From Non-Negative to General Operator Cost Partitioning
Proof Details
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This technical report contains detailed proofs for our AAAI paper (Pommerening et al. 2015). We use the same notation and concepts as described there.

Connection to Operator-Counting Constraints

We show that operator-counting constraints yield cost-partitioned heuristics even if auxiliary variables are used.

Theorem 1. *Let $\mathcal{C}$ be a set of operator-counting constraints for a state s. Then*

$$h^{\mathrm{LP}}_{\mathcal{C}}(\mathit{cost}) = h^{\mathrm{OCP}}((h^{\mathrm{LP}}_{\{C\}})_{C\in\mathcal{C}}, s).$$

In the following, we assume that there are n operator-counting variables $\mathbf{Count} = (\mathsf{Count}_{o_1}, \dots, \mathsf{Count}_{o_n})^\top$ and m_C auxiliary variables $\mathbf{Aux}^C = (\mathsf{Aux}^C_1, \dots, \mathsf{Aux}^C_{m_C})^\top$ for each constraint $C \in \mathcal{C}$. We further assume w.l.o.g. that all variables are non-negative and all constraints are greater-than-or-equal constraints. We later introduce equations that correspond to unbounded variables in the dual. We mark such dual variables with a star (e. g., ${}^*\mathsf{Cost}_o$).

An operator-counting constraint C over $\mathbf{Count}$ and $\mathbf{Aux}^C$ with k_C inequalities consists of an $(n \times k_C)$ matrix $\mathit{coeffs}_{\mathit{Count}}(C)$ (influence on the operator-counting variables), an $(m_C \times k_C)$ matrix $\mathit{coeffs}_{\mathit{Aux}}(C)$ (influence on the auxiliary variables), and a k_C-vector of bounds $\mathit{bounds}(C) \in \mathbb{R}^{k_C}$.

The integer program for an operator-counting constraint C using cost function *cost* is $\mathrm{IP}_{\{C\}}(\mathit{cost})$:

$$\text{Minimize} \sum_{o\in\mathcal{O}} \mathit{cost}(o)\mathsf{Count}_o \text{ subject to}$$
$$\mathit{coeffs}_{\mathit{Count}}(C)\mathbf{Count} + \mathit{coeffs}_{\mathit{Aux}}(C)\mathbf{Aux}^C \geq \mathit{bounds}(C)$$
$$\mathbf{Count} \geq \mathbf{0} \text{ and } \mathbf{Aux}^C \geq \mathbf{0}$$
$$\mathsf{Count}_o \text{ is an integer for all } o \in \mathcal{O}$$

We consider the LP relaxation ($\mathrm{LP}_{\{C\}}(\mathit{cost})$) of $\mathrm{IP}_{\{C\}}(\mathit{cost})$ which drops the integrality condition for Count_o. The dual of $\mathrm{LP}_{\{C\}}(\mathit{cost})$ has k_C variables $\mathbf{Dual} = (\mathsf{Dual}_1, \dots, \mathsf{Dual}_{k_C})^\top$ and $n+m_C$ constraints. Interpreting *cost* as the vector of costs we can write is as

$$\text{Maximize } \mathit{bounds}(C) \cdot \mathbf{Dual} \text{ subject to}$$
$$\mathit{coeffs}_{\mathit{Count}}(C)^\top \mathbf{Dual} \leq \mathit{cost}$$
$$\mathit{coeffs}_{\mathit{Aux}}(C)^\top \mathbf{Dual} \leq \mathbf{0}$$
$$\mathbf{Dual} \geq \mathbf{0}$$

We now consider a *set* $\mathcal{C}$ of operator-counting constraints for a state s. The linear program $\mathrm{LP}_{\mathcal{C}}(\mathit{cost})$ is:

$$\text{Minimize} \sum_{o\in\mathcal{O}} \mathit{cost}(o)\mathsf{Count}_o \text{ subject to}$$
$$\mathit{coeffs}_{\mathit{Count}}(C)\mathbf{Count} + \mathit{coeffs}_{\mathit{Aux}}(C)\mathbf{Aux}^C \geq \mathit{bounds}(C) \quad \text{for } C \in \mathcal{C}$$
$$\mathbf{Count} \geq \mathbf{0} \text{ and } \mathbf{Aux}^C \geq \mathbf{0} \text{ for } C \in \mathcal{C}$$

We now introduce new variables LCount^C_o and new equations $\mathsf{LCount}^C_o = \mathsf{Count}_o$ for every $C \in \mathcal{C}$ and $o \in \mathcal{O}$. This allows us to replace every occurrence of Count_o in the remaining inequalities by LCount^C_o:

$$\text{Minimize} \sum_{o\in\mathcal{O}} \mathit{cost}(o)\mathsf{Count}_o \text{ subject to}$$
$$\mathit{coeffs}_{\mathit{Count}}(C)\mathbf{LCount}^C + \mathit{coeffs}_{\mathit{Aux}}(C)\mathbf{Aux}^C \geq \mathit{bounds}(C)$$
$$\mathbf{Count} - \mathbf{LCount}^C = \mathbf{0} \quad \text{for } C \in \mathcal{C}$$
$$\mathbf{Count} \geq \mathbf{0} \text{ and } \mathbf{Aux}^C \geq \mathbf{0}, \mathbf{LCount}^C \geq \mathbf{0} \text{ for } C \in \mathcal{C}$$

For each constraint C, the dual of this LP contains one non-negative variable Dual^C_i for each of its inequalities $1 \leq i \leq k_C$ and one unbounded variable ${}^*\mathsf{Cost}^C_o$ for each of its equations. Bounds and objective function change their role, so objective coefficients are $\mathit{bounds}(C)$ for $\mathbf{Dual}^C$ and $\mathbf{0}$ for ${}^*\mathbf{Cost}^C$. The primal variables $\mathbf{Aux}^C$ only occur in the inequalities for constraint C and correspond to the dual constraints

$$\mathit{coeffs}_{\mathit{Aux}}(C)^\top \mathbf{Dual}^C \leq \mathbf{0}$$

The primal variables $\mathbf{LCount}^C$ only occur in the inequalities and equations for constraint C and correspond to the dual constraints

$$\mathit{coeffs}_{\mathit{Count}}(C)^\top \mathbf{Dual}^C - {}^*\mathbf{Cost}^C \leq \mathbf{0}$$

Each primal variable Count_o occurs in exactly one equation for each constraint C and corresponds to the dual constraints

$$\sum_{C \in \mathcal{C}} {}^*\mathsf{Cost}_o^C \leq \mathit{cost}(o)$$

Together, these constraints make up the dual of $\mathrm{LP}_{\mathcal{C}}(\mathit{cost})$:

$$\begin{aligned} \text{Maximize } \sum_{C \in \mathcal{C}} \mathit{bounds}(C) \cdot \mathbf{Dual}^C & \text{ subject to} \\ \mathit{coeffs}_{\mathit{Count}}(C)^\top \mathbf{Dual}^C & \leq {}^*\mathbf{Cost}^C \\ \mathit{coeffs}_{\mathit{Aux}}(C)^\top \mathbf{Dual}^C & \leq \mathbf{0} \\ \mathbf{Dual}^C & \geq \mathbf{0} \\ & \quad \text{for } C \in \mathcal{C} \\ \sum_{C \in \mathcal{C}} {}^*\mathsf{Cost}_o^C & \leq \mathit{cost}(o) \text{ for } o \in \mathcal{O} \end{aligned}$$

The first two inequalities are exactly the dual constraints of $\mathrm{LP}_{\{C\}}({}^*\mathbf{Cost}^C)$ and the objective function is exactly the sum of the dual objective functions for these LPs. The remaining inequality ensures that ${}^*\mathbf{Cost}^C$ defines a general cost partitioning. As we maximize the sum of individual heuristic values over all possible general cost partitionings, the result is the optimal general cost partitioning of the component heuristics.

Net-Change Constraints

In the paper, we show that the state equation heuristic is perfect for the projection on a goal variable V, i. e.

$$h^{\mathrm{SEQ}}(s, \mathit{cost}) = h^*(s, \mathit{cost}) \text{ in } \Pi^V \text{ for every } V \in \mathit{vars}(s_\star)$$

Here we show that non-goal variables can be transformed into goal variables without influencing the heuristic value. The statement then holds for every $V \in \mathcal{V}$ and the proof in the paper shows that $h^{\mathrm{LP}}_{\mathcal{C}_V}(\mathit{cost}) = h^V(s)$.

We extend the domain of each non-goal variable V with a new value v^* and the goal description with the fact $\langle V, v^* \rangle$. Additionally, we add a new operator o_V that is free of cost, has no preconditions and the single effect $\langle V, v^* \rangle$. This transformation obviously does not change the perfect heuristic value: o_V can be added for all non-goal variables to any plan for the original task to satisfy the new goals. Each plan for the transformed task is a plan for the original task if all occurrences of o_V are removed.

The transformation also does not change the value of the state equation heuristic. Because the new operator o_V can be applied independent of the value of V, it falls in the category of operators that *sometimes produce* $\langle V, v^* \rangle$ and *sometimes consume* $\langle V, v' \rangle$ for every $v' \in \mathit{dom}(V) \setminus \{v^*\}$. Net change constraints only mention operators that produce a fact or always consume it. Thus, existing constraints are not changed by adding o_V.

In the objective function of the LP each operator occurs weighted by its cost. Since the cost of o_V is 0, the objective function is unchanged.

The only modification to the linear program is the constraint that is added for the new fact $\langle V, v^* \rangle$. As operator o_V is the only one producing this fact and no operator consumes it, the new constraint is

$$\mathsf{Count}_{o_V} \geq 1$$

The variable Count_{o_V} occurs only in this constraint and does not influence the objective value, so it can safely be set to 1. The LP for the transformed task thus has the same solutions for the counting variables of the original operators.

Potential Heuristics

Admissible and consistent potential heuristics can be classified by a linear program and the paper shows how to do this for a restricted set of tasks. Here we generalize this definition for arbitrary planning tasks.

For a partial variable assignment p, we introduce the notation $\mathit{maxpot}(V, p)$ for the maximal potential that a state consistent with p can have for variable V:

$$\begin{aligned} \mathit{maxpot}(V, p) &= \max_{s \text{ consistent with } p} \mathrm{pot}(\langle V, s[V] \rangle) \\ &= \begin{cases} \mathrm{pot}(\langle V, p[V] \rangle) & \text{if } V \in \mathit{vars}(p) \\ \max_{v \in \mathit{dom}(V)} \mathrm{pot}(\langle V, v \rangle) & \text{otherwise} \end{cases} \end{aligned}$$

A potential heuristic h^{pot} is goal-aware if and only if $h^{\mathrm{pot}}(s) \leq 0$ for all states s consistent with $s_\star$. It is sufficient to require this condition only for a state that is consistent with $s_\star$ and has maximal potential among all goal states:

$$\max_{s \text{ consistent with } s_\star} h^{\mathrm{pot}}(s) = \sum_{V \in \mathcal{V}} \mathit{maxpot}(V, s_\star) \leq 0$$

The resulting inequality is not state-dependent and is necessary and sufficient for h^{pot} to be goal-aware.

A potential heuristic h^{pot} is consistent if and only if $h^{\mathrm{pot}}(s) \leq \mathit{cost}(o) + h^{\mathrm{pot}}(s[\![o]\!])$ for every state s and every operator o applicable in s. This condition can be simplified as follows because all facts that are not changed by an effect cancel out:

$$\begin{aligned} \mathit{cost}(o) &\geq \sum_{V \in \mathcal{V}} \mathrm{pot}(\langle V, s[V] \rangle) - \sum_{V \in \mathcal{V}} \mathrm{pot}(\langle V, s[\![o]\!][V] \rangle) \\ &= \sum_{V \in \mathit{vars}(\mathit{eff}(o))} (\mathrm{pot}(\langle V, s[V] \rangle) - \mathrm{pot}(\langle V, \mathit{eff}(o)[V] \rangle)) \end{aligned}$$

Again, it is sufficient to require the inequality only for a state that is consistent with the operator's precondition and that has maximal potential:

$$\mathit{cost}(o) \geq \sum_{V \in \mathit{vars}(\mathit{eff}(o))} (\mathit{maxpot}(V, \mathit{pre}(o)) - \mathrm{pot}(\langle V, \mathit{eff}(o)[V] \rangle))$$

The resulting inequality is no longer state-dependent and is necessary and sufficient for consistency.

Goal-aware and consistent potential heuristics can thus be compactly classified as a set of linear equations. A goal-aware and consistent heuristic is also admissible, so we can use an LP solver to optimize any linear combination of potentials and transform the solution into a consistent and admissible potential heuristic.

Definition 1. *Let f be a solution to the following LP:*

$$\textit{Maximize } \mathrm{opt} \textit{ subject to}$$

$$^{*}\mathsf{P}_{\langle V,v\rangle} \leq {}^{*}\mathsf{Max}_V \textit{ for all } V \in \mathcal{V} \textit{ and } v \in \textit{dom}(V)$$

$$\sum_{V \in \mathcal{V}} \textit{maxpot}(V, s_\star) \leq 0$$

$$\sum_{V \in \textit{vars}(\textit{eff}(o))} \left(\textit{maxpot}(V, \textit{pre}(o)) - {}^{*}\mathsf{P}_{\langle V, \textit{eff}(o)[V]\rangle}\right) \leq \textit{cost}(o)$$

$$\textit{for all } o \in \mathcal{O}$$

$$\textit{with } \textit{maxpot}(V,p) = \begin{cases} ^{*}\mathsf{P}_{\langle V,p[V]\rangle} & \textit{if } V \in \textit{vars}(p) \\ ^{*}\mathsf{Max}_V & \textit{otherwise} \end{cases}$$

where the objective function opt *can be chosen arbitrarily.*

Then the function $\mathrm{pot}_{\mathrm{opt}}(\langle V,v\rangle) = f(\mathsf{P}_{\langle V,v\rangle})$ *is the* potential function optimized for opt *and* $h^{\mathrm{pot}}_{\mathrm{opt}}$ *is the* potential heuristic optimized for opt.

Corollary 1. *The heuristic* $h^{\mathrm{pot}}_{\mathrm{opt}}$ *is goal-aware, consistent, and admissible for any linear combination of potentials* opt.

Potential Heuristic Estimate in the Initial State

We consider the potential heuristic optimized for the initial state, i. e. using the optimization criterion $\mathrm{opt}_{s_{\mathrm{I}}} = \sum_{V \in \mathcal{V}} \mathsf{P}_{\langle V, s_{\mathrm{I}}[v]\rangle}$.

Proposition 1. $h^{\mathrm{pot}}_{\mathrm{opt}_{s_{\mathrm{I}}}}(s_{\mathrm{I}}) = h^{\mathrm{SEQ}}(s_{\mathrm{I}})$.

The state equation estimate $h^{\mathrm{SEQ}}(s_{\mathrm{I}})$ can be written as an LP heuristic for an operator-counting constraint SEQ for s_{I} with coefficient matrix *coeffs*(SEQ) and bounds vector *bounds*(SEQ). The matrix *coeffs*(SEQ) has a column for each operator and a row for each fact. The entry for operator o and fact $\langle V,v\rangle$ is

$$(\textit{coeffs}(\mathrm{SEQ}))_{o,\langle V,v\rangle} = \begin{cases} 1 & \text{if } o \text{ always produces } \langle V,v\rangle \\ 1 & \text{if } o \text{ sometimes produces } \langle V,v\rangle \\ -1 & \text{if } o \text{ always consumes } \langle V,v\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \textit{eff}(o)[V] = v \\ -1 & \text{if } \textit{pre}(o)[V] = v \text{ and} \\ & \quad V \in \textit{vars}(\textit{eff}(o)) \\ 0 & \text{otherwise} \end{cases}$$

The vector of bounds contains the following entry for each fact $\langle V,v\rangle$:

$$\textit{bounds}(\mathrm{SEQ})_{\langle V,v\rangle} = 1\{\text{if } s_\star[V] = v\} - 1\{\text{if } s_{\mathrm{I}}[V] = v\}$$

The heuristic value $h^{\mathrm{SEQ}}(s_{\mathrm{I}})$ is the objective value of the following LP:

$$\text{Minimize} \sum_{o \in \mathcal{O}} \textit{cost}(o)\mathsf{Count}_o \text{ subject to}$$

$$\textit{coeffs}(\mathrm{SEQ})\mathbf{Count} \geq \textit{bounds}(\mathrm{SEQ})$$

$$\mathbf{Count} \geq \mathbf{0}$$

The dual of this LP has one non-negative variable $\mathsf{X}_{\langle V,v\rangle}$ for each fact $\langle V,v\rangle$ and one constraint for each operator:

$$\text{Maximize} \sum_{\substack{V \in \mathcal{V} \\ v \in \textit{dom}(V)}} \textit{bounds}(\mathrm{SEQ})_{\langle V,v\rangle}\mathsf{X}_{\langle V,v\rangle} \text{ subject to}$$

$$\textit{coeffs}(\mathrm{SEQ})^{\top}\mathbf{X} \leq \textit{cost}$$

$$\mathbf{X} \geq \mathbf{0}$$

The objective function is 0 for all facts that are neither contained in s_{I} nor in $s_\star$ so it can be simplified to:

$$\sum_{V \in \textit{vars}(s_\star)} \mathsf{X}_{\langle V, s_\star[V]\rangle} - \sum_{V \in \mathcal{V}} \mathsf{X}_{\langle V, s_{\mathrm{I}}[V]\rangle}$$

The constraint for operator o is 0 for all variables that are not affected by o:

$$\sum_{V \in \textit{vars}(\textit{eff}(o))} \mathsf{X}_{\langle V, \textit{eff}(o)[V]\rangle} - \sum_{\substack{V \in \textit{vars}(\textit{eff}(o)) \\ \cap \textit{vars}(\textit{pre}(o))}} \mathsf{X}_{\langle V, \textit{pre}(o)[V]\rangle} \leq \textit{cost}(o)$$

We call the resulting linear program LP(SEQ) and use LP(pot) for the LP calculated for $h^{\mathrm{pot}}_{\mathrm{opt}_{s_{\mathrm{I}}}}(s_{\mathrm{I}})$.

A solution f for LP(SEQ) can be converted into a solution g for LP(pot) that has the same objective value. This can be checked by setting

$$g(^{*}\mathsf{P}_{\langle V,v\rangle}) = \begin{cases} f(\mathsf{X}_{\langle V,s_\star[V]\rangle}) - f(\mathsf{X}_{\langle V,v\rangle}) & \text{if } V \in \textit{vars}(s_\star) \\ \qquad\qquad\quad - f(\mathsf{X}_{\langle V,v\rangle}) & \text{otherwise} \end{cases}$$

$$g(^{*}\mathsf{Max}_V) = \max_{v \in \textit{dom}(V)} g(^{*}\mathsf{P}_{\langle V,v\rangle})$$

The objective function of LP(pot) under g is identical to the one of LP(SEQ) under f:

$$\sum_{V \in \mathcal{V}} g(^{*}\mathsf{P}_{\langle V, s_{\mathrm{I}}[V]\rangle}) = \sum_{V \in \textit{vars}(s_\star)} f(\mathsf{X}_{\langle V, s_\star[V]\rangle}) - \sum_{V \in \mathcal{V}} f(\mathsf{X}_{\langle V, s_{\mathrm{I}}[V]\rangle})$$

The inequality $g(^{*}\mathsf{P}_{\langle V,v\rangle}) \leq g(^{*}\mathsf{Max}_V)$ is obviously satisfied. To show $\sum_{V \in \mathcal{V}} g(\textit{maxpot}(V, s_\star)) \leq 0$, we show that $g(\textit{maxpot}(V, s_\star)) \leq 0$ holds for all V. For a goal variable V we have $g(\textit{maxpot}(V, s_\star)) = g(^{*}\mathsf{P}_{\langle V, s_\star[V]\rangle}) = f(\mathsf{X}_{\langle V, s_\star[V]\rangle}) - f(\mathsf{X}_{\langle V, s_\star[V]\rangle}) = 0$. For non-goal variables there is some $v_{\max} \in \textit{dom}(V)$ with

$$g(\textit{maxpot}(V, s_\star)) = g(^{*}\mathsf{Max}_V) = \max_{v \in \textit{dom}(V)} g(^{*}\mathsf{P}_{\langle V,v\rangle})$$
$$= g(^{*}\mathsf{P}_{\langle V,v_{\max}\rangle}) = -f(\mathsf{X}_{\langle V,v_{\max}\rangle}) \leq 0$$

For the last inequality we consider each variable $V \in \textit{vars}(\textit{eff}(o))$ separately. If $V \in \textit{vars}(\textit{pre}(o))$:

$$g(\textit{maxpot}(V, \textit{pre}(o))) - g(^{*}\mathsf{P}_{\langle V, \textit{eff}(o)[V]\rangle})$$
$$= g(^{*}\mathsf{P}_{\langle V, \textit{pre}(o)[V]\rangle}) - g(^{*}\mathsf{P}_{\langle V, \textit{eff}(o)[V]\rangle})$$
$$= f(\mathsf{X}_{\langle V, \textit{eff}(o)[V]\rangle}) - f(\mathsf{X}_{\langle V, \textit{pre}(o)[V]\rangle})$$

For $V \notin \textit{vars}(\textit{pre}(o))$ there is some $v_{\max} \in \textit{dom}(V)$ with

$$g(\textit{maxpot}(V, \textit{pre}(o))) - g(^{*}\mathsf{P}_{\langle V, \textit{eff}(o)[V]\rangle})$$
$$= g(^{*}\mathsf{Max}_V) - g(^{*}\mathsf{P}_{\langle V, \textit{eff}(o)[V]\rangle})$$
$$= g(^{*}\mathsf{P}_{\langle V, v_{\max}\rangle}) - g(^{*}\mathsf{P}_{\langle V, \textit{eff}(o)[V]\rangle})$$
$$= f(\mathsf{X}_{\langle V, \textit{eff}(o)[V]\rangle}) - f(\mathsf{X}_{\langle V, v_{\max}\rangle}) \leq f(\mathsf{X}_{\langle V,v\rangle})$$

Summing over all variables in $\mathit{vars}(\mathit{eff}(o))$ then shows the desired inequality with the constraint from LP(SEQ).

We have shown that a solution from LP(SEQ) can be converted into a solution for LP(pot) with the same objective value. For the other direction, we show that a solution g for LP(pot) can be converted into a solution f for LP(SEQ) with an objective value that is at least as high. For this transformation, we set

$$f(\mathsf{X}_{\langle V,v\rangle}) = g({}^*\mathsf{Max}_V) - g({}^*\mathsf{P}_{\langle V,v\rangle}).$$

Since $g({}^*\mathsf{Max}_V) \geq g({}^*\mathsf{P}_{\langle V,v\rangle})$ for all values v, the assignment f is non-negative.

The objective value for assignment f is:

$$\begin{aligned}
&\sum_{V\in \mathit{vars}(s_\star)} f(\mathsf{X}_{\langle V,s_\star[V]\rangle}) - \sum_{V\in\mathcal{V}} f(\mathsf{X}_{\langle V,s_\mathrm{I}[V]\rangle})\\
&= \sum_{V\in \mathit{vars}(s_\star)} \big(g({}^*\mathsf{Max}_V) - g({}^*\mathsf{P}_{\langle V,s_\star[V]\rangle})\big)\\
&\quad - \sum_{V\in\mathcal{V}} \big(g({}^*\mathsf{Max}_V) - g({}^*\mathsf{P}_{\langle V,s_\mathrm{I}[V]\rangle})\big)\\
&= \sum_{V\in\mathcal{V}} g({}^*\mathsf{P}_{\langle V,v\rangle}) - \sum_{V\in\mathcal{V}\setminus \mathit{vars}(s_\star)} g({}^*\mathsf{Max}_V)\\
&\quad - \sum_{V\in \mathit{vars}(s_\star)} g({}^*\mathsf{P}_{\langle V,s_\star[V]\rangle})\\
&= \sum_{V\in\mathcal{V}} g({}^*\mathsf{P}_{\langle V,s_\mathrm{I}[V]\rangle}) - \sum_{V\in\mathcal{V}} g(\mathit{maxpot}(V,s_\star))\\
&\geq \sum_{V\in\mathcal{V}} g({}^*\mathsf{P}_{\langle V,s_\mathrm{I}[V]\rangle})
\end{aligned}$$

The last step uses the goal-awareness condition of LP(pot).

To show that the inequality of LP(SEQ) is satisfied by the assignment f, we again consider each variable $V \in \mathit{vars}(\mathit{eff}(o))$ separately. If $V \in \mathit{vars}(\mathit{pre}(o))$:

$$\begin{aligned}
&f(\mathsf{X}_{\langle V,\mathit{eff}(o)[V]\rangle}) - f(\mathsf{X}_{\langle V,\mathit{pre}(o)[V]\rangle})\\
&= g({}^*\mathsf{P}_{\langle V,\mathit{pre}(o)[V]\rangle}) - g({}^*\mathsf{P}_{\langle V,\mathit{eff}(o)[V]\rangle})\\
&= g(\mathit{maxpot}(V,\mathit{pre}(o)) - g({}^*\mathsf{P}_{\langle V,\mathit{eff}(o)[V]\rangle})
\end{aligned}$$

For $V \notin \mathit{vars}(\mathit{pre}(o))$:

$$\begin{aligned}
&f(\mathsf{X}_{\langle V,\mathit{eff}(o)[V]\rangle})\\
&= g({}^*\mathsf{Max}_V) - g({}^*\mathsf{P}_{\langle V,\mathit{eff}(o)[V]\rangle})\\
&= g(\mathit{maxpot}(V,\mathit{pre}(o))) - g({}^*\mathsf{P}_{\langle V,\mathit{eff}(o)[V]\rangle})
\end{aligned}$$

Summing over all variables in $\mathit{vars}(\mathit{eff}(o))$ then shows the desired inequality with the constraint from LP(pot).

We have seen that solutions can be converted back and forth between LP(SEQ) and LP(pot) without lowering the objective value, which means both LPs must calculate the same heuristic value.

References

Pommerening, F.; Helmert, M.; Röger, G.; and Seipp, J. 2015. From non-negative to general operator cost partitioning. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence (AAAI 2015)*. AAAI Press. To appear.