PHYSICAL REVIEW VOLUME 83, NUMBER 1 JULY 1, 1951

Irreversibility and Generalized Noise*

HERBERT B. CALLEN AND THEODORE A. WELTON†
Randal Morgan Laboratory of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

(Received January 11, 1951)

A relation is obtained between the generalized resistance and the fluctuations of the generalized forces in linear dissipative systems. This relation forms the extension of the Nyquist relation for the voltage fluctuations in electrical impedances. The general formalism is illustrated by applications to several particular types of systems, including Brownian motion, electric field fluctuations in the vacuum, and pressure fluctuations in a gas.

I. INTRODUCTION

THE parameters which characterize a thermodynamic system in equilibrium do not generally have precise values, but undergo spontaneous fluctuations. These thermodynamic parameters are of two classes: the "extensive" parameters,[1] such as the volume or the mole numbers, and the "intensive" parameters[2] or "generalized forces," such as the pressure or chemical potentials.

An equation relating particularly to the fluctuations in voltage (a "generalized force") in linear electrical systems was derived many years ago by Nyquist,[3] and such voltage fluctuations are generally referred to as Nyquist or Johnson "noise." The voltage fluctuations are related, not to the standard thermodynamic parameters of the system, but to the electrical resistance. The Nyquist relation is thus of a form unique in physics, correlating a property of a system in *equilibrium* (i.e., the voltage fluctuations) with a parameter which characterizes an irreversible process (i.e., the electrical resistance). The equation, furthermore, gives not only the mean square fluctuating voltage, but provides, in addition, the frequency spectrum of the fluctuations. The proof of the relation is based on an ingenious union of the second law of thermodynamics and a direct calculation of the fluctuations in a particular simple system (an ideal transmission line).

It has frequently been conjectured that the Nyquist relation can be extended to a general class of dissipative systems other than merely electrical systems. Yet, to our knowledge, no proof has been given of such a generalization, nor have any criteria been developed for the type of system or the character of the "forces" to which the generalized Nyquist relation may be applied. The development of such a proof and of such criteria is the purpose of this paper (Secs. II, III, and IV). The general theorem thus establishes a relation between the "impedance" in a general linear dissipative system and the fluctuations of appropriate generalized "forces."

Several illustrative applications are made of the general theorem. The viscous drag of a fluid on a moving body is shown to imply a fluctuating force, and application of the general theorem immediately yields the fundamental result of the theory of Brownian motion. The existence of a radiation impedance for the electromagnetic radiation from an oscillating charge is shown to imply a fluctuating electric field in the vacuum, and application of the general theorem yields the Planck radiation law. Finally, the existence of an acoustic radiation impedance of a gaseous medium is shown to imply pressure fluctuations, which may be related to the thermodynamic properties of the gas. The theorem thus correlates a number of known effects under one general principle and is able to predict a class of new relations.

In the final section of the paper, we discuss an intuitive interpretation of the principles underlying the theorem.

It is felt that the relationship between equilibrium fluctuations and irreversibility which is here developed provides a method for a general approach to a theory of irreversibility, using statistical ensemble methods. We are currently investigating such an approach.

II. THE DISSIPATION

A system may be said to be dissipative if it is capable of absorbing energy when subjected to a time-periodic perturbation (as an electrical resistor absorbs energy from an impressed periodic voltage). The system may be said to be linear if the power dissipation is quadratic in the magnitude of the perturbation. For a linear dissipative system, an impedance may be defined, and the proportionality constant between the power and the square of the perturbation amplitude is simply related to the impedance [in the electrical case, Power $=(\text{voltage})^2\cdot R/|Z|^2$].

In the present section we treat the applied perturbation by the usual quantum mechanical perturbation

* This work was supported in part by the ONR.
† Now at Oak Ridge National Laboratory, Oak Ridge, Tennessee.
[1] For the theory of fluctuations of extensive parameters see Fowler, *Statistical Mechanics* (Cambridge University Press, London, 1936), second edition; or Tolman, *Principles of Statistical Mechanics* (Oxford University Press, London, 1938). A recent development of the theory is given by M. J. Klein and L. Tisza, Phys. Rev. **76**, 1861 (1949).
[2] A statistical mechanical theory of fluctuations of intensive parameters will be given in a subsequent paper by R. F. Greene and H. B. Callen.
[3] H. Nyquist, Phys. Rev. **32**, 110 (1928). A very neat derivation and an interesting discussion is given by J. C. Slater, Radiation Laboratory Report; "Report on Noise and the Reception of Pulses," February 3, 1941, unpublished.

methods and thus relate the power dissipation to certain matrix elements of the perturbation operator. We thereby show that for small perturbations, a system with densely distributed energy levels is dissipative and linear, and we obtain certain pertinent information relative to the impedance function.

Let the hamiltonian of the system in the absence of the perturbation be H_0, a function of the coordinates $q_1\cdots q_K\cdots$ and momenta $p_1\cdots p_K\cdots$ of the system. In the presence of the perturbation, the hamiltonian is

$$H=H_0(\cdots q_K\cdots p_K\cdots)+VQ(\cdots q_K\cdots p_K\cdots). \tag{2.1}$$

where Q is a function of the coordinates and momenta, and V is a function of time which measures the instantaneous magnitude of the perturbation.

Again invoking the electrical case as a clarifying example, we may have V as the impressed voltage and $Q=\sum e_i x_i/L$, where e_i is the charge on the ith particle, x_i is its distance from one end of the resistor, and L is the total length of the resistor.

If the applied perturbation varies sinusoidally with time, we have

$$V=V_0\sin\omega t. \tag{2.2}$$

We may now employ standard time-dependent perturbation theory to compute the power dissipation. Let $\psi_1, \psi_2\cdots\psi_n\cdots$ be the set of eigenfunctions of the unperturbed hamiltonian H_0, so that

$$H_0\psi_n=E_n\psi_n, \tag{2.3}$$

and let the true wave function be ψ. Expanding ψ in terms of the ψ_n,

$$\psi=\sum_n a_n(t)\psi_n, \tag{2.4}$$

and substituting into the Schroedinger equation for ψ,

$$H_0\psi+V_0\sin\omega t Q\psi=i\hbar\partial\psi/\partial t, \tag{2.5}$$

one obtains a set of first-order equations for the coefficients $a_n(t)$, which may readily be integrated. If the energy levels of the system are densely distributed, one thus finds that the total induced transition probability of a system initially in the state ψ_n is

$$\tfrac{1}{2}\pi V_0^2\hbar^{-1}\{|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) + |\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)\}, \tag{2.6}$$

where the symbol $\langle E_n+\hbar\omega|Q|E_n\rangle$ indicates the matrix element of the operator corresponding to Q between the state with eigenvalue $E_n+\hbar\omega$ and the state with eigenvalue E_n. The symbol $\rho(E)$ indicates the density-in-energy of the quantum states in the neighborhood of E, so that the number of states between E and $E+\delta E$ is $\rho(E)\delta E$.

Each transition from the state ψ_n to the state with eigenvalue $E_n+\hbar\omega$ is accompanied by the absorption of energy $\hbar\omega$, and each transition from ψ_n to the state with eigenvalue $E_n-\hbar\omega$ is accompanied by the emission of energy $\hbar\omega$. Thus the rate of absorption of energy by a system initially in the nth state is

$$\tfrac{1}{2}\pi V_0^2\omega\{|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) - |\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)\}. \tag{2.7}$$

To predict the behavior of a real thermodynamic system, we must average over-all initial states, weighting each according to the Boltzmann factor $\exp(-E_n/kT)$. Let the weighting factor be $f(E_n)$, so that

$$f(E_n+\hbar\omega)/f(E_n)=f(E_n)/f(E_n-\hbar\omega) =\exp(-\hbar\omega/kT). \tag{2.8}$$

The power dissipation is, then,

$$\text{Power}=\tfrac{1}{2}\pi V_0^2\omega\sum_n\{|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) - |\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)\}f(E_n). \tag{2.9}$$

The summation over n may be replaced by an integration over energy

$$\sum_n(\quad)\rightarrow\int_0^\infty(\quad)\rho(E)dE, \tag{2.10}$$

whence

$$\text{Power}=\tfrac{1}{2}\pi V_0^2\omega\int_0^\infty\rho(E)f(E) \times\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega) - |\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE. \tag{2.11}$$

We thus find that a small periodic perturbation applied to a system, the eigenstates of which are densely distributed in energy, leads to a power dissipation quadratic in the perturbation. For such a linear system it is possible to define an impedance $Z(\omega)$, the ratio of the force V to the response $\dot{Q}$, where all quantities are now assumed to be written in standard complex notation,

$$V=Z(\omega)\dot{Q}. \tag{2.12}$$

The instantaneous power is $V\dot{Q}R/|Z|$, and the average power is

$$\text{Power}=\tfrac{1}{2}V_0^2R(\omega)/|Z(\omega)|^2, \tag{2.13}$$

where $R(\omega)$, the resistance, is the real part of $Z(\omega)$.

If the applied perturbation is not sinusoidal, but some general function of time $V(t)$, and if $v(\omega)$ and $\dot{q}(\omega)$ are the fourier transforms of $V(t)$ and $\dot{Q}(t)$, the impedance is defined in terms of the fourier transforms:

$$v(\omega)=Z(\omega)\dot{q}(\omega). \tag{2.14}$$

In this notation we then obtain, for the general linear dissipative system,

$$R/|Z|^2=\pi\omega\int_0^\infty\rho(E)f(E)\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega) - |\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE. \tag{2.15}$$

III. THE FLUCTUATION

We have, in the previous section, considered a system to which is applied a force V, eliciting a response $\dot{Q}$. We now consider the system to be left in thermal equilibrium, with no applied force. We may expect, even in this isolated condition, that the system will exhibit a spontaneously fluctuating $\dot{Q}$, which may be associated with a spontaneously fluctuating force. We shall see, in this section, that such a spontaneously fluctuating force does in fact exist, and we shall find its magnitude.

Let $\langle V^2\rangle$ be the mean square value of the spontaneously fluctuating force, and let $\langle\dot{Q}^2\rangle$ be the mean square value of the spontaneously fluctuating $\dot{Q}$. Although we shall be primarily interested in $\langle V^2\rangle$, we shall find it convenient to compute $\langle\dot{Q}^2\rangle$ and to obtain $\langle V^2\rangle$ from Eq. (2.14).

Consider that the system is known to be in the nth eigenstate. The hermitian property of H_0 causes the expectation value of $\dot{Q}$, $\langle E_n|\dot{Q}|E_n\rangle$, to vanish. The mean square fluctuation of $\dot{Q}$ is therefore given by the expectation value of $\dot{Q}^2$ or $\langle E_n|\dot{Q}^2|E_n\rangle$. Then

$$\begin{aligned}\langle E_n|\dot{Q}^2|E_n\rangle &= \textstyle\sum_m\langle E_n|\dot{Q}|E_m\rangle\langle E_m|\dot{Q}|E_n\rangle \\ &= \hbar^{-2}\textstyle\sum_m\langle E_n|H_0Q-QH_0|E_m\rangle \\ &\qquad\times\langle E_m|H_0Q-QH_0|E_n\rangle \\ &= \hbar^{-2}\textstyle\sum_m(E_n-E_m)^2|\langle E_m|Q|E_n\rangle|^2. \end{aligned}\tag{3.1}$$

Introducing a frequency ω by

$$\hbar\omega=|E_n-E_m|,\tag{3.2}$$

the summation over m may be replaced by two integrals over ω (one for $E_n<E_m$ and one for $E_n>E_m$):

$$\begin{aligned}\langle E_n|\dot{Q}^2|E_n\rangle &= \hbar^{-2}\int_0^\infty(\hbar\omega)^2|\langle E_n+\hbar\omega|Q|E_n\rangle|^2 \\ &\qquad\times\rho(E_n+\hbar\omega)\hbar d\omega+\hbar^{-2}\int_0^\infty(\hbar\omega)^2 \\ &\qquad\times|\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)\hbar d\omega. \\ &= \int_0^\infty\hbar\omega^2\{|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) \\ &\qquad+|\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)\}d\omega. \end{aligned}\tag{3.3}$$

The fluctuation actually observed in a real thermodynamic system is obtained by multiplying the fluctuation in the nth state by the weighting factor $f(E_n)$, and summing

$$\begin{aligned}\langle\dot{Q}^2\rangle=\sum_n f(E_n)\int_0^\infty\hbar\omega^2\{|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) \\ +|\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)\}d\omega.\end{aligned}\tag{3.4}$$

As in Eq. (2.10), the summation over n may be replaced by an integration over the energy spectrum if we introduce the density factor $\rho(E)$. Thus we finally obtain

$$\begin{aligned}\langle\dot{Q}^2\rangle=\int_0^\infty\hbar\omega^2\Big[\int_0^\infty\rho(E)f(E)\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega) \\ +|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE\Big]d\omega,\end{aligned}\tag{3.5}$$

or, utilizing the definition (2.14) of the impedance,

$$\begin{aligned}\langle V^2\rangle=\int_0^\infty|Z|^2\hbar\omega^2\Big[\int_0^\infty\rho(E)f(E) \\ \times\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega) \\ +|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE\Big]d\omega.\end{aligned}\tag{3.6}$$

IV. THE GENERALIZED NYQUIST RELATION

In the two previous sections we have computed $R/|Z|^2$ and $\langle V^2\rangle$. These quantities involve the constructs

$$\begin{aligned}\int_0^\infty\rho(E)f(E)\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega) \\ \pm|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE,\end{aligned}\tag{4.1}$$

the negative sign being associated with $R/|Z|^2$ and the positive sign with $\langle V^2\rangle$. We shall now see that the two values of (4.1) are simply related, and thus establish the desired relation between $\langle V^2\rangle$ and $R(\omega)$.

Consider first the value of (4.1) corresponding to the negative sign, which we denote by $C(-)$.

$$\begin{aligned}C(-)=\int_0^\infty f(E)|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\rho(E)dE \\ -\int_0^\infty f(E)|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E)\rho(E-\hbar\omega)dE.\end{aligned}\tag{4.2}$$

In the second integral we note that $\langle E-\hbar\omega|Q|E\rangle$ vanishes for $E<\hbar\omega$, and making the transformation $E\rightarrow E+\hbar\omega$ in the integration variable, we obtain

$$\begin{aligned}C(-)=\int_0^\infty|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\rho(E)f(E) \\ \times\{1-f(E+\hbar\omega)/f(E)\}dE.\end{aligned}\tag{4.3}$$

By Eq. (2.8) this becomes

$$\begin{aligned}C(-)=\{1-\exp(-\hbar\omega/kT)\}\int_0^\infty|\langle E+\hbar\omega|Q|E\rangle|^2 \\ \times\rho(E+\hbar\omega)\rho(E)f(E)dE.\end{aligned}\tag{4.4}$$

If $C(+)$ denotes the value of (4.1) corresponding to the positive sign, we obtain, in an identical fashion,

$$C(+)=\{1+\exp(-\hbar\omega/kT)\}\int_0^\infty |\langle E+\hbar\omega|Q|E\rangle|^2 \times\rho(E+\hbar\omega)\rho(E)f(E)dE. \quad (4.5)$$

With these alternative expressions for (4.1), we can write, from Eq. (2.15),

$$R(\omega)/|Z(\omega)|^2=\pi\omega\{1-\exp(-\hbar\omega/kT)\} \times\int_0^\infty |\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\rho(E)f(E)dE, \quad (4.6)$$

and from Eq. (3.6),

$$\langle V^2\rangle=\int_0^\infty |Z|^2\hbar\omega^2\{1+\exp(-\hbar\omega/kT)\} \times\int_0^\infty |\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\rho(E)f(E)dEd\omega. \quad (4.7)$$

Comparison of these equations yields directly our fundamental theorem:

$$\langle V^2\rangle=(2/\pi)\int_0^\infty R(\omega)E(\omega,T)d\omega, \quad (4.8)$$

where

$$E(\omega,T)=\tfrac{1}{2}\hbar\omega+\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}. \quad (4.9)$$

It may be recognized that $E(\omega, T)$ is, formally, the expression for the mean energy at the temperature T of an oscillator of natural frequency ω.

At high temperatures, $E(\omega, T)$ takes its equipartition value

$$E(\omega,T)\simeq kT, \quad (kT\gg\hbar\omega) \quad (4.10)$$

and the generalized Nyquist relation takes its most familiar form

$$\langle V^2\rangle\simeq(2/\pi)kT\int R(\omega)d\omega. \quad (4.11)$$

To reiterate then, a system with a generalized resistance $R(\omega)$ exhibits, in equilibrium, a fluctuating force given by Eq. (4.8) or, at high temperature, by Eq. (4.11).

We shall now consider a few specific applications of this theorem. The application to the electrical case is obvious, the general Eq. (4.8) being identical with the Nyquist relation if the force V is interpreted as the voltage. The content of the general theorem is, however, clarified by considering certain less trivial applications.

V. APPLICATION TO BROWNIAN MOTION

The fundamental result of the theory of the Brownian motion of a small particle immersed in a fluid is that the particle moves in response to a randomly fluctuating force $F(t)$ (with components F_x, F_y, F_z) such that

$$\langle F_x^2\rangle=(2/\pi)kT\eta\int d\omega. \quad (5.1)$$

Here η is a frictional constant, so defined that the viscous drag on a particle moving with velocity v is

$$\text{Frictional force}=-\eta v. \quad (5.2)$$

(If, in particular, the particle is spherical, η is known by Stokes' law as $6\pi\cdot$(viscosity)$\cdot$(radius).)

It is interesting to recall briefly the rather complicated and circuitous chain of reasoning by which the above result is obtained. One first makes the *assumption* that the particle moves in response to a randomly fluctuating force which has a *constant*, but unknown, spectral density. (The spectral density is, in actuality, not constant, and Eq. (5.1) is not valid at high frequencies.) By application of the theory of stochastic processes, one is then able to predict the distribution functions for either the displacement or the velocity of the particle.[4] The distribution function for displacement yields the diffusion constant, which in turn may be related by the Einstein relation[5] to the frictional constant η, thus evaluating the spectral density.[6] Alternatively, the distribution function for velocity yields the energy, which is known by the equipartition theorem and which therefore evaluates the spectral density, yielding Eq. (5.1).

We now apply our general formalism to the Brownian motion. We assume the existence of a viscous force as given by Eq. (5.2). The system of a particle in a fluid, the particle being acted on by an external force, is then dissipative and linear. The real part of the impedance is simply η (the inertial mass of the particle giving a pure reactance of $m\omega$). We conclude immediately, in accordance with Eq. (4.8), that a particle in a fluid is acted upon by a spontaneously fluctuating force for which

$$\langle F_x^2\rangle=(2/\pi)\eta\int_0^\infty E(\omega,T)d\omega. \quad (5.3)$$

For high temperatures or low frequencies, $(\hbar\omega\ll kT)$; this reduces to Eq. (5.1).

VI. ELECTRIC DIPOLE RADIATION RESISTANCE AND ELECTRIC FIELD FLUCTUATIONS IN THE VACUUM

An oscillating electric charge radiates energy, leading to a radiation resistance. We shall see that this radiation resistance implies a fluctuating electric field as given by the Planck radiation law.

[4] See M. C. Wang and G. E. Uhlenbeck, Revs. Modern Phys. **17**, 323 (1945); and J. L. Doob, Ann. Math. **43**, 351 (1942).

[5] See A. Einstein, *Investigations on the Theory of the Brownian Movement* (Dutton and Company, New York); or A. Einstein, Ann. Physik **17**, 549 (1905).

[6] A similar analysis has been applied to the flow of heat by L. S. Ornstein and J. M. W. Milatz, Physica **6**, 1139 (1939).

Consider a dipole, of charge e, displacement x, and dipole moment $p=ex$. Let one charge be fixed and let the other oscillate so that

$$P=P_0 \sin\omega t. \tag{6.1}$$

It is well known that the electric dipole radiation leads to a dissipative force[7]

$$F_d=-\tfrac{2}{3}e^2c^{-3}d^2v/dt^2, \tag{6.2}$$

where v is the velocity of the moving charge. The equation of motion of this charge is

$$mdv/dt+m\omega_0^2x+F_d=F, \tag{6.3}$$

where F is the applied force, and ω_0 is the natural frequency associated with the intra-dipole binding force. Inserting (6.1) in (6.3) we get

$$F=mP_0e^{-1}(\omega_0^2-\omega^2)\sin\omega t+\tfrac{2}{3}e\omega^3c^{-3}P_0\cos\omega t. \tag{6.4}$$

One may note that the average rate of radiation of energy $\langle Fv\rangle$ is

$$\langle Fv\rangle=\tfrac{1}{2}(\tfrac{2}{3}e\omega^3c^{-3}P_0)(\omega P_0e^{-1})=\tfrac{1}{3}\omega^4c^{-3}P_0^2. \tag{6.5}$$

The real part of the impedance is obtained by taking the ratio of the in-phase component of F to v. Thus

$$R(\omega)=(\tfrac{2}{3}e\omega^3c^{-3}P_0)/(\omega P_0e^{-1})=\tfrac{2}{3}e^2c^{-3}\omega^2. \tag{6.6}$$

According to our general theorem, we now deduce that there exists a randomly fluctuating force $e\mathcal{E}_x$ on the charge, and hence a randomly fluctuating electric field $\mathcal{E}_x$, such that

$$\langle e^2\mathcal{E}_x^2\rangle=(2/\pi)\int_0^\infty E(\omega,T)\tfrac{2}{3}e^2c^{-3}\omega^2d\omega,$$

or

$$\langle\mathcal{E}_x^2\rangle=(4/3)\pi^{-1}c^{-3}\times\int_0^\infty\{\tfrac{1}{2}\hbar\omega+\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}\}\omega^2d\omega. \tag{6.7}$$

This conclusion can be put into a more familiar form by utilizing the fact that the energy density in an isotropic radiation field is simply

$$\text{Energy density}=\langle\mathcal{E}^2\rangle/4\pi=3\langle\mathcal{E}_x^2\rangle/4\pi \tag{6.8}$$

whence

$$\text{Energy density}=\pi^{-2}c^{-3}\int_0^\infty\{\tfrac{1}{2}\hbar\omega+\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}\}\omega^2d\omega. \tag{6.9}$$

The first term in this equation gives the familiar infinite "zero-point" contribution, and the second term gives the Planck radiation law.[8]

[7] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1936).

[8] The interaction of free electron and radiation field has been discussed from a somewhat different point of view by W. Pauli, Z. Physik **18**, 272 (1923); A. Einstein and P. Ehrenfest, Z. Physik **19**, 301 (1923).

VII. ACOUSTIC RADIATION RESISTANCE AND PRESSURE FLUCTUATIONS IN A GAS

We now consider the acoustic radiation from a small oscillating sphere in a gaseous medium. This radiation leads to a radiation impedance which, in accordance with our general theorem, implies a fluctuating pressure in the gas.

The wave equation for the propogation of pressure waves in the gas is

$$\nabla^2P=c^{-2}\partial^2P/\partial t^2, \tag{7.1}$$

where c is the velocity of sound in the gas. Let the radius of the sphere be a, and let

$$a=a_0+e^{-i\omega t}\delta a \tag{7.2}$$

so that the sphere expands and contracts sinusoidally. The boundary condition to be satisfied by the pressure waves at $r=a_0$ is

$$\rho\partial^2a/\partial t^2=-\partial P/\partial r \quad \text{at} \quad r=a_0, \tag{7.3}$$

where ρ is the equilibrium value of the density. The solution of these equations is readily found to be

$$P=r^{-1}P_0\exp(iKr-i\omega t), \tag{7.4}$$

where

$$K=\omega/c \tag{7.5}$$

and

$$P_0=-\rho\omega^2a_0^2\delta a[1+iKa_0]\cdot[1+(Ka_0)^2]^{-1}\times\exp(-iKa_0). \tag{7.6}$$

Thus, the compressive force acting on the surface of the sphere is

$$F=4\pi a_0P_0\exp(iKa_0-i\omega t), \tag{7.7}$$

and defining the radiation impedance as the ratio of complex force to complex velocity, we find

$$\begin{aligned}Z&=F/[-i\omega\exp(-i\omega t)\delta a]\\ Z&=[4\pi a_0^2\rho c(Ka_0)^2-i4\pi a_0^2\rho cKa_0]/[1+(Ka_0)^2].\end{aligned} \tag{7.8}$$

The generalized Nyquist relation now states that a sphere immersed in a gas will experience a fluctuating compressive force, such that

$$\langle F^2\rangle=(2/\pi)\int E(\omega,T)4\pi a_0^2\rho c(\omega a_0/c)^2\times[1+(\omega a_0/c)^2]^{-1}d\omega. \tag{7.9}$$

The fluctuating pressure is the compressive force per unit area on a vanishingly small sphere.

$$\langle P^2\rangle=\lim_{a_0\to0}\langle F^2\rangle/(4\pi a_0^2)^2, \tag{7.10}$$

or

$$\langle P^2\rangle=\tfrac{1}{2}\pi^{-2}c^{-1}\rho\int E(\omega,T)\omega^2d\omega. \tag{7.11}$$

This result may be checked by a direct computation paralleling the standard derivation of the Planck radiation law for the electromagnetic modes in a vacuum. The number of acoustic modes with frequency between ω and $\omega+d\omega$ is $\frac{1}{2}\pi^{-2}c^{-3}\omega^2 d\omega$, and the acoustic energy density is

$$\text{Energy density}=\int E(\omega, T)\tfrac{1}{2}\pi^{-2}c^{-3}\omega^2 d\omega. \quad (7.12)$$

Employing the relation that the acoustic energy density is proportional to the mean square excess pressure

$$\text{Energy density}=\rho^{-1}c^{-2}\langle P^2\rangle, \quad (7.13)$$

we again obtain Eq. (7.11).

It is interesting to compare the above result with the pressure fluctuations at a boundary of the gas. The proximity to the boundary, and the shape of the boundary, may be expected to influence the radiation impedance and hence the pressure fluctuations. We consider the pressure fluctuations immediately contiguous to a plane rigid boundary, and we shall find that for this simple case, the mean square pressure fluctuation is just twice that in the volume of the gas.

Consider a plane wall bounding a semi-infinite region containing the gas. If the wall contains a circular piston of radius a_0, the radiation resistance is[9]

$$R=\pi a_0^2\rho c[1-ca_0^{-1}\omega^{-1}J_1(2\omega a_0/c)], \quad (7.14)$$

where J_1 indicates the first order bessel function. The fluctuating force acting on a circular area in a plane boundary is therefore

$$\langle F^2\rangle=(2/\pi)\int E(\omega, T)\pi a_0^2\rho c \times[1-ca_0^{-1}\omega^{-1}J_1(2\omega a_0/c)]d\omega, \quad (7.15)$$

and the fluctuating pressure is

$$\langle P^2\rangle=\lim_{a_0\to 0}\langle F^2\rangle/(\pi a_0^2)^2 \quad (7.16)$$

or

$$\langle P^2\rangle=\rho\pi^{-2}c^{-1}\int E(\omega, T)\omega^2 d\omega. \quad (7.17)$$

Thus the mean square fluctuating wall pressure, as given by (7.17), is just twice the mean square fluctuating volume pressure, as given by (7.11). This factor of two clearly arises from the fact that the pressure waves in the gas must have velocity nodes at the wall. Fluctuations in the neighborhood of the wall may be found by treating the radiation from an oscillating sphere near a reflecting boundary.

Finally, it will be noted that the above equations for pressure fluctuations involve the velocity of sound in the gas, which is not a usual thermodynamic parameter. This quantity may, however, be expressed in terms of standard thermodynamic quantities. Thus for frequencies which are sufficiently high that the compressions in the acoustic waves may be considered to be adiabatic, we have[9]

$$c^2=C_P C_V^{-1}\rho^{-1}\mathcal{K}_T^{-1}, \quad (7.18)$$

where C_P and C_V are the specific heats at constant pressure and volume, ρ is the density, and $\mathcal{K}_T$ is the isothermal compressibility. For these frequencies, the pressure fluctuations in the volume of the gas are thus given by

$$\langle P^2\rangle=\tfrac{1}{2}\pi^{-2}\rho^2\mathcal{K}_T C_V C_P^{-1}\int E(\omega, T)\omega^2 d\omega. \quad (7.19)$$

VIII. CONCLUSION

The generalized Nyquist relation establishes a quantitative correlation between dissipation, as described by the resistance, and certain fluctuations. It seems to be possible to give an intuitive interpretation of such a connection.

A dissipative process may be conveniently considered to involve the interaction of two systems, which we characterize as the "source system" and the "dissipative system." The dissipative system, explicitly considered in Secs. II and III, is necessarily a system with densely distributed energy levels and is capable of absorbing energy when acted upon by a periodic force. The source system is the system which provides this periodic force and which delivers energy to the dissipative system.

Assume the source system to be first isolated from the dissipative system and to be given some internal energy. If the source system is a simple dynamical system, its subsequent dynamics will be periodic (as, for instance, the oscillations of a pendulum or of a polyatomic molecule). The system may be thought of as possessing a sort of internal coherence. If, now, the source system is allowed to act on the dissipative system, this internal coherence is destroyed, the periodic motion vanishes and the energy is sapped away, and the source system is left at last with only the random disordered energy ($\simeq kT$) characteristic of thermal equilibrium. This loss of coherence within the source system may be thought of as being *caused* by the random fluctuations generated by the dissipative system and acting on the source system. The dissipation thus appears as the macroscopic manifestation of the disordering effect of the Nyquist fluctuations and, as such, is necessarily quantitatively correlated with the fluctuations.

An analogy which is perhaps useful is provided by the historical development of the theory of spontaneous radiation from excited atoms. After the initial development of quantum mechanics, it was found impossible to compute the spontaneous transition probabilities for an isolated excited atom, and this dissipative process appeared to be outside the existing structure of dynamics. With the development of quantum electrodynamics, however, the dissipation could be computed,

[9] P. M. Morse, *Vibration and Sound* (McGraw-Hill Book Company, Inc., New York, 1936).

and it was found that the "spontaneous" transitions could be consistently considered to be induced by the random fluctuations of the electromagnetic field in the vacuum. In this case, of course, the excited atom plays the role of the source system, and the "vacuum" plays the role of the dissipative system.

It would thus appear that a reasonable approach to the development of a theory of linear irreversible processes is through the development of the theory of fluctuations in equilibrium systems. Certain results in this connection will be given in subsequent papers by Richard F. Greene and one of the authors (H.B.C.).

PHYSICAL REVIEW VOLUME 83, NUMBER 1 JULY 1, 1951

The Disintegration of Nd^{147}

W. S. EMMERICH AND J. D. KURBATOV
Ohio State University, Columbus, Ohio
(Received March 12, 1951)

Three groups of monochromatic electrons corresponding to gamma-rays of 91.5±1.0 kev, 320±3 kev, and 534±4 kev have been identified in the disintegration of Nd^{147}. These gamma-rays are ascribed to transitions in Pm^{147}. Evidence has been obtained for a complex beta-spectrum of Nd^{147}. On the basis of coincidences, a partial scheme of disintegration for Nd^{147} is proposed. The total energy decrease from the ground state of Nd^{147} to the ground state of Pm^{147} nuclei is 1.425±0.015 Mev.

I. INTRODUCTION

SINCE the original observation of an 11-day period radioactive neodymium[1] several papers have been published on the radiations emitted by this species.[2–5] When fission products became available, it was possible to identify the 11-day period as mass number 147. By absorption technique, the beta-emission was found to be 0.9 Mev and ~0.4 Mev with intensities of 60 and 40 percent, respectively.[4] Low energy electrons, x-rays, and gamma-rays of ~0.58 Mev with an intensity of 40 percent were also observed. Coincidences were obtained between high energy betas and x-rays, also between lower energy betas and gammas. In a recent letter[5] beta-energies of 780 kev and 175 kev were reported, complex beta-gamma-coincidences were found, and the absence of gamma-gamma-coincidences noted.

II. PROCEDURE

For the present study, commercially supplied neodymium in the form of neodymium oxide was irradiated with neutrons at the Oak Ridge National Laboratory. The activated material was aged to allow for the decay of 12-min Pm^{151} and 47-hr Pm^{149}. Corrections were applied in various phases of this investigation for the growth of 0.22-Mev beta-rays of the daughter product, Pm^{147}. Spectrometer sources were prepared on Cellophane tape. They were not covered, and a radiogram showed that the distribution of the activity was practically uniform. Since neutron-activated material was used, inert neodymium was present, and a method of obtaining a correction for scattering in the source (at low energies) is mentioned below in connection with the correction for counter window absorption. Sources for the coincidence counter were mounted on Cellophane and covered with zapon.

TABLE I. Internal conversion electrons in Nd^{147}.

Energy of electrons (kev)	Estimated intensity	Conversion shell	Energy of gamma-ray (kev)
46.0±0.5	strong	*K*	
84.5±0.5	strong	*L*	91.5
89.9±0.5	medium	*M*	
275 ±3	weak	*K*	320
315 ±4	very weak	*L*	
489 ±4	medium	*K*	534
528 ±5	weak	*L*	

Measurements were carried out with the aid of a permanent magnet electron spectrograph, a thick lens beta-spectrometer, and coincidence counters.

The electron spectrograph is of the semicircular type using photographic plates as detectors. Although large sources had to be used to attain sufficient intensity for the measurement of internal conversion electrons in this instrument, a resolution of ~ one percent could be obtained for energies over 100 kev. The magnetic field strength between the pole pieces was determined with internal conversion electrons of I^{131} and Cs^{137}.

The ring focusing was efficiently attained in the beta-spectrometer by using a coil that extends along the total path length of the electrons. The position of the defining baffle was found from electron trajectories as determined by the empirically measured magnetic field distribution inside the coil. Additional baffles were installed to minimize scattering from the walls and to eliminate the higher order focusing of slow electrons. A 2-mg/cm^2 mica window G-M counter was used as detector. The spectrometer was operated with a resolution of ~4

[1] Law, Pool, Kurbatov, and Quill, Phys. Rev. **59**, 936 (1941).
[2] W. Bothe, Z. Naturforsch. **1**, 179 (1946).
[3] Cork, Shreffler, and Fowler, Phys. Rev. **74**, 240 (1948).
[4] Marinsky, Glendenin, and Coryell, J. Am. Chem. Soc. **69**, 2781 (1947).
[5] C. E. Mandeville and E. Shapiro, Phys. Rev. **79**, 391 (1950).