Bounded Intention Planning Revisited: Proof

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We claim that each applicable operator partition induces a strong semistubborn set such that the applicable operators are the same as the operators in the partition.

Theorem 1. *Let s be a state, $X \in P_s$ be an applicable partition. Then $T_s := X \cup \{o \mid o$ interferes with $o' \in X\}$ is a strong semistubborn set with the same applicable operators as X.*

We split the proof for Theorem 1 into three parts, proving the claim for each possible type of operator partitions separately.

We will use the following notation: an operator $o \in \bar{\mathcal{O}}$ *affects* a variable $v \in \bar{\mathcal{V}}$ if and only if $v \in \mathit{vars}(\mathit{eff}_o)$ (and hence also $v \in \mathit{vars}(\mathit{pre}_o)$).

Proposition 1. *First case of Theorem 1. Let s be a state, $X \in P_s$ be an applicable operator partition of type* Fire_o*, for the original operator $o \in \mathcal{O}$. Then the set $T_s := X \cup \{o' \mid o'$ interferes with $o'' \in X\}$ is a strong semistubborn set with the same applicable operators as X.*

Proof. We first note that Fire_o contains exactly one operator, namely $\mathit{Fire}(o)$. Wlog. we assume that o affects $v \in \mathcal{V}$, i. e. $v \in \mathit{vars}(\mathit{pre}_o)$ and $v \in \mathit{vars}(\mathit{eff}_o)$. Furthermore, o possibly has a prevail-condition on some variable $w \in \mathcal{V}$, $w \neq v$, i. e. $w \in \mathit{vars}(\mathit{prv}_o)$. By definition $\mathit{Fire}(o)$ has the following properties:

$$
\begin{aligned}
\mathit{pre}_{\mathit{Fire}(o)}[v] &= \mathit{pre}_o[v] \\
\mathit{eff}_{\mathit{Fire}(o)}[v] &= \mathit{eff}_o[v] \\
\mathit{pre}_{\mathit{Fire}(o)}[O_v] &= o \\
\mathit{eff}_{\mathit{Fire}(o)}[O_v] &= \mathit{free} \\
\mathit{prv}_{\mathit{Fire}(o)}[w] &= \mathit{prv}_o[w] && \forall w \in \mathit{vars}(\mathit{prv}_o) \\
\mathit{pre}_{\mathit{Fire}(o)}[O_w] &= \mathit{frozen} && \forall w \in \mathit{vars}(\mathit{prv}_o) \\
\mathit{eff}_{\mathit{Fire}(o)}[O_w] &= \mathit{free} && \forall w \in \mathit{vars}(\mathit{prv}_o) \\
\mathit{pre}_{\mathit{Fire}(o)}[C_w] &= v && \forall w \in \mathit{vars}(\mathit{prv}_o) \\
\mathit{eff}_{\mathit{Fire}(o)}[C_w] &= \mathit{free} && \forall w \in \mathit{vars}(\mathit{prv}_o)
\end{aligned}
$$

Second, note that we know the following about state s, considering that $\mathit{Fire}(o)$ is applicable in s:

$$
\begin{aligned}
s[v] &= \mathit{pre}_o[v] \\
s[w] &= \mathit{prv}_o[w] && \forall w \in \mathit{vars}(\mathit{prv}_o) \\
s[O_v] &= o \\
s[O_w] &= \mathit{frozen} && \forall w \in \mathit{vars}(\mathit{prv}_o) \\
s[C_w] &= v && \forall w \in \mathit{vars}(\mathit{prv}_o)
\end{aligned}
$$

We show that all operators o' that interfere with $\mathit{Fire}(o)$ are not applicable in s. Thus $\mathit{Fire}(o)$ is the only applicable operator in T_s. Second, we show that for all these operators $o' \in T_s$ (except for $\mathit{Fire}(o)$), T_s already contains a necessary enabling set for o' in s.

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Let $u \neq \mathit{Fire}(o)$ be an arbitrary operator interfering with $\mathit{Fire}(o)$. Whenever we mention o' in the following, we refer to an operator $o' \in \mathcal{O}$, $o' \neq o$.

1. If u disables $\mathit{Fire}(o)$, we must distinguish the following cases.
   (a) $v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[v] \neq \mathit{pre}_{\mathit{Fire}(o)}[v]$. By definition, only fire operators can affect variables in $\mathcal{V}$. Let $u = \mathit{Fire}(o')$. From the definition of augmented operators, a fire operator affecting variable v also affects O_v. We conclude $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = o' \neq o = s[O_v]$. Therefore $\mathit{Fire}(o')$ is not applicable in s.

       $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$ in s: only $\mathit{Fire}(o)$ can change O_v from o to *free*, which is required because all *SetO* operators (which can set O_v to o') require $O_v = \mathit{free}$ as precondition.
   (b) $O_v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[O_v] \neq \mathit{pre}_{\mathit{Fire}(o)}[O_v]$. We have to distinguish three possible types for u:
       i. $u = \mathit{Fire}(o')$. If $v \in \mathit{vars}(\mathit{eff}_{o'})$, Case 1(a) applies. If $v \in \mathit{vars}(\mathit{prv}_{o'})$, we have $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = \mathit{frozen} \neq o = s[O_v]$. Thus $\mathit{Fire}(o')$ is not applicable. $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$ for the same reasons as in Case 1(a).
       ii. $u = \mathit{SetO}(o')$. We have $\mathit{pre}_{\mathit{SetO}(o')}[O_v] = \mathit{free} \neq o = s[O_v]$ and Case 1(a) applies.
       iii. $u = \mathit{Freeze}(v, x)$ for $x \in \mathcal{D}(v)$. We have $\mathit{pre}_{\mathit{Freeze}(v,x)}[O_v] = \mathit{free} \neq o = s[O_v]$ and Case 1(a) applies.
   (c) $w \in \mathit{vars}(\mathit{eff}_u)$ for variable $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{eff}_u[w] \neq \mathit{prv}_{\mathit{Fire}(o)}[w]$. Only fire operators can affect original variables from $\mathcal{V}$. Let $u = \mathit{Fire}(o')$. We conclude $\mathit{pre}_{\mathit{Fire}(o')}[O_w] = o' \neq \mathit{frozen} = s[O_w]$. Therefore $\mathit{Fire}(o')$ is not applicable.

       We claim that $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$ in s: we observe that in order to apply $\mathit{Fire}(o')$, O_w must not have the value *frozen*. Consider an operator $\tilde{o}$ that changes the value of O_w from *frozen* to *free*. Note that only a fire operator $\tilde{o} := \mathit{Fire}(o'')$ with $w \in \mathit{vars}(\mathit{prv}_{o''})$ can achieve this, because exactly for such fire operators, we have $\mathit{pre}_{\mathit{Fire}(o'')}[O_w] = \mathit{frozen}$ and $\mathit{eff}_{\mathit{Fire}(o'')}[O_w] = \mathit{free}$. If $v \in \mathit{vars}(\mathit{eff}_{\mathit{Fire}(o'')})$, then $\mathit{pre}_{\mathit{Fire}(o'')}[O_v] = o'' \neq o = s[O_v]$ and thus $\mathit{Fire}(o)$ must be applied first, as argued in Case 1(a). If $v \notin \mathit{vars}(\mathit{eff}_{\tilde{o}})$, then for some variable v', $v' \in \mathit{vars}(\mathit{eff}_{\mathit{Fire}(o'')})$. By definition of $\mathit{Fire}(o'')$, $\mathit{pre}_{\mathit{Fire}(o'')}[C_w] = v' \neq v = s[C_w]$. All operators that set C_w from v to *free* also affect v and thus have a precondition on O_v, and thus $\mathit{Fire}(o)$ needs to be applied before them.
   (d) $O_w \in \mathit{vars}(\mathit{eff}_u)$ for variable $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{eff}_u[O_w] \neq \mathit{pre}_{\mathit{Fire}(o)}[O_w]$. We have to distinguish two possible types for u (freeze operators for variable w do not disable $\mathit{Fire}(o)$):

i. $u = \mathit{Fire}(o')$. If $w \in \mathit{vars}(\mathit{eff}_{o'})$, we have $w \in \mathit{vars}(\mathit{eff}_{\mathit{Fire}(o')})$ and Case 1(c) applies. If $w \in \mathit{vars}(\mathit{prv}_{o'})$, there must be a variable $v' \in \mathcal{V}$ for which $v' \in \mathit{vars}(\mathit{eff}_{o'})$. If $v' = v$, then $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = o' \neq o = s[O_v]$ and $\mathit{Fire}(o')$ is not applicable in s. $\{\mathit{Fire}(o)\}$ is a necessary enabling set in s for $\mathit{Fire}(o')$ as argued in Case 1(a). If $v' \neq v$, then $\mathit{pre}_{\mathit{Fire}(o')}[C_w] = v' \neq v = s[C_w]$ and thus $\mathit{Fire}(o')$ is not applicable in s. $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$, because all operators that set C_w from v to *free* also affect v and thus have a precondition on O_v, and thus $\mathit{Fire}(o)$ needs to be applied before them, because $s[O_v] = o$.

ii. $u = \mathit{SetO}(o')$. We have $\mathit{pre}_{\mathit{SetO}(o')}[O_w] = \mathit{free} \neq \mathit{frozen} = s[O_w]$ and thus $\mathit{SetO}(o')$ is not applicable in s. $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{SetO}(o')$: Case 1(c) applies.

(e) $C_w \in \mathit{vars}(\mathit{eff}_u)$ for variable $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{eff}_u[C_w] \neq \mathit{pre}_{\mathit{Fire}(o)}[C_w]$. We have to distinguish two possible types for u:

i. $u = \mathit{Fire}(o')$. We know that $\mathit{pre}_{\mathit{Fire}(o')}[C_w] = v'$ for some variable $v' \in \mathcal{V}$. The case reduces to Case 1(d)i, the case "If $w \in \mathit{vars}(\mathit{prv}_{o'})$".

ii. $u = \mathit{SetC}(w, c)$ for some $c \in CG(w)$. We have $\mathit{pre}_{\mathit{SetC}(w,c)}[C_w] = \mathit{free} \neq v = s[C_w]$ and thus $\mathit{SetC}(w, c)$ is not applicable in s. $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{SetC}(w, c)$, because only fire operators affecting v can change C_w from v to *free* and $s[O_v] = o$ (see also Case 1(d)i, second case "If $v' \neq v$").

2. If $\mathit{Fire}(o)$ disables u, we must distinguish the following cases.

(a) $v \in \mathit{vars}(\mathit{pre}_u)$ and $\mathit{pre}_u[v] \neq \mathit{eff}_{\mathit{Fire}(o)}[v]$. Only fire operators $u = \mathit{Fire}(o')$ can have original variables from $\mathcal{V}$ as a precondition. From the definition of fire operators, we conclude $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = o' \neq o = s[O_v]$. Hence $\mathit{Fire}(o')$ is not applicable in s.

$\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$ because only $\mathit{Fire}(o)$ can change O_v from o to *free*, which is precondition for all *SetO* operators, which are in turn needed to set O_v to o' (see also Case 1(a)).

(b) $v \in \mathit{vars}(\mathit{prv}_u)$ and $\mathit{prv}_u[v] \neq \mathit{eff}_{\mathit{Fire}(o)}[v]$. We must distinguish three possible types for u:

i. $u = \mathit{Fire}(o')$. From $v \in \mathit{vars}(\mathit{prv}_{o'})$, we know that $\mathit{pre}_{o'}[O_v] = \mathit{frozen} \neq o = s[O_v]$. Hence $\mathit{Fire}(o')$ is not applicable in s. Furthermore, $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$ because only $\mathit{Fire}(o)$ can change O_v from o to *free*, which is required because all *Freeze* operators (which can set O_v to *frozen*) require $O_v = \mathit{free}$ as precondition.

ii. $u = \mathit{SetO}(o')$. From the definition of *SetO* operators, we know that $\mathit{pre}_{\mathit{SetO}(o')}[O_v] = \mathit{free} \neq o = s[O_v]$. Hence $\mathit{Fire}(o')$ is not applicable in s. Furthermore, $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$ because only $\mathit{Fire}(o)$ can change O_v from o to *free*, which is the condition for $\mathit{SetO}(o')$ to become applicable.

iii. $u = \mathit{Freeze}(v, x)$ for some $x \in \mathcal{D}(v)$. We have $\mathit{pre}_{\mathit{Freeze}(v,x)}[O_v] = \mathit{free} \neq o = s[O_v]$ and thus Case 2(b)ii applies.

(c) $O_v \in \mathit{vars}(\mathit{pre}_u)$ and $\mathit{pre}_u[O_v] \neq \mathit{eff}_{\mathit{Fire}(o)}[O_v]$. As $\mathit{eff}_{\mathit{Fire}(o)}[O_v] = \mathit{free}$, only fire operators $u = \mathit{Fire}(o')$ can be disabled (and not operators of type *SetO* or *Freeze*, as these could only have preconditions $O_v = \mathit{free}$). If $v \in \mathit{vars}(\mathit{eff}_{o'})$, Case 2(a) applies. If $v \notin \mathit{vars}(\mathit{eff}_{o'})$, $v \in \mathit{vars}(\mathit{prv}_{o'})$ and Case 2(b)i applies.

(d) $O_w \in \mathit{vars}(\mathit{pre}_u)$ for some $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{pre}_u[O_w] \neq \mathit{eff}_{\mathit{Fire}(o)}[O_w]$. As $\mathit{eff}_{\mathit{Fire}(o)}[O_w] = \mathit{free}$, only fire operators $u = \mathit{Fire}(o')$ can be disabled (and not operators of type *SetO* or *Freeze*, as these could only have preconditions $O_w = \mathit{free}$). If $w \in \mathit{vars}(\mathit{eff}_{o'})$, Case 2(a) applies. If $w \notin \mathit{vars}(\mathit{eff}_{o'})$, $w \in \mathit{vars}(\mathit{prv}_{o'})$ and Case 2(b)i applies.

(e) $C_w \in \mathit{vars}(\mathit{pre}_u)$ for some $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{pre}_u[C_w] \neq \mathit{eff}_{\mathit{Fire}(o)}[C_w]$. Only a fire operator $u = \mathit{Fire}(o')$ can be disabled by $C_w = \mathit{free}$ (and no *SetC* operator, because they have *free* as precondition). Let $\mathit{pre}_{\mathit{Fire}(o')}[C_w] := v'$. If $v' = v$, then we have $v \in \mathit{vars}(\mathit{pre}_{\mathit{Fire}(o')})$ and thus Case 2(a) applies. If $v' \neq v$, then $\mathit{pre}_{\mathit{Fire}(o')}[C_w] = v' \neq v = s[C_w]$ and hence $\mathit{Fire}(o')$ is not applicable. Furthermore, $\{\mathit{Fire}(o)\}$ is a necessary enabling set for $\mathit{Fire}(o')$, because any (fire) operator that can change C_w from v to *free* (which is required for *SetC* operators to set C_w to v') must also affect v and thus O_v. As $s[O_v] = o$, $\mathit{Fire}(o)$ must necessarily be applied before $\mathit{Fire}(o')$.

3. If $\mathit{Fire}(o)$ and u have conflicting effects, we must distinguish the following cases.

(a) $v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[v] \neq \mathit{eff}_{\mathit{Fire}(o)}[v]$. This case mirrors Case 1(a).

(b) $O_v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[O_v] \neq \mathit{eff}_{\mathit{Fire}(O)}[O_v]$. This case mirrors Case 1(b), with the exclusion of sub-case i., because $\mathit{Fire}(o)$ and $\mathit{Fire}(o')$ cannot disable each other via (intention) variables, as both set such variables to *free*.

(c) $O_w \in \mathit{vars}(\mathit{eff}_u)$ for some variable $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{eff}_u[O_v] \neq \mathit{eff}_{\mathit{Fire}(o)}[O_v]$. Same case as the previous one, i. e. Case 3(b).

(d) $C_w \in \mathit{vars}(\mathit{eff}_u)$ for some variable $w \in \mathit{vars}(\mathit{prv}_{\mathit{Fire}(o)})$ and $\mathit{eff}_u[O_v] \neq \mathit{eff}_{\mathit{Fire}(o)}[O_v]$. This case mirrors Case 1(e), with the exclusion of sub-case i., because $\mathit{Fire}(o)$ and $\mathit{Fire}(o')$ cannot disable each other via (intention) variables, as both set such variables to *free*.

□

Proposition 2. *Second case of Theorem 1. Let s be a state, $X \in P_s$ be an applicable operator partition of type* $\mathrm{SetO}_{v=x}$ *for a variable $v \in \mathcal{V}$. Then the set $T_s := X \cup \{o' \mid o'$ interferes with $o'' \in X\}$ is a strong semistubborn set with the same applicable operators as X.*

Proof. We first note that $\mathrm{SetO}_{v=x}$ contains operators $\mathit{SetO}(o)$ for operators $o \in \mathcal{O}$ with $\mathit{pre}_o[v] = x$ and one operator $\mathit{Freeze}(v, x)$. By definition, we have:

$$\begin{aligned} \mathit{pre}_{\mathit{SetO}(o)/\mathit{Freeze}(v,x)}[O_v] &= \mathit{free} \\ \mathit{prv}_{\mathit{SetO}(o)/\mathit{Freeze}(v,x)}[v] &= x \\ \mathit{eff}_{\mathit{SetO}(o)}[O_v] &= o \\ \mathit{eff}_{\mathit{Freeze}(v,x)}[O_v] &= \mathit{frozen} \end{aligned}$$

Second, note that we know the following about state s, considering that $\mathit{SetO}(o)$ and $\mathit{Freeze}(v, x)$ are applicable in s:

$$\begin{aligned} s[v] &= x \\ s[O_v] &= \mathit{free} \end{aligned}$$

We show that all operators o' that interfere with the operators from the partition X are not applicable in s. Thus the operators from X are the only applicable operators in T_s. Second, we show that for all these operators $o' \in T_s$, T_s already contains a necessary enabling set for o' in s.

Let $u \neq \mathit{SetO}(o), \mathit{Freeze}(v, x)$ be an arbitrary operator interfering with $\mathit{SetO}(o)$ or $\mathit{Freeze}(v, x)$.

1. If u disables $\mathit{SetO}(o)$ and $\mathit{Freeze}(v, x)$, we only have one case: $O_v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[O_v] \neq \mathit{pre}_{\mathit{SetO}(o)/\mathit{Freeze}(v,x)}[O_v]$. Note that the case $v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[v] \neq \mathit{prv}_{\mathit{SetO}(o)/\mathit{Freeze}(v,x)}[v]$ can be reduced to the same case, because only fire operators can affect variables $v \in \mathcal{V}$ and if they affect variable v, they also affect O_v.
   We must distinguish three possible types for u.
   (a) $u = \mathit{Fire}(o')$. We have $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = o' \neq \mathit{free} = s[O_v]$ and hence $\mathit{Fire}(o')$ is not applicable in s. Furthermore, $\mathit{SetO}(o')$ must be applied before $\mathit{Fire}(o')$ can be applied. We claim that X is a necessary enabling set for $\mathit{Fire}(o')$: either $\mathit{SetO}(o') \in X$ in which case we are done or $\mathit{prv}_{\mathit{SetO}(o')}[v] = x'$ for $x' \neq x$ and thus any operator $\mathit{SetO}(o)$ from X must be applied in order to allow to change the value of v through the intended operator o. Note that it is possible that $\mathit{Freeze}(v, x)$ is required to allow the application of an operator which in turn fulfills a prevail-condition of $\mathit{Fire}(o')$. It is therefore not enough to choose the subset $\mathit{SetO}(o) \subseteq X$ as a necessary enabling set for $\mathit{Fire}(o')$ in s, but the set X is.
   (b) $u = \mathit{SetO}(o')$. We know $\mathit{prv}_{\mathit{SetO}(o')}[v] = x'$. If $x' = x$, $\mathit{SetO}(o')$ is applicable in s and $\mathit{SetO}(o') \in X$. If $x' \neq x$, $\mathit{prv}_{\mathit{SetO}(o')}[v] = x' \neq x = s[v]$ and hence $\mathit{SetO}(o')$ is not applicable in s. In the latter case, X is a necessary enabling set for $\mathit{SetO}(o')$ in s for the same arguments as shown in the previous Case 1(a).
   (c) $u = \mathit{Freeze}(v, x')$ (By assumption $u \neq \mathit{Freeze}(v, x)$ and hence $x' \neq x$). We have $\mathit{prv}_{\mathit{Freeze}(v,x')}[v] = x' \neq x = s[v]$ and thus $\mathit{Freeze}(v,')$ is not applicable in s. X is a necessary enabling set for $\mathit{Freeze}(v, x')$ in s for the same reasons shown in Case 1(a).
2. If $\mathit{SetO}(o)$ disables u, we only have one case: $O_v \in \mathit{vars}(\mathit{pre}_u)$ and $\mathit{pre}_u[O_v] \neq \mathit{eff}_{\mathit{SetO}(o)}[O_v]$. We must distinguish three possible types for u.
   (a) $u = \mathit{Fire}(o')$. If $v \in \mathit{vars}(\mathit{eff}_{o'})$, Case 1(a) applies. If $v \notin \mathit{vars}(\mathit{eff}_{o'})$, $v \in \mathit{vars}(\mathit{prv}_{o'})$ and we have $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = \mathit{frozen} \neq \mathit{free} = s[O_v]$ and thus $\mathit{Fire}(o')$ is not applicable. We claim that X is a necessary enabling set for $\mathit{Fire}(o')$ in s. We observe that $\mathit{Freeze}(v, x')$ for $x' = \mathit{prv}_{o'}[v]$ must be applied before $\mathit{Fire}(o')$ can be applied. If $x' = x$, $\mathit{Freeze}(v, x') \in X$ and we are done. If $x' \neq x$, we must first apply SetO operators in X to enable other operators to change v to x' eventually, which is prevail-condition for applying $\mathit{Freeze}(v, x')$, in turn requirement for application of $\mathit{Fire}(o')$.
   (b) $u = \mathit{SetO}(o')$. We have $\mathit{prv}_{\mathit{SetO}(o')}[v] = x'$ and thus Case 1(b) applies.
   (c) $u = \mathit{Freeze}(v, x')$. We have $\mathit{prv}_{\mathit{Freeze}(v,x')}[v] = x'$ and Case 1(c) applies.
3. If $\mathit{Freeze}(v, x)$ disables u, we only have one case: $O_v \in \mathit{vars}(\mathit{pre}_u)$ and $\mathit{pre}_u[O_v] \neq \mathit{eff}_{\mathit{SetO}(o)}[O_v]$. We must distinguish three possible types for u.
   (a) $u = \mathit{Fire}(o')$. If $v \in \mathit{vars}(\mathit{eff}_{o'})$, Case 1(a) applies. If $v \notin \mathit{vars}(\mathit{eff}_{o'})$, $v \in \mathit{vars}(\mathit{prv}_{o'})$ and we have $\mathit{prv}_{\mathit{Fire}(o')}[v] = x'$. If $x' = x$, $\mathit{Freeze}(v, x)$ does not disable $\mathit{Fire}(o')$. If $x' \neq x$, we have $\mathit{prv}_{\mathit{Fire}(o')}[v] = x' \neq x = s[v]$ and hence $\mathit{Fire}(o')$ is not applicable in s. For $\mathit{Fire}(o')$ to become applicable, v must change its value from x to x'. The remaining argumentation mirrors the one of Case 2(a).
   (b) $u = \mathit{SetO}(o')$. We have $\mathit{prv}_{\mathit{SetO}(o')}[v] = x'$ and thus Case 1b) applies.
   (c) $u = \mathit{Freeze}(v, x')$. We have $\mathit{prv}_{\mathit{Freeze}(v,x')}[v] = x'$ and Case 1(c) applies.
4. If $\mathit{SetO}(o)$ and u have conflicting effects, we only have one case: $O_v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[O_v] \neq \mathit{eff}_{\mathit{SetO}(o)}[O_v]$. We must distinguish three possible types for u.
   (a) $u = \mathit{Fire}(o')$. We have $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = o'$. This case mirrors Case 1(a).
   (b) $u = \mathit{SetO}(o')$. We have $\mathit{prv}_{\mathit{SetO}(o')}[v] = x'$. This case mirrors Case 1(b).
   (c) $u = \mathit{Freeze}(v, x')$. We have $\mathit{pre}_{\mathit{Freeze}(v,x')}[v] = x'$. This case mirrors Case 1(c).
5. If $\mathit{Freeze}(v, x)$ and u have conflicting effects, we only have one case: $O_v \in \mathit{vars}(\mathit{eff}_u)$ and $\mathit{eff}_u[O_v] \neq \mathit{eff}_{\mathit{Freeze}(v,x)}[O_v]$. We must distinguish two possible types for u (there is only one *Freeze* operator per variable and *Freeze* operators for different variables cannot interfere).
   (a) $u = \mathit{Fire}(o')$. We have $\mathit{pre}_{\mathit{Fire}(o')}[O_v] = o'$. This case mirrors Case 1(a).
   (b) $u = \mathit{SetO}(o')$. We have $\mathit{prv}_{\mathit{SetO}(o')}[v] = x'$. This case mirrors Case 1b).

□

Proposition 3. *Third case of Theorem 1. Let s be a state, $X \in P_s$ be an applicable operator partition of type* SetC_v *for a variable $v \in \mathcal{V}$. Then the set $T_s := X \cup \{o' \mid o'$ interferes with $o'' \in X\}$ is a strong semistubborn set with the same applicable operators as X.*

Proof. We first note that SetC_v contains operators $\mathit{SetC}(v, c)$ for $c \in CG(v)$. By definition, we have:

$$\begin{aligned} \mathit{pre}_{\mathit{SetC}(v,c)}[C_v] &= \mathit{free} \\ \mathit{eff}_{\mathit{SetC}(v,c)}[C_v] &= c \end{aligned}$$

Second, note that we know the following about state s, considering that $\mathit{SetC}(v, c)$ is applicable in s:

$$s[C_c] = \mathit{free}$$

We show that all operators o' that interfere with the operators from the partition X are not applicable in s. Thus the operators from X are the only applicable operators in T_s. Second, we show that for all these operators $o' \in T_s$, T_s already contains a necessary enabling set for o' in s.

We first observe that for an operator $\mathit{SetC}(v, c) \in X$, only other *SetC* operators for the same variable v and *Fire* operators may share the variable C_v with $\mathit{SetC}(v, c)$. Furthermore, all operators

SetC(v, c') for all $c' \in CG(v)$ are already included in X (and applicable). We thus only need to consider *Fire* operators for possible interference in the following.

Let *Fire*(o') be an arbitrary operator interfering with *SetC*(v, c). We observe that *Fire*(o') cannot disable *SetC*(v, c) because it could only set C_v to *free*. There are two remaining cases.

1. If *SetC*(v, c) disables *Fire*(o'), we have only one case: $C_v \in$ *vars*$(pre_{Fire(o')})$ and $pre_{Fire(o')}[C_v] \neq eff_{SetC(v,c)}$. We have $pre_{Fire(o')}[C_v] = c'$ for a variable $c' \in$ *vars*$(prv_{o'})$. If $c' = c$, *SetC*(v, c) does not disable *Fire*(o'). If $c' \neq c$, we have $pre_{Fire(o')}[C_v] = c' \neq$ *free* $= s[C_v]$ and hence *Fire*(o') is not applicable in s. We observe that *SetC*(v, c') must be applied before *Fire*(o') can be applied, and because *SetC*$(v, c') \in X$, X is a necessary enabling set for *Fire*(o') in s.
2. If *SetC*(v, c) and *Fire*(o') have conflicting effects, we know $pre_{Fire(o')}[C_v] = c'$ for a variable $c' \in$ *vars*$(prv_{o'})$. Hence Case 1 applies.

□

We have shown for all operators o' that interfere with the operators of an operator partition X applicable in s that they are not applicable in s. Furthermore, X is a necessary enabling set for all such o' in s. The criteria for $T_s = X \cup \{o \mid o \text{ interferes with } o' \in X\}$ to be a strong semistubborn set in s are thus met. The overall proof for Theorem 1 follows from Propositions 1, 2, and 3.

□