University of Basel

Encoding Diverse Sudoku Variants as SAT Problems

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July 28, 2022

Normal Sudoku

							1	
2	1				3	4	8	
	3	9	8			2		
	6		3		4	9		
		1	6		7		4	
		8			2	1	7	
	2	6	7				9	8
	9							

Normal Sudoku Puzzle
by 'Red' Ed Russell

6	8	5	4	2	9	7	1	3
2	1	7	5	6	3	4	8	9
4	3	9	8	7	1	2	6	5
8	6	2	3	1	4	9	5	7
9	7	4	2	5	8	6	3	1
3	5	1	6	9	7	8	4	2
5	4	8	9	3	2	1	7	6
1	2	6	7	4	5	3	9	8
7	9	3	1	8	6	5	2	4

Corresponding Solution

Encoding Normal Sudoku

Proposed by Lynce and Ouaknine in 2006

At least one number from 1 to 9 appears in each grid cell.

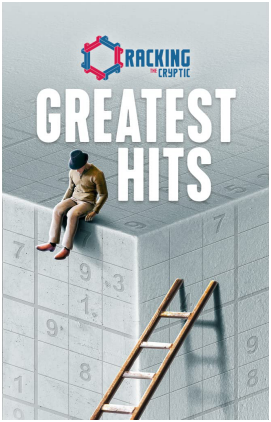
$$\bigwedge_{x=1}^{9} \bigwedge_{y=1}^{9} \bigvee_{z=1}^{9} s_{x,y,z}$$

Every number appears at most once per row.

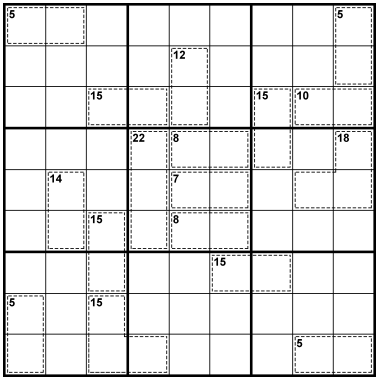
$$\bigwedge_{y=1}^{9} \bigwedge_{z=1}^{9} \bigwedge_{x=1}^{9} \bigwedge_{i=x+1}^{9} \neg s_{x,y,z} \vee \neg s_{i,y,z}$$

...

Diverse Sudoku Variants



Cracking The Cryptic Greatest Hits (CTCGH)



Killer Sudoku Example
by Phistomefel

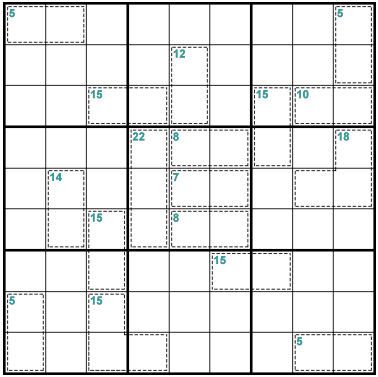
Pseudo-Boolean Constraints - PBCs

In General:

$$w_1 v_1 + w_2 v_2 + ... + w_n v_n \leq K$$

For Killer Cages:

$$\sum_{(x,y) \in Cage} \sum_{z=1}^{9} s_{x,y,z} * z = target\ sum$$



Killer Sudoku Example
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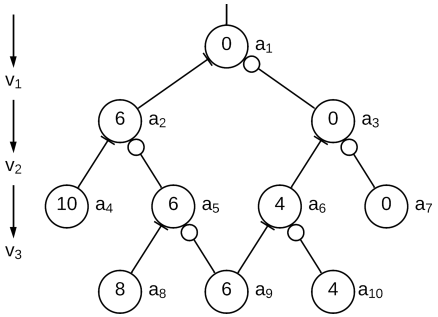
Binary Decision Diagrams - BDDs

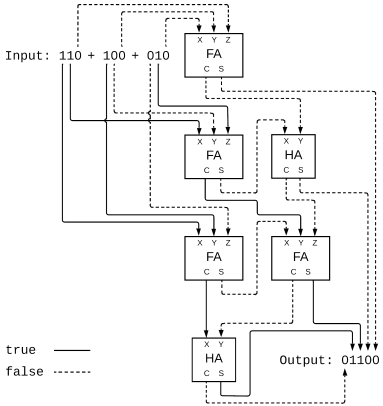


BDD to encode $6 * v_1 + 4 * v_2 + 2 * v_3 = 6$

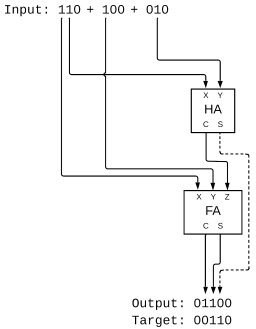
Formulae:

0. (a_1)
1. $(\ a_1 \wedge \ v_1 \rightarrow \ a_2)$
   $(\ a_1 \wedge \neg v_1 \rightarrow \ a_3)$
   $(\neg a_1 \wedge \ v_1 \rightarrow \neg a_2)$
   $(\neg a_1 \wedge \neg v_1 \rightarrow \neg a_3)$
   $(\ a_2 \wedge \ a_3 \rightarrow \ a_1)$
   $(\neg a_2 \wedge \neg a_3 \rightarrow \neg a_1)$
2. ...

Adder Networks

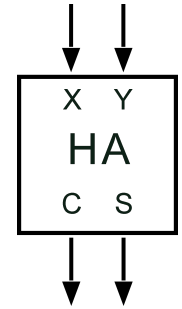


Adder Network to compute
$6 + 4 + 2 = 12$



Adder Network to encode
$6 * v_1 + 4 * v_2 + 2 * v_3 = 6$

Adder Networks



Half Adder Node

Formulae:

$$
\begin{aligned}
(\neg x \wedge \neg y &\rightarrow \neg s)\\
(\neg x \wedge \ \ y &\rightarrow \ \ s)\\
(\ \ x \wedge \neg y &\rightarrow \ \ s)\\
(\ \ x \wedge \ \ y &\rightarrow \neg s)\\
(\neg x \wedge \neg y &\rightarrow \neg c)\\
(\neg x \wedge \ \ y &\rightarrow \neg c)\\
(\ \ x \wedge \neg y &\rightarrow \neg c)\\
(\ \ x \wedge \ \ y &\rightarrow \ \ c)
\end{aligned}
$$

Optimization of Killer Sudoku Encoding

Example: How to achieve a sum of 8 in a cage with size three?
$\rightarrow 1+2+5=8$ or $1+3+4=8$

PBCs + Combinations, use only possible values in LHS of PBCs:

- In the example these would be $\{1,2,3,4,5\}$.

Combinations, do not use PBCs:

- At least one combination C is used per cage.
- At most one combination C is used per cage.
- If a cage uses combination C every cell must contain a value of C.
- The values 1 to 9 appear at most ones per cage.

Comparing Puzzles from CTCGH

Rule | Instance	9 Marks The Spot	Chess Sudoku	Fawlty Towers	Frozen Picnic	Mark 1	Nurikabe Sudoku	Sudoku Man Of Mystery	The Miracle Thermo	The Original Sandwich	The Pyramid	Thermo 2020	Thermo Couples	Thermo Squares	The Road To Genius
Normal Sudoku	×	×	×	×	×	×	×	×	×	×	×	×	×	×
Anti-Knight		×			×					×		×		
Killer Sudoku										×				
Arrowheads				×										
Thermometers		×		×	×			×			×	×	×	
Sandwich Sum				×					×					
Secret Direction	×													
Fawlty Towers			×											
Nurikabe Sudoku						×								

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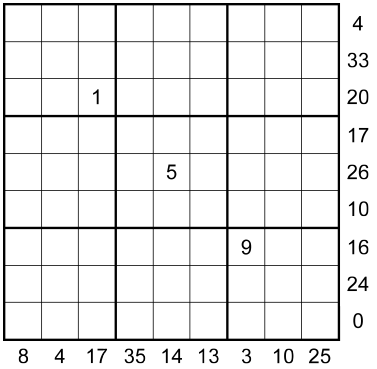
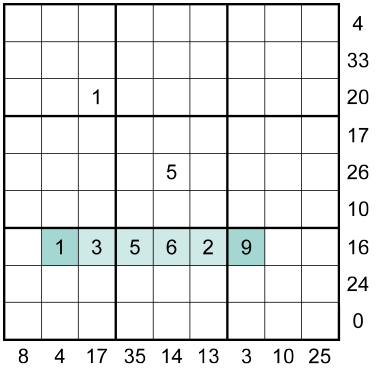
Rule | Instance	9 Marks The Spot	Chess Sudoku	Fawlty Towers	Frozen Picnic	Mark 1	Nurikabe Sudoku	Sudoku Man Of Mystery	The Miracle Thermo	The Original Sandwich	The Pyramid	Thermo 2020	Thermo Couples	Thermo Squares	The Road To Genius
Normal Sudoku	×	×	×	×	×	×	×	×	×	×	×	×	×	×
Anti-Knight		×			×					×		×		
Killer Sudoku										×				
Arrowheads				×										
Thermometers		×		×	×			×			×	×	×	
Sandwich Sum				×					×					
Secret Direction	×													
Fawlty Towers			×											
Nurikabe Sudoku						×								

Sandwich Sum



"The Original Sandwich"
by Matus Deminger

Sandwich Sum



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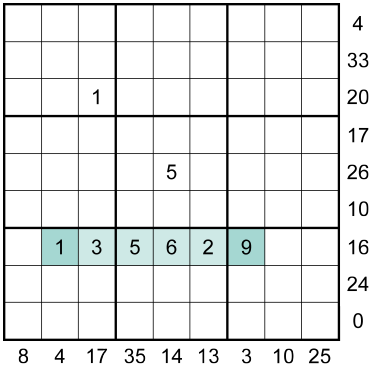


"The Original Sandwich"
by Matus Deminger

- The values 1 and 9 must be positioned correctly.
- There is exactly one Sandwich per row/column.
- The cell values of a Sandwich must have the correct sum.

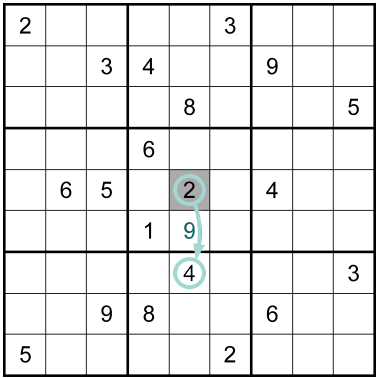
$$\bigwedge_{y=1}^{9} \bigwedge_{\ell=0}^{7} \bigwedge_{x=1}^{8-\ell} \bigwedge_{\varphi \in PBC} \varphi \vee \neg \mathcal{S}(x, y, \ell)$$

Secret Direction

2					3			
		3	4			9		
				8				5
			6					
	6	5		2		4		
			1					
				4				3
		9	8			6		
5					2			

"9 Marks The Spot"
by Ricky Cruz

Secret Direction



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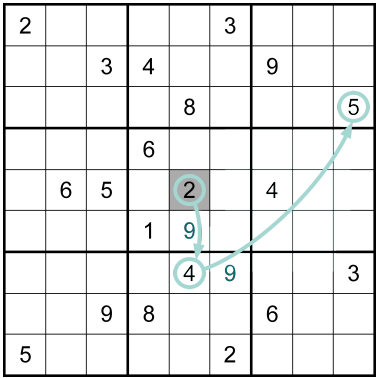

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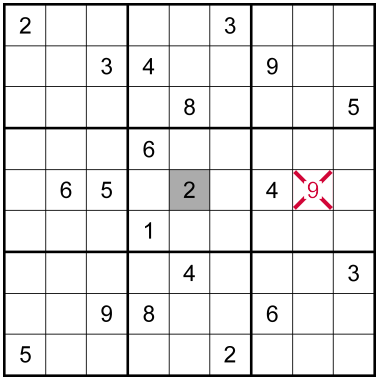

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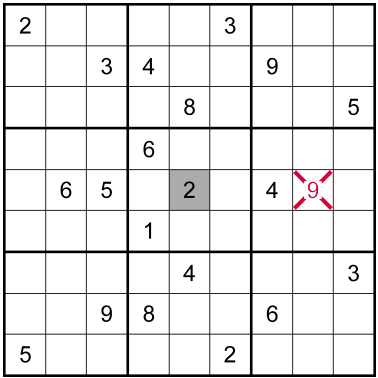


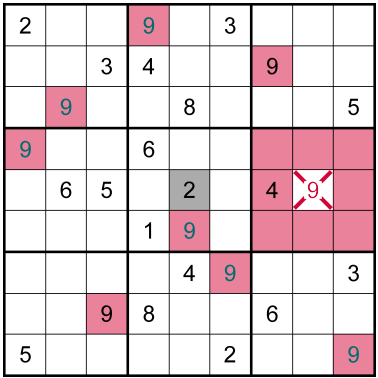
"9 Marks The Spot"
by Ricky Cruz

Variable for each cell (x, y) to indicate if it is in depth d of path.

Constraints:

- Center 9
- Start and goal
- Depths per cell and cells per depth
- Cells not part of path
- Direction-Implication

Secret Direction



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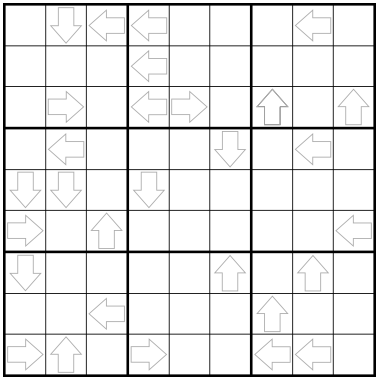
Secret Direction

Direction-Implication:

$$\left(\begin{array}{c}(x,y) \text{ is in depth } d \text{ of path}\\ \wedge\\ 9 \text{ points in direction of } (x_s,y_s)\\ \wedge\\ \text{value of } (x,y) \text{ equals its distance to } (x_s,y_s)\end{array}\right) \rightarrow (x_s,y_s) \text{ is in depth } d+1 \text{ of path}$$

Nurikabe Sudoku

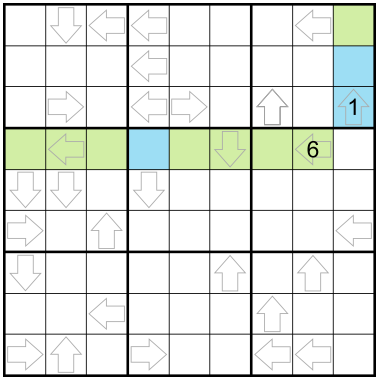


"Nurikabe Sudoku"
by Matyas Martinkas

Rules:

- Island or ocean
- Only one Ocean
- No 2×2 square of ocean cells
- Islands consist of at least three cells
- No value repetition within islands
- Islands only touch diagonally
- Cell values of arrow-cells

Nurikabe Sudoku



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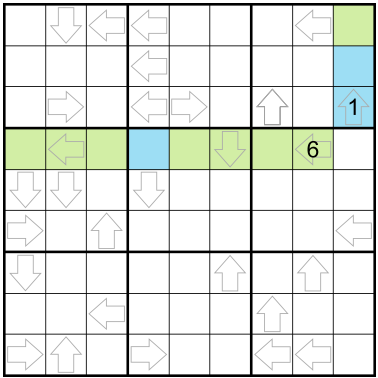

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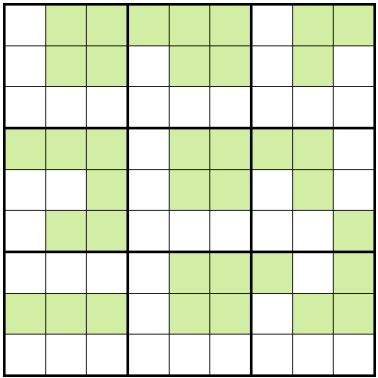
Nurikabe Sudoku - One Ocean



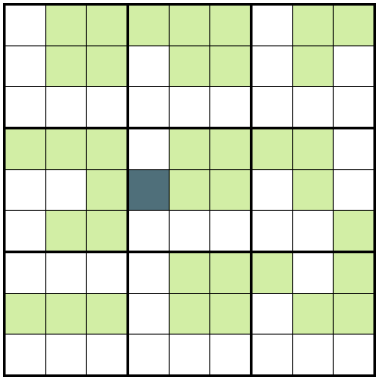
Flood Visualization

All ocean cells reachable from each other

Flood fill

Variable for each cell (x, y) to indicate if it is in depth d of a flood with source (x_s, y_s)

Nurikabe Sudoku - One Ocean



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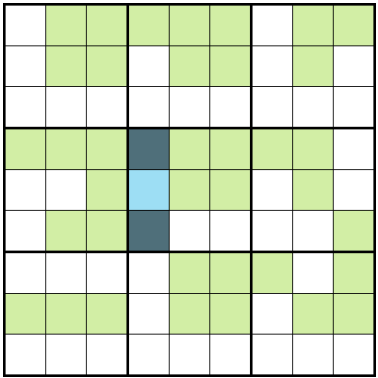

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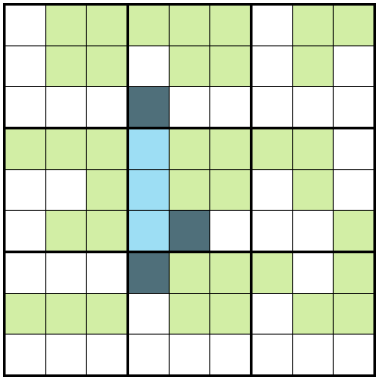

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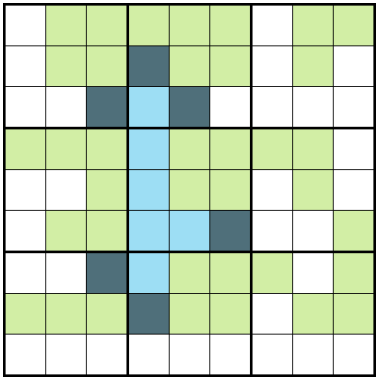
Nurikabe Sudoku - One Ocean



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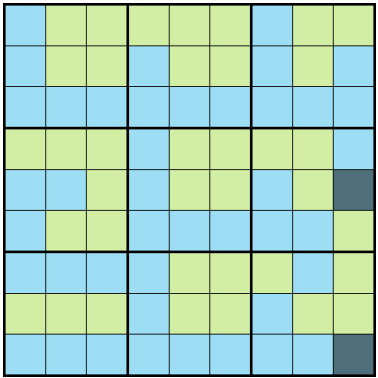
Nurikabe Sudoku - One Ocean



Flood Visualization

All ocean cells reachable from each other

Flood fill

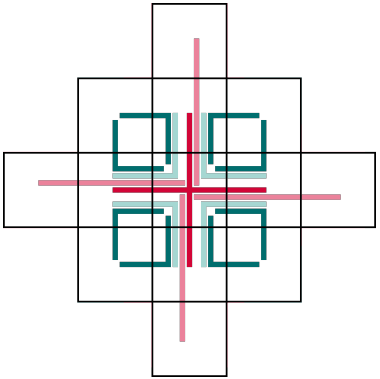
Variable for each cell (x, y) to indicate if it is in depth d of a flood with source (x_s, y_s)

Nurikabe Sudoku - One Ocean

Flood-Implication:

$$\left(\begin{array}{c}\text{(x,y) is at depth } d \text{ of flood from } (x_s, y_s) \\ \wedge \\ (x_s, y_s) \text{ is ocean} \\ \wedge \\ (x, y) \neq (x_s, y_s)\end{array}\right) \rightarrow \left(\begin{array}{l}\text{orthogonally adjacent cell of } (x, y) \\ \text{is in depth } d \text{ of flood from } (x_s, y_s)\end{array}\right)$$

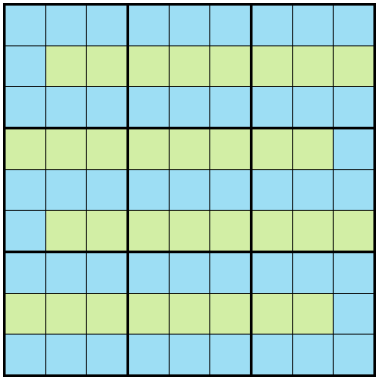
Nurikabe Sudoku - Islands



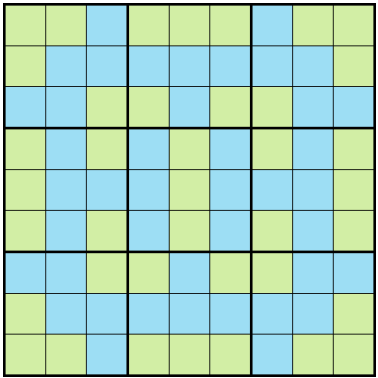
Possible Constellations

- Need a variable for each cell (x, y) to indicate if it is part of island n
- Must enforce continuousness of cells that belong to an island n

Nurikabe Sudoku - Flood Depth and Number of Islands



Example with max. flood depth

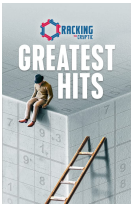


Example with max. number of islands

Experiments

Puzzle instances:

- 14 instances of different variants from CTCGH
- 10 instances of Killer Sudoku from "The Times Killer Su Doku Book 18"
- 10 instances of Killer Sudoku from "The Times Ultimate Killer Su Doku Book 14"



Experiments

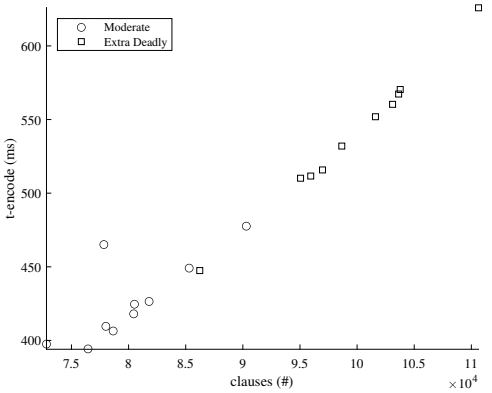
Tested configurations for each puzzle instance differ by:

- SAT-solver: MiniSat or Sat4j
- PBC-Encoding method: BDDs or Adder Networks
- Level of optimization (for Killer Sudokus)

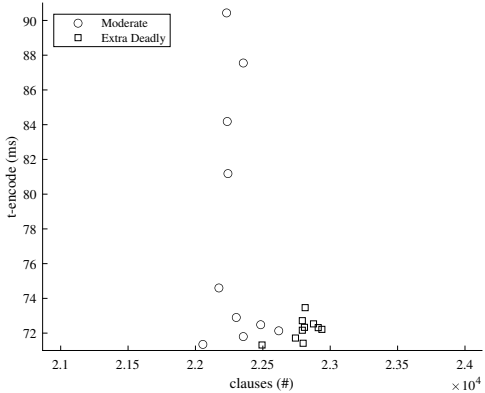
Experiment Methodology:

- 8-Core Processor (3.80 GHz), 28 GB of RAM available to JVM
- 60 runs for every configuration
- Sorted clauses

Encoding Times - Killer Sudoku

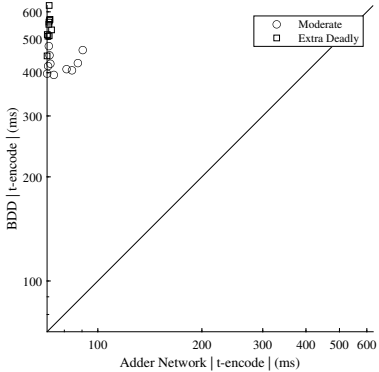


Encoding with Binary Decision Diagrams

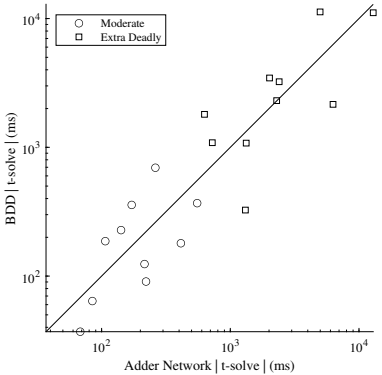


Encoding with Adder Networks

Binary Decision Diagrams vs. Adder Networks - Killer Sudoku

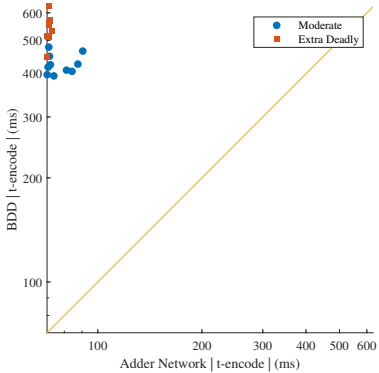


Avg. Encoding Times

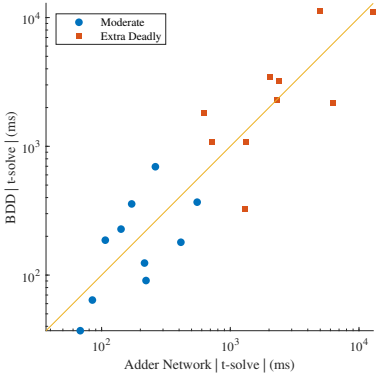


Avg. Solving Times

Binary Decision Diagrams vs. Adder Networks - Killer Sudoku

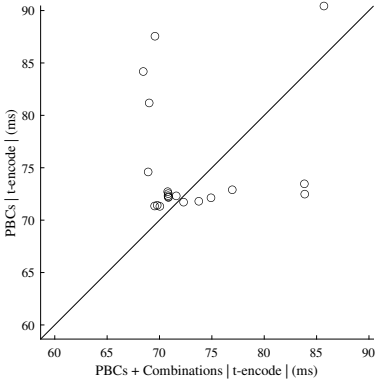


Avg. Encoding Times

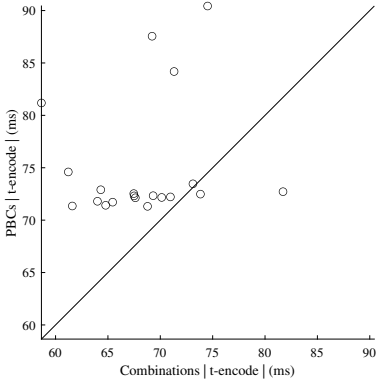


Avg. Solving Times

Optimization of Killer Sudoku Encoding

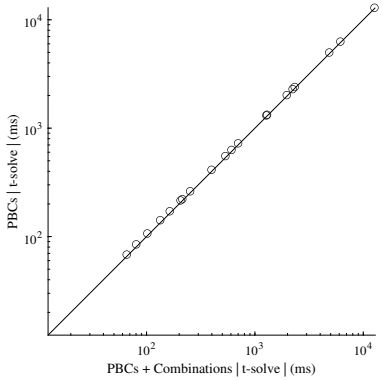


Encoding Time Comparison
(with only allowed values on LHS of PBCs)

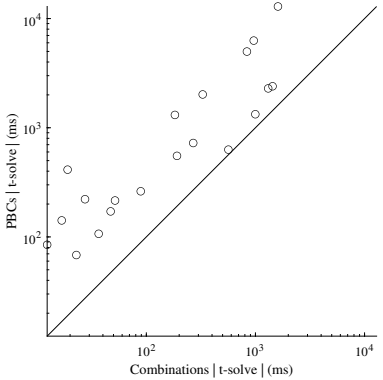


Encoding Time Comparison
(without PBCs)

Optimization of Killer Sudoku Encoding



Solving Time Comparison
(with only allowed values on LHS of PBCs)



Solving Time Comparison
(without PBCs)

Comparing Puzzles from CTCGH

Variant	t-avg. (ms)			#clauses	#variables
	encode	solve			
		Sat4j	MiniSat		
9 Marks The Spot	4428.02	170.90	1179.53	727636	5994
Chess Sudoku	29.05	3.23	1061.05	8912	729
Fawlty Towers	65.38	11.10	904.62	17632	2186
Frozen Picnic	168.32	14.92	886.33	40519	5387
Mark 1	28.60	8.18	903.05	8839	729
Nurikabe Sudoku	77406.68	4145268.40	413259.82	13904145	1169013
Sudoku Man Of Mystery	23.18	1.97	876.67	7399	729
The Miracle Thermo	739.77	446.32	1671.62	138743	5265
The Original Sandwich	1677.18	15408.00	2059.25	302640	42635
The Pyramid	48.68	219.07	1434.53	15233	1679
Thermo 2020	24.23	1.97	1355.67	7659	729
Thermo Couples	27.98	3.80	1354.52	8884	729
Thermo Squares	23.92	17.95	1362.25	7692	729
The Road To Genius	22.95	1.97	1364.20	7401	729

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Conclusion

Results:

- Encodings for diverse Sudoku Variants
- Encodings for Pseudo-Boolean Constraints
- Encodings for Killer Sudokus

Future Work:

- Craft new Sudoku Puzzles
- Encode and solve further variants from CTCGH

Questions?

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