensionality of the system and leads to exponential convergence of the error:

$$u(x) = \sum_{j=0}^{P} u_j(x)\Phi_j, \tag{12}$$

where $\{\zeta_n\}_{n=1}^{N}$ denotes a set of (non-Gaussian) random variables and P is the number of generalized PC expansion terms. Numerical examples showed substantial reduction of the computational cost compared to Monte Carlo simulations for low dimensional stochastic inputs e.g. [57,170,189,190].

2.2.3. Other methods

Recently, a new spectral representation-based model has been developed for the *direct* simulation of a class of non-Gaussian processes [82]. The model is based on the spectral representation theorem for weakly stationary processes and can match the second moment properties along with several higher order moments of any non-Gaussian process. The model consists of a superposition of harmonics with uncorrelated but dependent random amplitudes and is useful for both Monte Carlo simulation and analytical studies for the response of linear and nonlinear systems to non-Gaussian noise [84].

Elishakoff et al. [47] proposed a conditional simulation technique for a non-Gaussian stochastic field. This was an extension of the unconditional simulation technique by Yamazaki and Shinozuka [192] into a conditional stochastic field. In their technique, non-Gaussian random variables (values of the stochastic field at specific points) are transformed to Gaussian ones without taking into account correlation, and sample simulations are carried out in the Gaussian stochastic field. A correlation coefficient between samples that have been transformed to a given stochastic field is then obtained and computations are iterated until the coefficient converges to a target value. This approach has been verified only through numerical simulations. An improved version of the technique by Elishakoff et al. is the method proposed by Hoshiya et al. [87] for the simulation of conditional *translation* stochastic fields, which has a better theoretical formulation.

In [17], a procedure based on Markov theory is developed, in which matching of the SDF is accomplished by adjusting the drift coefficient of the Fokker–Planck equation governing the probability density and the diffusion coefficient to match the PDF. The procedure is applicable to an arbitrary PDF, if the SDF is of a low-pass

(broad band) type, and to a large class of PDFs, if the SDF is of a narrow band type with its peak located at a nonzero frequency.

2.2.4. The case of non-Gaussian vector processes and fields

A relatively small number of simulation techniques have been developed so far for non-Gaussian *vector* (multi-variate) processes and fields [5,70,71,143,147]. These stochastic fields are useful for the representation of spatially correlated system properties or random loads in the stochastic finite element method (see Section 3). The algorithm of Popescu et al. [147] is an iterative spectral representation-based scheme extending the approach of Yamazaki and Shinozuka to the multi-variate case. The algorithm by Gioffrè et al. is based on translation vector processes and thus it is not iterative. Conditions on the cross-covariance matrix are given to assure the applicability of the model, which is calibrated on the basis of experimental results obtained from wind tunnel tests on a tall building [70,71]. The method is efficient for reproducing the non-Gaussian nature of pressure fluctuations on separated flow regions. An application of the model in the case of translation vectors with non-identically distributed components is presented by Arwade [5]. It is shown that the translation model has the ability to match exactly target marginal distributions and a broad variety of cross-correlation matrices and is well suited to the simulation of heterogeneous material properties. However, the generation of sample functions of vector processes with highly skewed non-Gaussian characteristics and weakly correlated components that accurately match the prescribed target cross-covariance (or cross-spectral density) matrix and marginal PDF, is still a challenge [21]. The accurate and efficient simulation of this kind of non-Gaussian processes remains an open area of research.

3. The stochastic finite element method (SFEM)

The second step in the analysis of uncertain systems is the propagation of uncertainty through the system and the assessment of its stochastic response. This is the most important issue in stochastic mechanics and is mainly addressed today in the framework of the stochastic finite element method (SFEM). SFEM is an extension of the classical deterministic approach for the solution of stochastic (static and dynamic) problems and has received considerable attention especially in the last two decades, due to the technological advances in the available computational power [46,64,67,94,96,111,176]. SFEM involves finite elements whose properties are random. From a mathematical point of view, SFEM is a powerful tool for the solution of stochastic partial differential equations (PDEs) and it has been treated as such in numerous publications where convergence and error estimation issues are examined in detail e.g. [10–12,30,41,58,113,124]. SFEM has been successfully applied in a wide variety of problems (e.g. solid, structural and fluid mechanics, acoustics, heat transfer) [1–4,13–15,18,20,22–29,31,32,37,40,42,44–46,48–51,57–69,72,74,89,91,92,94–96,99,103,104,106–108,111–113,116–124,126–139,148,154,155,158,160,162,166,173,175,188,196].

There are two main variants of SFEM in the literature: i) the perturbation approach [96,106,107], which is based on a Taylor series expansion of the response vector and, ii) the spectral stochastic finite element method – SSFEM [67] where each response quantity is represented using a series of random Hermite polynomials. Monte Carlo simulation – MCS [138] can also be added to these two variants. In the framework of MCS, a deterministic problem is solved a (large) number of times and the response variability is calculated using simple relationships of statistics. Due to its robustness and simplicity, MCS is often used in the literature as a reference method in order to check the accuracy of other approaches and is sometimes combined with the two aforementioned SFEM variants [63]. The SFEM comprises three basic steps: the discretization of the stochastic fields representing the uncertain system properties, the formulation of the stochastic matrix (first at the element and then at the global-system level) and finally, the response variability calculation (response statistics). These steps along with their computational aspects are described in the following sections.

3.1. Discretization of stochastic processes and fields

The first basic step of SFEM is the discretization of the stochastic processes/fields used to represent the uncertain mechanical and geometric system properties. The term "discretization" means the approximation (replacement) of the continuous stochastic field (see Section 2) by a finite number of random variables forming a random vector:

$$f(\mathbf{x}) \overset{\text{discretization}}{\rightarrow} \hat{f}(\mathbf{x}) = \{f_i\}. \tag{13}$$

The discretization methods existing in the literature can be split into two main categories: (i) *point discretization methods* where the final random variables are simply the values of the stochastic field at specific points of the system domain (element centroid, nodes, integration points) and, (ii) *average-type discretization methods* where the random variables are defined as (weighted) integrals of the stochastic field over each finite element. The main representatives of the first category are the *midpoint*, *nodal point*, *integration point* and *interpolation* methods [37,106,107], while the *local average* and

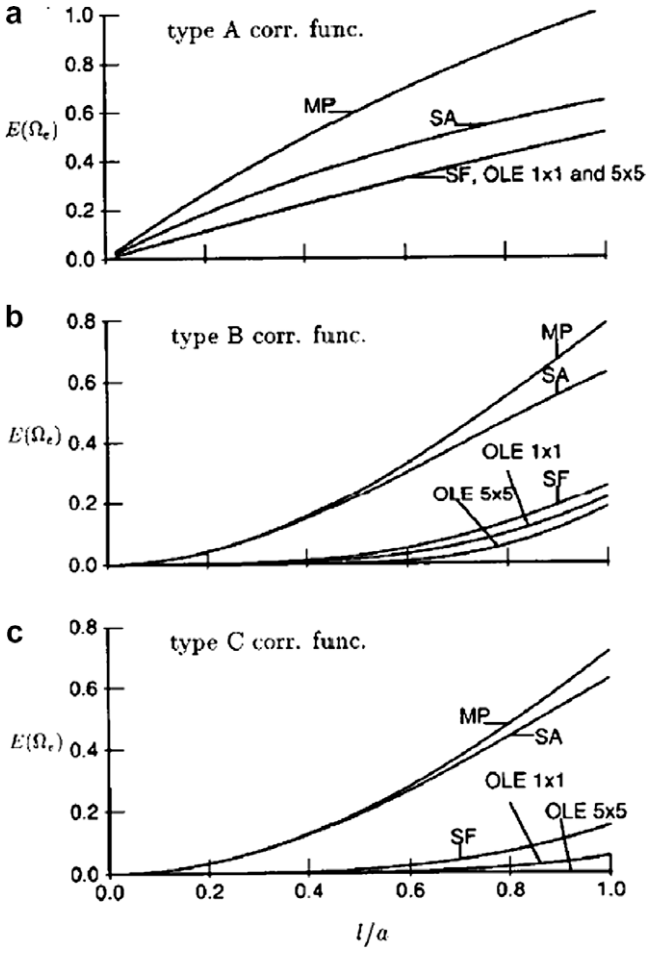

Fig. 12. Comparison of errors for the midpoint (MP), interpolation (SF) and local average (SA) methods for varying element size and correlation structure (exponential-A, square exponential-B and sinusoidal-C). Reprinted from [102], with permission from ASCE.

weighted integral methods [182,183,31,32] are the main representatives of the second group. These methods have been extensively used by several researchers in the framework of SFEM leading to results of different levels of accuracy e.g. [4,25,31,72,111,173,184]. A thorough comparison of the midpoint, interpolation and local average methods for the case of 2D–1V homogeneous Gaussian stochastic fields with three different correlation structures (exponential, square exponential and sinusoidal) and various correlation length parameters, has been presented in [102] (Fig. 12). A comparison between the local average and weighted integral methods in the calculation of the response variability of plane stress/strain, plate and thin shell structures can be found in [184,72,4,173], respectively (see also Fig. 13).

Another important issue in SFEM is the choice of the "stochastic mesh" used for the discretization of the stochastic fields and its relationship with the finite element mesh used for the analysis. The choice of the "stochastic mesh" is mainly determined by the correlation length parameter which is directly related to the variability of the random field, whereas the finite element mesh is usually defined by the geometry and the stress or flux gradients of the response. In [37] it is stated that, since the choice of these two meshes is based on different criteria, the use of two different meshes is possible and may be more efficient in practical problems. This can be explained by the fact that, for strongly correlated stochastic fields, the "stochastic mesh" can be significantly coarser than the finite element mesh thus leading to a substantial reduction of the random variables (dimension) of the problem. However, the use of the same mesh is simpler and sometimes more convenient. The ideal case is of course that of a mesh which accurately describes at the same time the geometry, the stress gradients of the response and the variability of the stochastic field, but this case does not appear often in engineering problems.

Concerning the size of the stochastic mesh, Der Kiureghian and Ke [37] proposed the value $L_{RF} \approx b/4 \div b/2$, where b is the correlation length parameter and L_{RF} is the typical element size in the random field mesh. This result has been obtained by repeatedly evaluating the reliability index of a beam with stochastic rigidity using meshes with decreasing element size. This range was confirmed by Li and Der Kiureghian [102] and Zeldin and Spanos [194] who arrived to this conclusion by comparing the power spectra of the random fields before and after discretization. Another rule for the size of the random field mesh is provided in [166]. To the author's knowledge, there are very few publications in the literature where use is made of two really independent meshes in conjunction with a general mapping procedure of the random field realization onto the finite element mesh [20,157]. However, this is the only general approach to be adopted in large-scale engineering applications where the finite element mesh is automatically generated and the elements have variable size and unprescribed orientation.

3.2. Formulation of the stochastic finite element matrix

The discretized stochastic fields are used for the formulation of the stochastic matrix of each finite element (e) which, in the case of a random spatial variation of Young modulus described by a zero-mean, homogeneous stochastic field $f(x,y,z)$, has the following form:

$$k^{(e)} = \int_{V^{(e)}} B^{(e)T} D_0^{(e)} B^{(e)} dV^{(e)} + \int_{V^{(e)}} B^{(e)T} D_0^{(e)} B^{(e)} f^{(e)}(x,y,z) dV^{(e)}, \qquad (14)$$

or

$$k^{(e)} = k_0^{(e)} + \Delta k^{(e)},$$

where $k_0^{(e)}$ and $\Delta k^{(e)}$ are the mean (deterministic) and fluctuating parts of the stochastic finite element matrix, respectively, $B^{(e)}$ is the (deterministic) strain–displacement matrix, $D_0^{(e)}$ is the mean

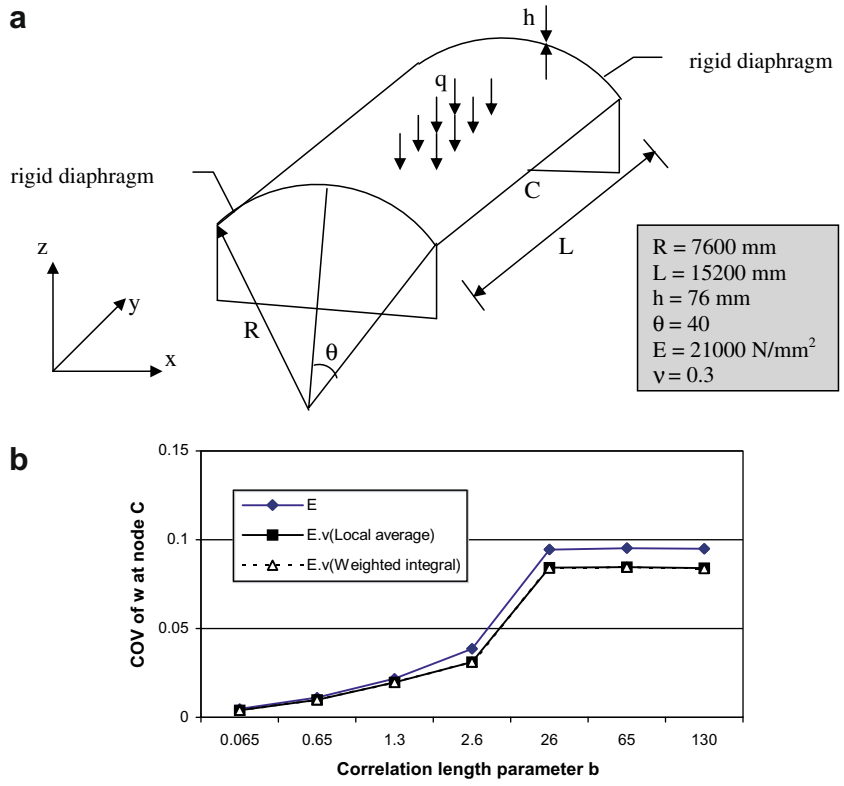

Fig. 13. (a) Scordelis-Lo shell; (b) comparison between local average and weighted integral methods (Scordelis-Lo shell) – COV of vertical deflection w_c as a function of correlation length parameter b for the case of combined Young modulus and Poisson ratio variation. Reprinted from [173], Copyright © 2003, with permission from Elsevier.

value of the constitutive matrix and $V^{(e)}$ is the volume of the finite element. Expressions for the stochastic matrix of several kinds of finite elements (2D and 3D beam, plane stress/strain, plate, shell) are existing in the literature [31,138,184,72,4,173]. It is worth noting that, in the case of dynamic problems, the mass and damping matrices can be random as well e.g. [49,50,181]. In this case, a random eigenvalue problem has to be solved in order to obtain the random eigenpairs of the system. Since the solution of this problem is often computationally demanding, several papers are devoted to this topic proposing different approaches for achieving an enhanced computational efficiency e.g. [68,151,178].

The global stochastic matrix of the system has a similar form and is formed using $k^{(e)}$ as follows:

$$K = \sum_{e=1}^{Ne} k^{(e)} = K_0 + \Delta K, \tag{15}$$

where N_e is the number of finite elements in the problem at hand. Finally, static analysis in the context of SFEM, results in the solution of the algebraic problem given below:

$$P = (K_0 + \Delta K)u, \tag{16}$$

where P and u are the loading and nodal displacement vectors, respectively. In the case of large-scale systems, the solution of this problem is computationally demanding and thus constitutes the crucial point in the applicability and efficiency of the SFEM as will be discussed in the next sections.

3.3. Monte Carlo simulation – MCS

3.3.1. Direct MCS

MCS is the simplest method for treating the response variability calculation in the framework of SFEM. In this method, *NSIM* samples of the stochastic system matrix are generated using a random number generator and the final equilibrium Eq. (16) is solved *NSIM* times, leading to a population (sample) of the response vector. Based on this population, the response variability of the system is calculated using simple relationships of statistics. For example, if u_i is the displacement at the i-th d.o.f., then the unbiased estimates of the mean value and variance of the sample are

$$E(u_i) = \frac{1}{NSIM}\sum_{j=1}^{NSIM} u_i(j), \tag{17}$$

$$\sigma^2(u_i) = \frac{1}{NSIM-1}\left[\sum_{j=1}^{NSIM} u_i^2(j) - NSIM\cdot E^2(u_i)\right]. \tag{18}$$

It is obvious that the accuracy of the estimation depends on the number of samples and, in particular, the estimate of standard deviation σ is inversely proportional to $\sqrt{NSIM}$. A small number of samples e.g. $NSIM \approx 50$ permits only a rough approximation of the mean value and variance of the response. With a larger sample size e.g $NSIM \approx 500$, it is possible to estimate the CDF of the response [161]. The solution of *NSIM* deterministic problems has a significant computational cost especially in the combined case of large-scale systems and of considerable stochastic dimension. It is therefore desirable to combine MCS with discretization methods that do not involve a large number of random variables such as the midpoint or the local average method, which lead to only one random variable per finite element. The weighted integral method is also advantageous because it permits an exact representation of the stochastic field using a small number of random variables ($\leqslant$6 in elasticity problems) e.g. [184,72].

The direct MCS described in this section is the basic version of the method. A variant of MCS called "fast MCS" has been recently used for the efficient numerical evaluation of the variability response function [165] needed to calculate spectral-probability distribution-free upper bounds of the response variability of structural systems [134]. Numerous other variants of this approach (importance sampling, subset simulation, line sampling) have been developed in the last decade especially for the efficient solution of reliability problems where the calculation of small failure probabilities requires a very large number of samples [161]. Even a few years ago, the application of the direct MCS to large-scale realistic problems was impossible due to its excessive computational cost. However, the development of robust and efficient solution algorithms in conjunction with the increasing availability of powerful computers and the suitability of the method to parallel processing with ideal efficiency, alleviate this limitation to a large extent. Thus, direct MCS is today a powerful (and perhaps the only universal) tool for treating complex SFEM applications. This is why it is often used as a reference approach for validating the results of other methods [4,9,44,51,138,173,176,177,179].

3.3.2. Variants of direct MCS for SFE-based reliability estimation

3.3.2.1. Importance sampling. Direct MCS becomes inefficient for the solution of reliability problems where a large number of low-probability realizations in the failure domain must be produced. In order to alleviate this problem without deteriorating the accuracy of the solution, numerous variants of this approach have been developed. An important class of improved MCS are variance–reduction techniques where the generation of samples of the basic random variables is controlled in an efficient way. The most prominent representative of this class of methods is importance sampling (IS), in which the generation of samples is controlled by a sampling distribution concentrated in the "important" (low-probability) region of the failure domain. The main challenge in the application of IS to physical problems is the determination of the sampling distribution, which depends on the specific system at hand and on the failure domain [161]. The optimal choice of the sampling distribution (for which the variance of the estimator of the probability of failure p_F vanishes) is practically infeasible since an a priori knowledge of p_F is required for this purpose [163]. Thus, several techniques based on kernel density estimators or design points have been proposed in order to produce a sampling distribution characterized by a reduced variance of the estimator of p_F

$$\hat{p}_F = \frac{1}{N}\sum_{i=1}^{N}\frac{1_F(\boldsymbol{\theta}^{(i)})h(\boldsymbol{\theta}^{(i)})}{f(\boldsymbol{\theta}^{(i)})}. \tag{19}$$

In this equation, N is the number of samples, 1_F denotes the indicator function of the failure domain, h is the joint probability distribution of the basic random variables and the samples $\{\boldsymbol{\theta}^{(i)}\}_{i=1}^{N}$ are generated according to the sampling distribution f.

IS is efficient for the reliability assessment of static linear and nonlinear systems characterized by a small number of basic random variables. However, for the dynamic reliability analysis of large nonlinear SFE systems in high stochastic dimensions, the computational effort needed to construct a suitable sampling distribution may exceed the effort required by the direct MCS [161].

3.3.2.2. Subset simulation. In order to overcome the inefficiency of direct MCS in calculating small failure probabilities, a novel approach called *subset simulation* (SS) has been recently proposed [7]. SS is a powerful tool, simple to implement and capable of solving a broad range of reliability problems e.g. [8]. The basic idea of SS is to express the failure probability p_F as a product of larger conditional probabilities by introducing a decreasing sequence of intermediate failure events (subsets) $\{F_i\}_{i=1}^{m}$ such that F_m=F and $F_1 \supset F_2 \supset \cdots \supset F_m = F$

$$p_F = P(F_m) = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1}/F_i). \tag{20}$$

With a proper choice of the intermediate events, the conditional failure probabilities can be made sufficiently large. Therefore, the original problem of computing a small failure probability is reduced to calculating a sequence of larger conditional probabilities, which can be efficiently estimated by means of direct MCS with a small number of samples.

A significant advantage of SS is that its convergence rate does not depend on the number of random variables (stochastic dimension) of the problem [163]. However, the convergence rate of SS strongly depends on the selection of the intermediate failure events as well as on the choice of the spread of the proposal PDF used to generate the conditional samples through a Markov chain procedure. Since no information is available in order to make optimum choices for the aforementioned parameters, the convergence of SS can be significantly delayed in some cases.

3.3.2.3. Line sampling. Another recently developed technique which permits the efficient treatment of high dimensional reliability problems is *line sampling* (LS) [98]. This technique takes advantage of an implicitly available performance function (data points on the limit state surface) obtained directly from FE analyses. A brief description of the procedure will be given herewith in the standard normal space but its generalization is straightforward. As already mentioned in the case of IS, the optimal choice of the sampling distribution is practically infeasible. However, something quite close to optimal sampling can be achieved by using LS and computing an *important direction* $\boldsymbol{\alpha}$ which points toward the failure domain nearest to the origin (Fig. 14). Neither the vector $\boldsymbol{\alpha}$ is required to point exactly to the design point, nor are any assumptions made with respect to the shape of the limit state surface. Under the condition that direct MCS will be used for the subspace $\boldsymbol{\theta}^{\perp}$ (Fig. 14), the estimator of Eq. (19) is reduced after some algebra to

$$\hat{p}_F = \frac{1}{N}\sum_{i=1}^{N} p_F^{(i)}, \tag{21}$$

where the conditional failure probabilities $p_F^{(i)}$ are computed using the standard normal CDF Φ as follows:

$$p_F^{(i)} = \int_{-\infty}^{+\infty} 1_F(\boldsymbol{\theta}^{(i)}) \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\theta_1^{(i)})^2}{2}\right] = \Phi(\beta_l^{(i)}) + \Phi(-\beta_u^{(i)}), \tag{22}$$

with the safe domain lying in the range $[\beta_l^{(i)}, \beta_u^{(i)}]$. It is worth noting that Eq. (21) is the best approximation (that with the smallest variance) of the estimator of Eq. (19).

A particular advantage of LS is its robustness. In contrast to IS, where an inappropriate choice of the sampling distribution leads to worse estimates compared to direct MCS, LS performs at least as well as direct MCS even in the worst possible case where the direction $\boldsymbol{\alpha}$ is selected orthogonal to the optimal direction [98]. In comparison to IS, it can be shown that LS requires far less performance evaluations (FE analyses) to obtain a similar accuracy. The advantages of LS become more pronounced in high stochastic dimensions as it is demonstrated in [163] where a comparison between different approaches for reliability estimation is presented. Finally, the application of a stepwise procedure proposed in [98] can lead to a further reduction of its computational cost.

3.4. The perturbation method – Taylor series expansion of the stochastic finite element matrix

The Taylor series expansion of the stochastic finite element matrix and of the resulting response vector of a physical system is known in the literature as the *perturbation method* e.g. [18,96,106,107,183]. In this approach, the stochastic field $f(x)$ representing an uncertain system property is discretized into N zero-mean random variables $\{a_i\}_{i=1}^{N}$. The Taylor series expansion of the stochastic system matrix is then expressed as

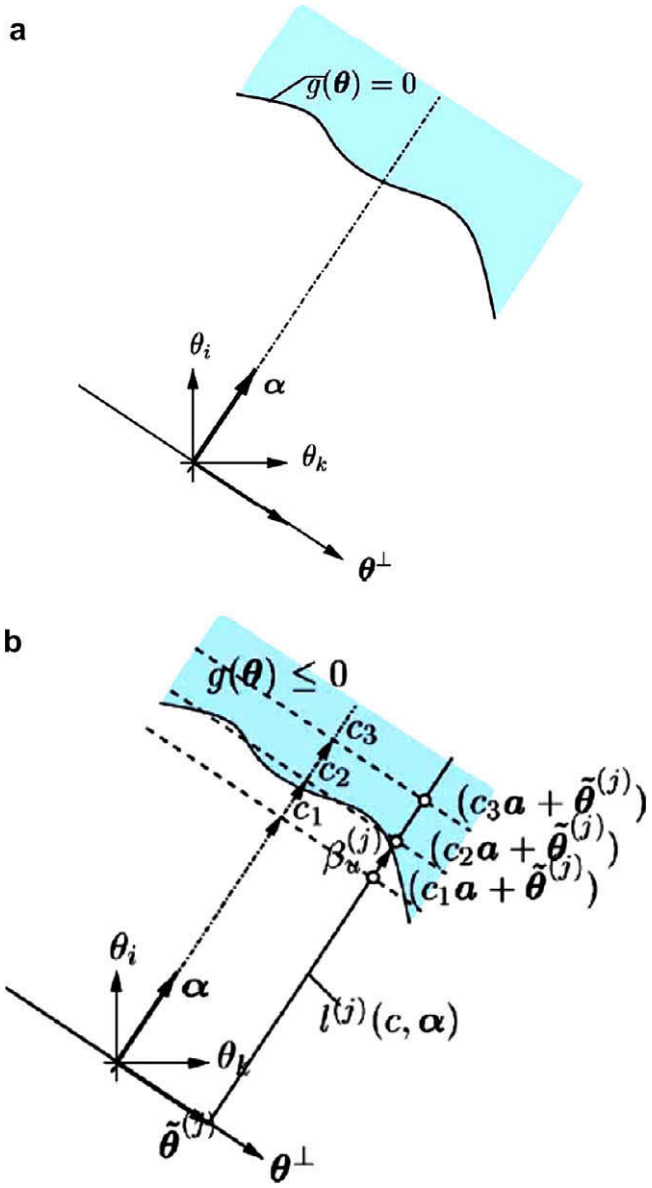

Fig. 14. (a) Limit state and important direction $\boldsymbol{\alpha}$; (b) line sampling procedure. Reprinted from [163], Copyright © 2004, with permission from Elsevier.

$$K = K_0 + \sum_{i=1}^{N} K_i^{\mathrm{I}} a_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_{ij}^{\mathrm{II}} a_i a_j + \cdots, \tag{23}$$

where

$$K_i^{\mathrm{I}} = \left.\frac{\partial K}{\partial a_i}\right|_{a=0} \quad \text{and} \quad K_{ij}^{\mathrm{II}} = \left.\frac{\partial^2 K}{\partial a_i \partial a_j}\right|_{a=0}, \tag{24}$$

and $a = [a_1 \quad a_2 \quad \ldots \quad a_N]^T$ is a random vector containing the random variables $\{a_i\}_{i=1}^{N}$.

The solution of the finite element Eq. (16) requires also a Taylor series expansion of the loading and response (e.g. displacement) vectors in a similar way:

$$P = P_0 + \sum_{i=1}^{N} P_i^{\mathrm{I}} a_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} P_{ij}^{\mathrm{II}} a_i a_j + \cdots, \tag{25}$$

$$u = u_0 + \sum_{i=1}^{N} u_i^{\mathrm{I}} a_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} u_{ij}^{\mathrm{II}} a_i a_j + \cdots \tag{26}$$

If the loading is considered as deterministic, it is obvious that $P_i^{\mathrm{I}} = P_{ij}^{\mathrm{II}} = 0$ and $P = P_0$. The displacement vector of Eq. (26) is calculated using the following iterative scheme:

$$\begin{aligned} u_0 &= K_0^{-1} P_0, \\ u_i^{\mathrm{I}} &= K_0^{-1}(P_i^{\mathrm{I}} - K_i^{\mathrm{I}} u_0), \\ u_{ij}^{\mathrm{II}} &= K_0^{-1}(P_{ij}^{\mathrm{II}} - K_i^{\mathrm{I}} u_j^{\mathrm{I}} - K_j^{\mathrm{I}} u_i^{\mathrm{I}} - K_{ij}^{\mathrm{II}} u_0). \end{aligned} \tag{27}$$

The quantities u_i^{I} and u_{ij}^{II} denote the sensitivity of the displacements with respect to the random variables a_i. An obvious drawback of the perturbation method is the need for calculation of the partial derivatives K_i^{I} and K_{ij}^{II} that increases significantly the computational cost of the approach especially in large stochastic dimensions.

3.4.1. First and second-order approximation of the response variability

The first order approximation of the displacement variability follows from Eq. (26) by omitting the higher order terms:

$$\text{Mean value}: \quad E_1(u) = u_0, \tag{28}$$

$$\begin{aligned} \text{Covariance matrix}: \quad \mathrm{Cov}_1(u,u) &= E\{[u - E_1(u)][u - E_1(u)]^T\} \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} u_i^{\mathrm{I}} (u_j^{\mathrm{I}})^T E(a_i a_j). \end{aligned} \tag{29}$$

The expectation $E(a_i a_j)$ is related to the autocorrelation function of the stochastic field.

The second-order approximation of the response variability can be computed in a similar way only in the case of Gaussian random variables a_i. For all other distributions, it is required the knowledge of the joint probability distribution function of the random variables. Since this is infeasible in practice, higher order approximations are limited to problems involving Gaussian random fields [111]. However, it should be noted that the improvement in accuracy obtained using higher order approximations is rather small compared to the disproportional increase of computational effort [132].

A second-order perturbation approach has been used in [106] leading to accurate results for a two d.o.f. dynamic problem. For a dynamic problem with geometric non-linearity, the results were satisfactory only for small coefficients of variation of the input stochastic field. Falsone and Impollonia [51] proposed an improved method based on the perturbation approach which overcomes its drawbacks. The accuracy of the method is remarkable in the evaluation of higher order moments and PDF of the response even for high amount of uncertainty of the input (Fig. 15). This method provides a valid alternative to the classical perturbation approach due to better accuracy and to MCS due to significantly reduced computational effort. A generalization of the method to geometrically nonlinear as well as to (linear) dynamic problems can be found in [91,50], respectively.

3.5. The spectral stochastic finite element method – SSFEM

The spectral stochastic finite element method – SSFEM has been introduced by Ghanem and Spanos [67] as an extension of the deterministic finite element method for the solution of boundary value problems with random material properties. In the initial version of the method presented in [67], the random spatial variation of the Young modulus of a structure is described by a Gaussian stochastic field which is represented using the Karhunen–Loève (K–L) expansion (see Section 2)

$$f(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\phi_i(\mathbf{x})\xi_i(\theta). \tag{30}$$

In this context, the stochastic matrix of a finite element (e) has the following form:

$$\mathbf{k}^{(e)}(\theta) = \mathbf{k}_0^{(e)} + \sum_{l=1}^{\infty} \mathbf{k}_l^{(e)} \xi_i(\theta), \tag{31}$$

where $\mathbf{k}_0^{(e)}$ is the mean value of $\mathbf{k}^{(e)}(\theta)$, $\mathbf{k}_i^{(e)}$ are deterministic matrices given by

$$\mathbf{k}_i^{(e)} = \sqrt{\lambda_i} \int_{\Omega_e} \phi_i(\mathbf{x}) \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega_e, \tag{32}$$

$\mathbf{B}$ is the strain–displacement matrix and $\mathbf{D}_0$ is the mean value of the constitutive matrix. Assuming deterministic loading, the finite element equilibrium equation has the form:

$$\left[\mathbf{K}_0 + \sum_{l=1}^{\infty} \mathbf{K}_i \xi_i(\theta)\right] \mathbf{U}(\theta) = \mathbf{F}, \tag{33}$$

where $\mathbf{U}(\theta)$ is the unknown vector of random nodal displacements. In the context of SSFEM, the vector $\mathbf{U}(\theta)$ is expanded in a series of random Hermite polynomials $\{\Psi_j(\theta)\}_{j=0}^{\infty}$ (polynomial or Wiener chaos, a terminology introduced by N. Wiener in the context of turbulence modeling [186]) as follows:

$$\mathbf{U}(\theta) = \sum_{j=0}^{\infty} \mathbf{U}_j \Psi_j(\theta), \tag{34}$$

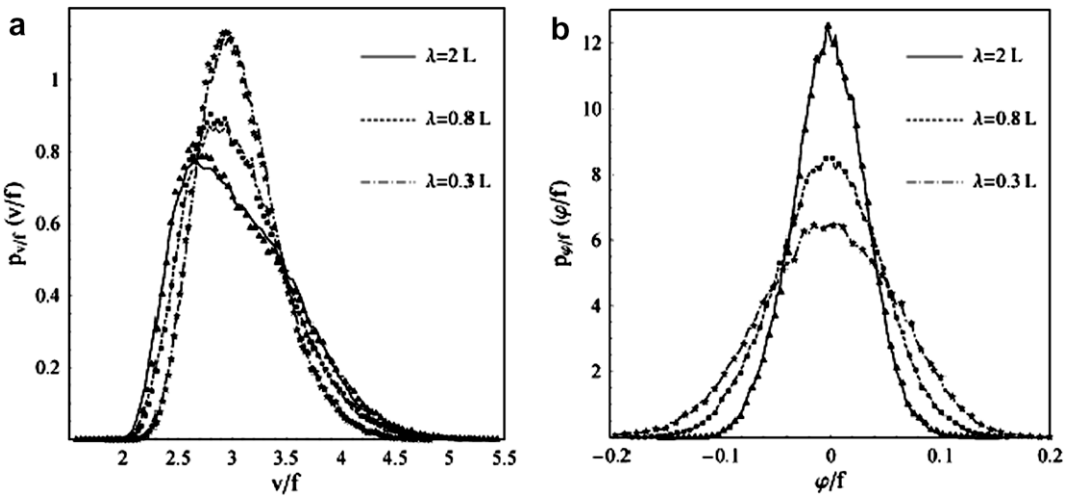

Fig. 15. PDF for the vertical displacement (a) and the rotation (b) at the midpoint of a clamped–clamped beam, compared with classical Monte Carlo simulation (symbols), for different values of the correlation length λ (Young modulus variation, $\sigma = 30\%$). Reprinted from [51], Copyright © 2002, with permission from Elsevier.

and the final equilibrium equation reads:

$$\left(\sum_{i=0}^{\infty}\mathbf{K}_i\xi_i(\theta)\right)\cdot\left(\sum_{j=0}^{\infty}\mathbf{U}_j\Psi_j(\theta)\right)-\mathbf{F}=0. \tag{35}$$

A finite number of terms is finally retained in both expansions (say M+1 terms in the K–L expansion and P terms in the polynomial chaos expansion – PCE) leading to a residual $\in_{M,P}$ that has to be minimized in the mean square sense in order to obtain the optimal approximation of the exact solution $\mathbf{U}(\theta)$ in the space H_P spanned by the polynomials $\{\Psi_k\}_{k=0}^{P-1}$ (Galerkin approach):

$$E[\in_{M,P}\cdot\Psi_k]=0,\quad k=0,1,\ldots,P-1, \tag{36}$$

where $P+1=\frac{(M+p)!}{M!p!}$ and p is the order of chaos polynomials. After some algebraic manipulations, the following $NP\times NP$ linear system of equations is finally obtained:

$$\begin{bmatrix}\mathbf{K}_{00} & \cdots & \mathbf{K}_{0,P-1}\\ \mathbf{K}_{10} & \cdots & \mathbf{K}_{1,P-1}\\ \vdots & & \vdots\\ \mathbf{K}_{P-1,0} & \cdots & \mathbf{K}_{P-1,P-1}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}_0\\ \mathbf{U}_1\\ \vdots\\ \mathbf{U}_{P-1}\end{bmatrix}=\begin{bmatrix}\mathbf{F}_0\\ \mathbf{F}_1\\ \vdots\\ \mathbf{F}_{P-1}\end{bmatrix}, \tag{37}$$

$\mathcal{K}U=\mathcal{F}$.

3.5.1. Computational aspects of SSFEM

Eq. (37) shows that the dimension of the resulting linear system in SSFEM depends directly on the number of terms P retained in the PCE of the random nodal displacement vector. Since P is multiplied by N(the number of degrees of freedom (d.o.f.)) as shown in Eq. (37), it is obvious that the computational cost required for the solution of this system is much larger than that of the corresponding deterministic problem. When direct solution techniques are used for this purpose, the required computing time is prohibitive especially in the case of finely discretized large-scale systems. This is why the use of SSFEM has been limited in the past to the solution of uncertain systems with a small number of degrees of freedom.

The particular form of the global matrix $\mathcal{K}$ in Eq. (37) can be exploited in order to obtain a more efficient solution of the system. $\mathcal{K}$ is block diagonal-sparse (Fig. 16) and Krylov-type iterative techniques such as the preconditioned conjugate gradient method (PCG) are particularly suitable in this case. A number of PCG variants with various preconditioning matrices led in most cases to a substantial reduction of the number of iterations (and thus of the computational cost) irrespectively of the coefficient of variation of the input random field which affects the condition number of matrix $\mathcal{K}$ and thus the convergence behaviour of the iterative algorithms e.g. [66,139,29,95,43,22]. Recently, a generalization of the classical spectral decomposition (truncated K–L expansion) for the solution of the problem interpreted as an "extended" eigenproblem has been proposed together with ad hoc iterative solution techniques inspired by classical techniques for solving the eigenproblem [122,123]. This method leads to further computational savings and reduction of memory requirements compared to Krylov-type techniques in the solution of linear problems.

3.5.2. Accuracy and range of applicability of SSFEM

3.5.2.1. Accuracy of SSFEM. It is observed that the coefficients $\mathbf{U}_j$ in the approximation of the displacement vector (Eq. (34)) result from a Galerkin minimization of the residual of Eq. (36). General convergence properties to the exact solution are also valid in this procedure. When the number of retained terms in Eq. (34) tends to infinity, SSFEM tends to be "exact". However, when a large number of terms are retained in PCE (say 35 for $p=3$ and $M=4$), the computational cost of the method may become prohibitive in large-scale problems. Convergence analyses and error estimators quantifying the accuracy of the method can be found in [64]. Another observation is that, in SSFEM, the response quantities are represented by a PCE in terms of the standard normal random variables of the input random field. Thus, this method can be considered as a polynomial response surface approach in which the coefficients are calculated using the Galerkin method [15].

In most applications, SSFEM is used in conjunction with a K–L expansion of the Gaussian random field(s) describing the uncertain parameters of the problem (see e.g. [160] for a discussion on Gaussian system properties). This random field must be characterized by a correlation length sufficiently large in order for the corresponding K–L expansion to yield a good approximation for a small number M of (<20) terms and a reasonable stochastic dimension is preserved (see also Section 2.1). For non-Gaussian properties, it has been suggested to use PCE for the representation of the input as well [64–66]. In the case of a lognormal distribution, this leads to closed-form expressions since a lognormal random field can be defined by a simple transformation of a Gaussian field [64,65]. However, the use of PCE for the representation of both input and output can lead to a loss of accuracy [176,177]. The use of the generalized PCE seems to be the best solution in the case of a general non-Gaussian input [57,108,189,190] (Fig. 17).

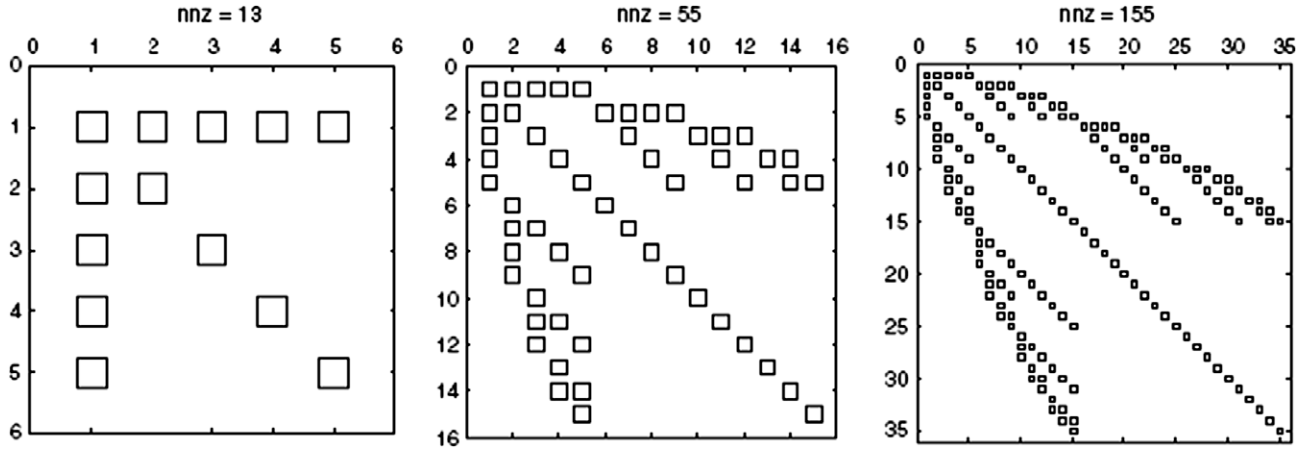

Fig. 16. Sparsity pattern of the global Galerkin matrix for $M=4$ and PC degree $p=1$ (left), $p=2$ (middle) and $p=3$ (right). With kind permission from Springer Science + Business Media: [43], © Springer-Verlag 2007.

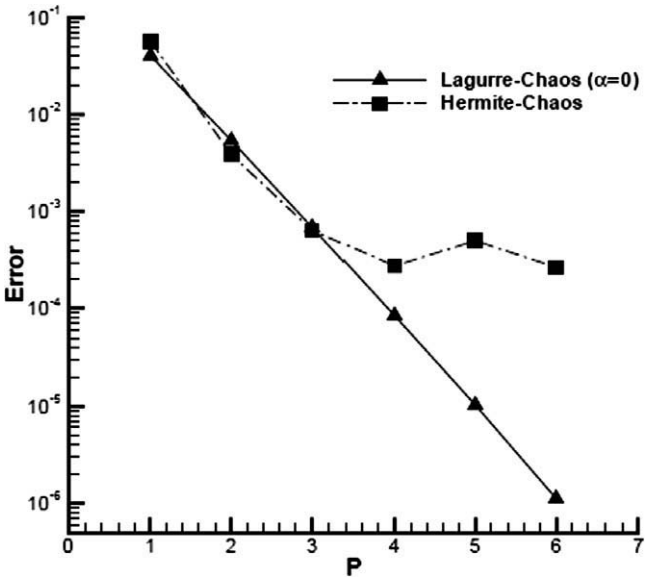

Fig. 17. Error convergence of the mean solution of a stochastic ordinary differential equation with random input following the exponential distribution, obtained using the Laguerre-chaos and Hermite-chaos [190]. Copyright © 2002, Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.

3.5.2.2. Range of applicability of SSFEM. The application of SSFEM is practically limited to linear problems. The first application of the method to elasto-plastic problems can be found in [3] where plastic and failure analysis of earth faults is attempted by introducing some simplifying assumptions. Geometrical and material non-linearity cannot be taken into account efficiently since PCE has been found to perform poorly in problems involving sharp non-linearities, discontinuities, slope changes or bifurcations [1,55] (see also Section 2.2). In these cases of non-smooth solutions, the choice of other basis functions such as wavelets or the use of an adaptive multi-element generalized PCE has been suggested as a remedy to the problem [101,185,131]. More recently, a novel stochastic response surface approach has been proposed for solving nonlinear mechanical problems [13,14]. The approach is based on a Hilbert approximation of the nonlinear mechanical function representing the uncertain system using Hermite or Lagrange polynomials. The coefficients of the approximation are calculated through a cubic B-spline interpolation of the response function. The method gives accurate results (a comparison with MCS is given in Fig. 18) but its application is limited to problems involving a small number of uncertain parameters. Alternatively, the concept of enrichment of standard finite element bases used in a deterministic context to achieve convergence acceleration in problems involving discontinuities such as cracks [115], could be extended to the stochastic case. In this context, additional basis functions (enrichment functions) are added to the polynomial bases in order to capture the peculiar behaviour of the solution [69].

3.5.2.3. Recently proposed methods. The most recent developments in spectral-Galerkin-based SFEM include the stochastic reduced basis methods (SRBMs) introduced in [118,154,116], the non-intrusive approaches proposed in [13–15], the use of the method in a multi-scale setting [191] and the extension to the stochastic framework of the eXtended finite element method (X-FEM) [124]. The SRBMs constitute an efficient alternative which is also limited to the analysis of random linear systems (at least in its present formulation). In contrast to the PC approach, the response process is represented using basis vectors spanning the preconditioned stochastic Krylov subspace. What is interesting here is that: (i)

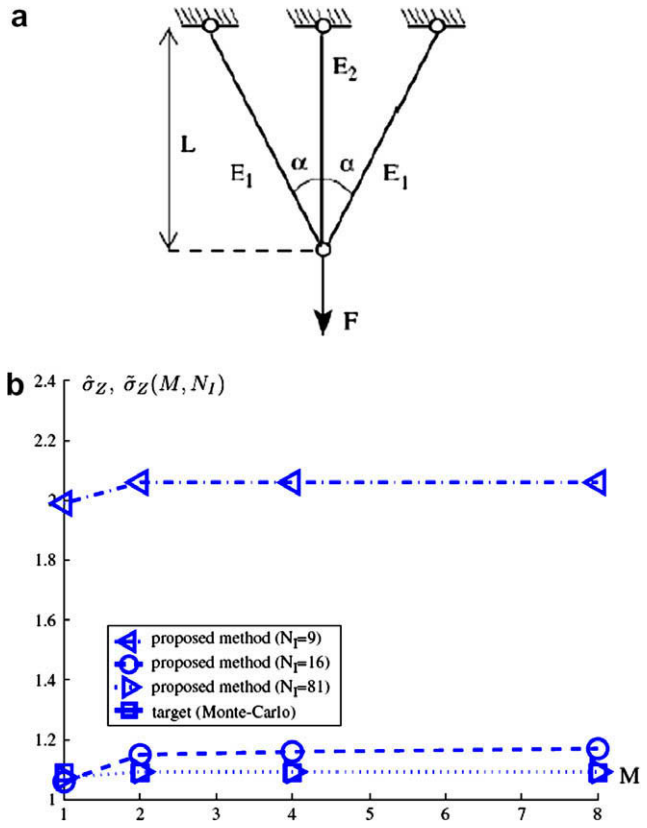

Fig. 18. (a) Three-bar elasto-plastic truss; (b) standard deviation of the vertical displacement Z at the bottom node of the truss as a function of the approximation order M (COV of Young moduli E_1, E_2=0.3, N_I: number of B-spline interpolation points). Reprinted from [13], Copyright © 2006, with permission from Elsevier.

the basis vectors are problem dependent in contrast to PC expansions where the choice of basis functions depends only on the input distribution and, (ii) subsequent application of the Galerkin scheme leads to a reduced-order deterministic system of equations to be solved for the unknown coefficients in the stochastic reduced basis representation. As a result, SRBMs are computationally efficient with regard to PCE at a comparable level of accuracy and are thus suitable for solving large-scale linear problems (for an exhaustive comparison of these projection schemes see [154]), as can be seen in Fig. 19. In the non-intrusive approaches presented in [13–15,26], based on a stochastic response surface, the PCE is used to create a surrogate model through a response surface without interfering with the FE procedure i.e. without directly modifying the element matrix. This is why these methods can take advantage of powerful deterministic FE codes and using them as a black-box. The multi-scale SFEM developed in [191] combines a stochastic variational approach and scale-bridging multi-scale shape functions in order to solve stochastic elliptic problems. The applicability and efficiency of the method are demonstrated with the analysis of a simplified benchmark multi-scale model of groundwater flow in porous media. Finally, the X-SFEM proposed in [124] is an approach relying on two major points: the implicit representation of complex geometries using random level-set functions and the use of a Galerkin approximation at both stochastic and deterministic (FE) levels (Fig. 20). This extension is important since there is no other efficient strategy available in the literature for dealing with uncertainties in the geometry. However, in its present formulation, it is valid only in the context of linear elasticity.

As a conclusion, it can be stated that SSFEM is a rather new but promising technique and many advances remain to be achieved for its successful and computationally efficient application to nonlin-

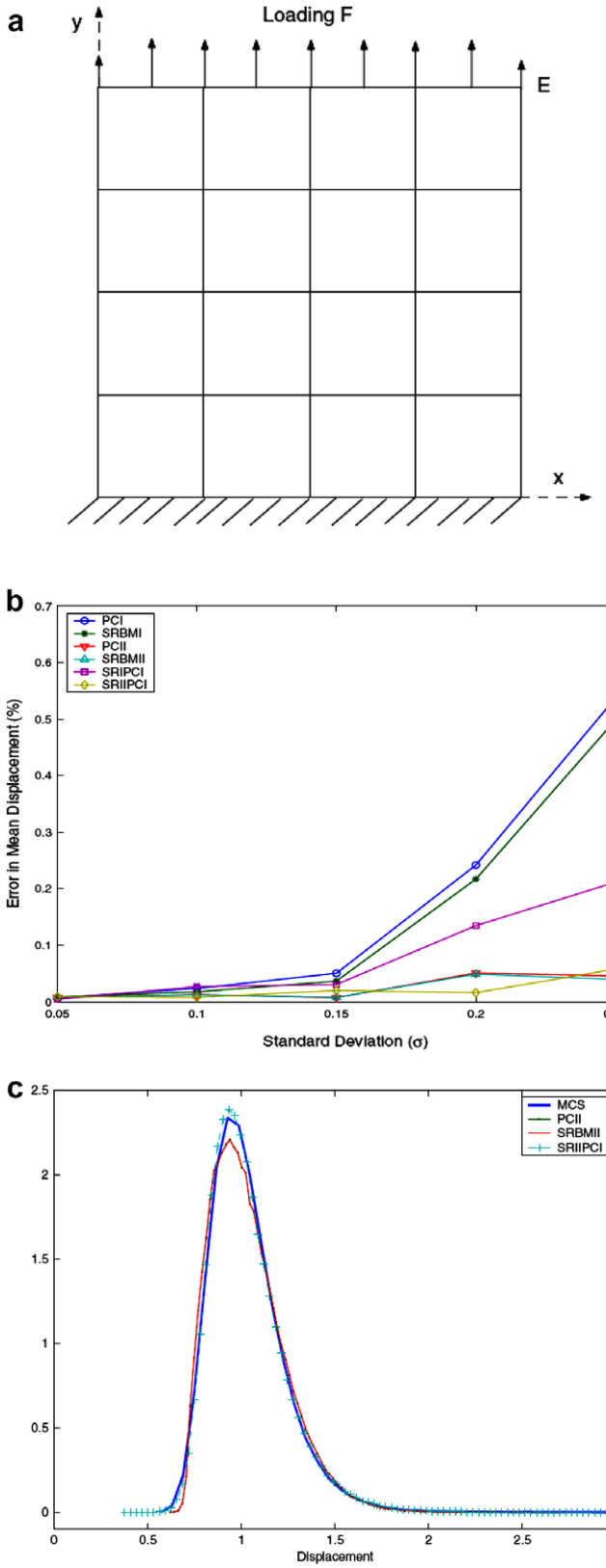

Fig. 19. (a) Clamped square plate subjected to uniform in-plane tension; (b) error in mean displacement at point E of the plate for various projection schemes as a function of the standard deviation of the random Young modulus; (c) PDF of displacement at point E computed using MCS and second-order projection schemes. Reprinted from [154], Copyright © 2005, with permission from Elsevier.

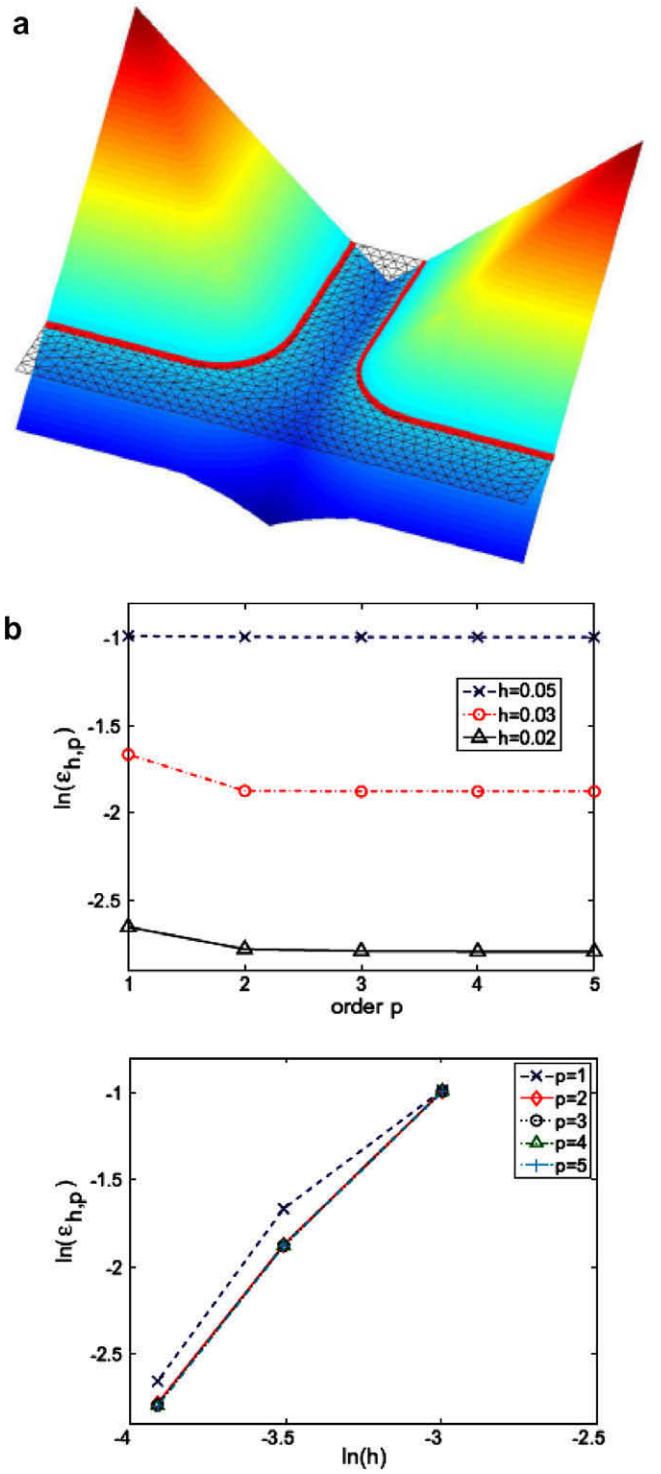

Fig. 20. (a) Random level-set representing a random welded joint of elliptical shape; (b) convergence of the X-SFEM approximation: error indicator $\varepsilon_{h,p}$ with respect to the order p of generalized PC (top) and the average FE mesh size h (bottom). Reprinted from [124], Copyright © 2008, with permission from Elsevier.

ear and inverse problems with stochastic data as well as to cases with time-dependence.

3.6. SFEM specialized software

Although many interesting variants of SFEM are available and continue to appear in the literature, this is not accompanied by an analogous rapid development of relative computer software. This is mainly due to the scepticism of a part of the engineering community with respect to stochastic methods and to the

persisting continuous refinement of existing deterministic FE software. In [161], it is stated that: "An important aspect for software development in probabilistic mechanics is the deterministic solver of the code; indeed, the necessity of having a good mechanical model in the first place must never be overlooked". This is true and the ability of combination of some SFEM variants as the non-intrusive techniques with well established powerful deterministic FE codes is very important and will further enhance the development and dissemination of SFEM and permit the solution of large-scale stochastic problems [155]. The incorporation of a probabilistic toolbox to the recent releases of ANSYS software can be cited as an example [153]. However, its uncertainty modeling capabilities are limited to the use of random variables. A stochastic FE library has also been coupled with ANSYS for response variability calculation [94].

Some representative examples of specialized SFEM software systems are COSSAN [158], NESSUS [155] and FERUM [36]. Finite element reliability using Matlab (FERUM) provides implementation of the SSFEM method for system response and reliability analysis but its capability of interaction with external (third-party) codes is limited. Numerical evaluation of stochastic structures under stress (NESSUS) is more attractive since it has this capability and is applicable to large-scale engineering problems with uncertainties in loading, geometry and material behavior. In its framework, the uncertain parameters are modeled using random variables and probabilistic sensitivity measures can be calculated. Finally, computational stochastic structural analysis (COSSAN) is an open system, designed to be easily adjustable and expandable to include new computational tasks. It contains several module groups each performing a different task such as stochastic finite elements, reliability assessment, response surface computation, system identification, nonlinear programming techniques etc. The current developments in COSSAN are focused on the construction of communication tools making this software capable of interacting with highly developed, third-party FE codes.

4. Conclusions

The article aimed at providing a state-of-the-art review of past and recent developments in the SFEM area, indicating future directions and addressing some open issues to be considered by the engineering community in the future:

- An overview of accurate and efficient simulation techniques of Gaussian and some kinds of non-Gaussian stochastic processes and fields, e.g. *scalar* processes with highly skewed non-Gaussian characteristics, has been presented. The techniques based on the translation field concept are very promising because they combine accuracy and computational efficiency with a number of properties (analytical calculation of crossing rates and extreme value distributions), which are useful for the reliability assessment of uncertain physical systems using MCS. The necessity of developing efficient methods for the simulation of non-Gaussian *vector* processes and fields has also been outlined.
- The three most important alternative formulations for SFE analysis (perturbation approach, MCS with its variants and SSFEM) have been critically reviewed and summarized. The topics of the discretization of the stochastic fields representing the uncertain system properties and of the formulation of the stochastic finite element matrix have also been treated in detail. The capability of using two independent meshes in conjunction with a general mapping procedure of the random field realization onto the finite element mesh appears to be essential in large-scale applications for computing time savings.
- The efficient application of SFEM to nonlinear and inverse problems with stochastic data as well as to cases with time-dependence remains a challenge. MCS is the only universal tool for treating such complex SFE problems at the expense of a prohibitive computational cost. SSFEM emerges as a powerful alternative in some cases with the potential of further improvements in its formulation.
- Rigorous proofs of convergence properties and error estimation studies are needed in order to strengthen the theoretical background of SFEM and thus lead to its wider acceptance by the scientific community.
- The development of robust and efficient solution techniques suitable to a parallel processing environment properly adjusted to solve the particular problem at hand will further enhance the potential of SFEM.
- Equally important is the development of user-friendly specialized SFEM software capable of interacting with powerful third-party codes and treating large-scale stochastic problems in tractable computing times.

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