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On Generalized Gaussian Quadratures for Exponentials and Their Applications

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We introduce new families of Gaussian-type quadratures for weighted integrals of exponential functions and consider their applications to integration and interpolation of bandlimited functions.

We use a generalization of a representation theorem due to Carathéodory to derive these quadratures. For each positive measure, the quadratures are parameterized by eigenvalues of the Toeplitz matrix constructed from the trigonometric moments of the measure. For a given accuracy ϵ, selecting an eigenvalue close to ϵ yields an approximate quadrature with that accuracy. To compute its weights and nodes, we present a new fast algorithm.

These new quadratures can be used to approximate and integrate bandlimited functions, such as prolate spheroidal wave functions, and essentially bandlimited functions, such as Bessel functions. We also develop, for a given precision, an interpolating basis for bandlimited functions on an interval.

1. INTRODUCTION

In this paper we relate the Carathéodory representation of finite sequences in terms of exponential sums with the computation of generalized Gaussian quadratures for exponentials. Generalized Gaussian quadratures were investigated by Markov [17, 18], Krein and Nudel'man [13], Karlin and Studden [12] and, more recently, by Yarvin and Rokhlin [30]. In [30] the authors introduce practical algorithms for computing the nodes and weights of generalized Gaussian quadratures. The resulting approximations have a number of important applications in a variety of fast algorithms [3, 31].

The Carathéodory representation theorem asserts existence and uniqueness of the representation of a finite sequence of complex numbers $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)$, $\boldsymbol{c} \neq 0$, in

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the form

$$c_k = \sum_{j=1}^{M} \rho_j e^{i\pi\theta_j k}, \tag{1.1}$$

for $k = 1, 2, \ldots, N$ and $M \leq N$, where $-1 < \theta_j \leq 1$ and $\rho_j > 0$. Carathéodory representation (1.1) has been the foundation for a number of algorithms for spectral estimation; in particular, [20] is known in electrical engineering literature as the Pisarenko method. In this paper we develop a fast algorithm for finding M, the phases $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$ with $|\theta_j| \leq 1$, and the weights $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_M)$. Our algorithm differs from that described in [20], although the basic approach is similar. We achieve finite but arbitrary accuracy and our algorithm requires $O(N(\log N)^2)$ operations ($O(N^2)$ in a simpler version).

One can view finding M, the phases $\boldsymbol{\theta}$, and the weights $\boldsymbol{\rho}$ as a nonlinear inverse problem for the unequally spaced discrete Fourier transform [1, 5]. It is interesting to note that the associated linear problem, namely the problem where M and $|\theta_j| \leq 1$ in (1.1) are given, can be arbitrarily ill conditioned. In other words, the condition number of the Vandermonde matrix $\{e^{i\pi\theta_j k}\}_{k,j=1,\ldots,M}$ can be arbitrarily large. On the other hand, the nonlinear problem is well posed and we will show the l^2-norm estimate

$$\|\boldsymbol{\rho}\|_2 \leq \sqrt{2}\|\boldsymbol{c}\|_2. \tag{1.2}$$

The main goal of this paper is to extend Carathéodory representation and use it to compute quadratures for integrals involving exponentials, as well as the Bessel and the prolate spheroidal wave functions (PSWF). These bandlimited or essentially bandlimited functions play a central role in many problems of signal processing and numerical analysis. We also consider the associated interpolation problem involving these functions. Specifically, we develop methodology to represent bandlimited functions on an interval using exponentials $\{e^{icxt_l}\}_{l=1}^{M}$, where the bandlimit c is a positive real constant and t_l, $|t_l| \leq 1$, are phases computed for a given c and accuracy ϵ. With this approach, we are no longer limited to representing periodic functions, as is the case with Fourier series.

In order to consider bandlimited functions on an interval, the PSWF were introduced in [24] and [15]. Recently, a method for computing generalized Gaussian quadratures for PSWF and, as a consequence, for bandlimited exponentials, was introduced in [29]. In [29] the authors first construct quadratures for the PSWF using the fact that the first n of these functions form a Chebyshev system, for any n. The approach for computing generalized Gaussian quadratures in [30] relies on a variant of Newton's method in conjunction with a continuation procedure. As a method it is quite general but is computationally intensive, although in many applications speed is not a limitation.

Our approach differs from that in [29]. We first develop a method for constructing optimal nodes and weights for integrals involving exponentials and then show that the same nodes and weights also provide quadratures for other essentially bandlimited functions, e.g., the PSWF.

The reader familiar with Gaussian quadratures should be warned that our methodology for generating quadratures is substantially different from the existing methods. The resulting quadrature formulas do not coincide with any other existing quadrature but, numerically, they are close to those in [29].

As a method for constructing generalized Gaussian quadratures, our results are limited to integrals (with a fairly arbitrary measure) involving exponentials. Our algorithm involves finding eigenvalues and eigenvectors of a Toeplitz matrix constructed from trigonometric moments of the measure and then computing the roots on the unit circle for appropriate eigenpolynomials. In particular, each eigenpolynomial with distinct roots gives rise to an identity which, for small eigenvalues, provides us with a Gaussian-type quadrature and also with a representation of positive definite Hermitian Toeplitz matrices. In these identities the size of the eigenvalue determines the accuracy of the quadrature formula.

It turns out that in the case of the weight leading to PSWF, the nodes of the corresponding Gaussian quadratures are zeros (appropriately scaled to the interval $[-1, 1]$) of discrete PSWF corresponding to small eigenvalues.

As an application, we use the new quadratures to obtain efficient approximations of nonperiodic bandlimited functions in terms of linear combinations of exponentials. In fact, we consider integrals of the form

$$u(x) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}ctx}\, d\mu(t), \tag{1.3}$$

where $d\mu(t)$ is a measure, typically $d\mu(t) = w(t)\,dt$, with w a weight function (w is a real, nonnegative, integrable function with $\int_{-1}^{1} w(\tau)\,d\tau > 0$). For a given bandlimit $c > 0$ and accuracy $\epsilon > 0$, our goal is to approximate $u(x)$ on the interval $[-1, 1]$ using the sum

$$\tilde{u}(x) = \sum_{k=1}^{M} w_k \mathrm{e}^{\mathrm{i}c\theta_k x}, \tag{1.4}$$

where $w_k > 0$ and $M = M(c, \epsilon)$, so that

$$|u(x) - \tilde{u}(x)| \leq \epsilon, \qquad \text{for } x \in [-1, 1]. \tag{1.5}$$

Since it is appropriate to view (1.4) as a quadrature, we will refer to θ_k as nodes and w_k as weights.

In order to find $\tilde{u}$ as in (1.4)–(1.5), we sample the function u in (1.3) at equally spaced points so that we exceed the sampling rate dictated by the Nyquist criterion. The equally spaced samples of u can be viewed as the trigonometric moments of the (rescaled) measure in (1.3). We extend Carathéodory representation to find M, the nodes $\{\theta_k\}_{k=1}^{M}$, and the weights $\{w_k\}_{k=1}^{M}$. We then use the same M, nodes, and weights to define $\tilde{u}$ in (1.4) for all $x \in (-1, 1)$ and show that (1.5) holds if u in (1.3) was sufficiently oversampled.

For the interpolation problem for bandlimited functions, we consider the linear space of functions $\mathcal{E}_c = \{f \in L_\infty[-1, 1]\colon\ f(x) = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}b_k x}\colon\ \{a_k\} \in l^1,\ |b_k| \leq c,\ \forall k\}$. These functions are not necessarily periodic in $[-1, 1]$. The interpolation problem amounts to representing, with accuracy ϵ, the functions in $\mathcal{E}_c$ by a fixed set of exponentials $\{\mathrm{e}^{\mathrm{i}c\, t_k x}\}_{k=1}^{M}$, where M is as small as possible. We show that by finding quadrature nodes $\{t_k\}_{k=1}^{M}$ for exponentials with bandlimit $2c$ and accuracy ϵ^2, we in fact obtain, with accuracy ϵ, a basis for bandlimited functions with bandlimit c. The connection between the generalized Gaussian quadratures for exponentials and the interpolation problem was first described in [29]. We use similar results to construct interpolatory bases for arbitrary accuracy ϵ.

The paper is organized as follows. We present a brief description of the Pisarenko method to obtain the classical Carathéodory representation and we derive the estimate (1.2) in Section 2. In Section 3 we discuss generalized Gaussian quadratures for weighted integrals and prove some of their properties for weights supported inside $[-1/2, 1/2]$. In Section 4 we introduce new families of Gaussian-type quadratures. We develop a fast algorithm in Section 5 to compute the nodes and weights of these quadratures. We solve the approximation problem (1.3)–(1.5) in Section 6 and use it in the next two sections to obtain quadratures and interpolating bases for bandlimited functions. We also discuss various examples to illustrate these results. Finally, conclusions are presented in Section 9.

2. CARATHÉODORY REPRESENTATION

Carathéodory representation solves the trigonometric moments problem and can be stated as follows (see [8, Chap. 4]).

THEOREM 2.1. *Given N complex numbers $\boldsymbol{c} = (c_1, c_2, \dots, c_N)$, not all zero, there exist unique $M \leq N$, positive numbers $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_M)$, and distinct real numbers $\theta_1, \theta_2, \dots, \theta_M$, $-1 < \theta_j \leq 1$, such that*

$$c_k = \sum_{j=1}^{M} \rho_j \mathrm{e}^{\mathrm{i}\pi\theta_j k}, \qquad \textit{for} \quad k = 1, 2, \dots, N. \tag{2.1}$$

Although the theorem applies to all finite sequences $\boldsymbol{c}$ of complex numbers, it is useful in practical applications if there is a reason to seek representations of the form (2.1) with positive weights ρ_j. For example, if the sequence $\boldsymbol{c}$ is the values of a covariance function, then this theorem provides the foundation for several spectral estimation algorithms in signal processing, e.g., the so-called Pisarenko method (see [20] for more details). In this paper, we are interested in the case where the sequence $\boldsymbol{c}$ contains the trigonometric moments of a positive weight.

Given $\boldsymbol{c}$, finding M, the phases $\boldsymbol{\theta} = (\theta_1, \dots, \theta_M)$, $|\theta_j| \leq 1$, and the positive weights $\boldsymbol{\rho}$ can be viewed as a nonlinear inverse problem for the unequally spaced discrete Fourier transform [1, 5].

As discussed in the introduction, the problem of finding $\boldsymbol{\rho}$, where $\boldsymbol{c}$, M, and $|\theta_j| \leq 1$ are given, can be arbitrarily ill conditioned. In contrast, the phases θ_j in Carathéodory representation are related to the vector $\boldsymbol{c}$ and we have a stability estimate:

THEOREM 2.2. *Vectors $\boldsymbol{c}$ and $\boldsymbol{\rho}$ as in Theorem 2.1 satisfy*

$$\|\boldsymbol{\rho}\|_2 \leq \sqrt{2}\, \|\boldsymbol{c}\|_2.$$

For the proof see the Appendix.

Grenander and Szegő's proof of Carathéodory representation [8] provides a method to obtain M, the phases $\boldsymbol{\theta}$, and the weights $\boldsymbol{\rho}$. It is also the foundation for Pisarenko's method for spectral estimation [20]. We now outline its main points.

2.1. *Algorithm I: Method to Obtain M, $\boldsymbol{\theta}$, and $\boldsymbol{\rho}$*

(1) Given $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)$, we extend the definition of c_k to negative k as $c_{-k} = \overline{c_k}$ and we define c_0 so that the $(N+1) \times (N+1)$ Toeplitz matrix $\boldsymbol{T}_N$ of elements $(\boldsymbol{T}_N)_{kj} = c_{j-k}$, has nonnegative eigenvalues and at least one eigenvalue is equal to zero.

(2) Define M as the rank of $\boldsymbol{T}_N$. By construction, we have $M \leq N$. We also say that M is the *rank* of the representation (2.1).

(3) Let $\boldsymbol{T}_M$ be the top left principal submatrix of order $M+1$ of $\boldsymbol{T}_N$. That is, the matrix $\boldsymbol{T}_M$ has elements $(c_{j-k})_{0 \leq k, j \leq M}$. Find the eigenvector $\boldsymbol{q}$ corresponding to the zero eigenvalue of $\boldsymbol{T}_M$.

(4) Construct the polynomial (eigenpolynomial) whose coefficients are the entries of the eigenvector $\boldsymbol{q}$. As shown in [8, p. 58], the M roots of this eigenpolynomial are distinct and have absolute value 1. The phases of these roots are the numbers θ_j.

(5) Find the weights $\boldsymbol{\rho}$ by solving the Vandermonde system (2.1) for $k = 1, \ldots, M$. They will, in addition, satisfy $\sum_k \rho_k = c_0$.

Remark 2.1. With the extension of the sequence c_k, (2.1) is valid for $|k| \leq N$. If $\boldsymbol{q} = (q_0, \ldots, q_M)$ is the eigenvector obtained in part (3) of Algorithm 2.1, then

$$\sum_{k=0}^{M} c_{k+s} q_k = 0, \tag{2.2}$$

for all s, $-N \leq s \leq 0$. In other words, we have found an order-M recurrence relation for the original sequence $\{c_k\}_{k=1}^{N}$.

Remark 2.2. In practice, we are interested in using Carathéodory representation if M is small compared with N, or more generally, if most weights are smaller than the accuracy sought. However, in such cases, $\boldsymbol{T}_N$ has a large (numerical) null subspace that causes severe numerical problems in determining c_0, the rank M, and the eigenvector $\boldsymbol{q}$.

Nevertheless, if the sequence $\boldsymbol{c}$ is the trigonometric moments of an appropriate weight, we will be able to modify the previous method in order to obtain the phases θ_j in an efficient manner. In this setting, the phases and weights in Carathéodory representation can be thought of as the nodes and weights of a Gaussian-type quadrature for weighted integrals. Once the phases are obtained, Theorem 2.2 assures that the computation of the weights is a well-posed problem. In Section 5.2 we present a fast algorithm to obtain the weights by evaluating certain polynomials at the nodes $\mathrm{e}^{\mathrm{i}\pi\theta_j}$.

Remark 2.3. Given any Hermitian Toeplitz matrix $\boldsymbol{T}$, let us consider its smallest eigenvalue $\lambda^{(N)}$. It is easy to see that Carathéodory representation implies the following representation of $\boldsymbol{T}$ as a sum of rank-one Hermitian Toeplitz matrices,

$$(\boldsymbol{T} - \lambda^{(N)} \boldsymbol{I})_{kl} = \sum_{j=1}^{M} \rho_j \mathrm{e}^{\mathrm{i}\pi\theta_j(l-k)}, \tag{2.3}$$

where ρ_j are positive and $\mathrm{e}^{\mathrm{i}\theta_j}$ are the zeros of the eigenpolynomial corresponding to the eigenvalue $\lambda^{(N)}$.

3. GENERALIZED GAUSSIAN QUADRATURES FOR EXPONENTIALS

3.1. Preliminaries: Chebyshev Systems

In this section we collect some definitions and results related to Chebyshev systems. We follow mostly Karlin and Studden [12] (see also [13]). Readers familiar with this topic may skip this section.

A family of $n+1$ real-valued functions $u_0, \ldots, u_n$ defined on an interval $I = [a, b]$ is a *Chebyshev system* (T-system) if any nontrivial linear combination

$$u(t) = \sum_{j=0}^{n} \alpha_j u_j(t) \tag{3.1}$$

has at most n zeros on the interval I. This property of a T-system can be viewed as a generalization of the same property for polynomials. Indeed, the family $\{1, t, t^2, \ldots, t^n\}$ provides the simplest example of a Chebyshev system.

Alternatively, a T-system over $[a, b]$ may be defined by the condition that the $n+1$ order determinant is nonvanishing,

$$\det \begin{bmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ \cdots & \cdots & \cdots & \cdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{bmatrix} \neq 0, \tag{3.2}$$

whenever $a \leq t_0 < t_1 < \cdots < t_n \leq b$. Without loss of generality, the determinant can be assumed positive.

Let $u_0, \ldots, u_n$ be a T-system on the interval I. The moment space $\mathcal{M}_{n+1}$ with respect to $u_0, \ldots, u_n$ is defined as the set

$$\mathcal{M}_{n+1} = \left\{ \boldsymbol{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1} \,\middle|\, c_j = \int_I u_j(t)\, d\mu(t),\ j = 0, \ldots, n \right\}, \tag{3.3}$$

where the measure $\mu(t)$ ranges over the family of nondecreasing right-continuous functions of bounded variation on the interval I. It can be shown that the moment space is a closed convex cone. We will denote the *interior* of the moment space $\mathrm{Int}(\mathcal{M}_{n+1})$.

Let us consider a representation of a point $\boldsymbol{c} = (c_0, \ldots, c_n) \in \mathcal{M}_{n+1}$

$$c_j = \sum_{k=1}^{m} \rho_k u_j(t_k), \qquad j = 0, \ldots, n, \tag{3.4}$$

where $\rho_k > 0$, $a \leq t_k \leq b$, $k = 1, \ldots, m$. The *index* $\mathcal{I}(\boldsymbol{c})$ of a point $\boldsymbol{c} \in \mathcal{M}_{n+1}$ is defined as the minimum number of points t_k that are used in the representation (3.4), where the boundary points $t_k = a$ and $t_k = b$ are counted as $1/2$ and the points in the interior of the interval $a < t_k < b$ are counted as 1.

The representation (3.4) induces a generalized Gaussian quadrature for the integral with the measure that defines the point $\boldsymbol{c}$ in the moment space while the index describes the number of nodes necessary for the quadrature. The following theorems (see [12] and [13]) generalize to any T-system the usual Gaussian quadratures for the polynomials $\{1, t, \ldots, t^n\}$.

THEOREM 3.1. *A point* $\boldsymbol{c} \in \mathcal{M}_{n+1}$, $\boldsymbol{c} \neq 0$, *is a* boundary *point of* $\mathcal{M}_{n+1}$ *if and only if* $\mathcal{I}(\boldsymbol{c}) < (n+1)/2$.

THEOREM 3.2. *Let* $u_0, \ldots, u_{2m}$ *be a T-system on* $[a, b]$ *and let* $\boldsymbol{c}$ *be a boundary point of* $\mathcal{M}_{2m+1}$. *Then there exists a unique representation with the index less than* $m+1$ *which involves no more than* $m+1$ *nodes.*

THEOREM 3.3. *Let* $u_0, \ldots, u_{2m}$ *be a T-system on* $[a, b]$ *and let* $\boldsymbol{c}$ *be an interior point of* $\mathcal{M}_{2m+1}$. *Then there exist at least two representations with the index* $m + 1/2$ *(with* $m+1$ *terms). Both of them have* $m+1$ *nodes, one of which is the end point of the interval.*

If the functions $u_0, \ldots, u_n$ are periodic on the interval I and satisfy (3.2), then they define a periodic T-system. A periodic T-system always involves an odd number of functions. Indeed, since the system is defined on a circle, the equally spaced values $(t_0, \ldots, t_n)$ can be continuously rotated into $(t_n, \ldots, t_0)$. If the number of functions were even, such rotation would change the sign of the determinant in (3.2) and, due to the continuous dependence on $(t_0, \ldots, t_n)$, would force the determinant to vanish at some intermediate point.

For periodic systems holds [12, 13]:

THEOREM 3.4. *Let* $u_0, \ldots, u_{2m}$ *be a periodic T-system on* $[-1, 1]$ *and let* $\boldsymbol{c}$ *be an interior point of* $\mathcal{M}_{2m+1}$. *Then for each point* t_0, $-1 \le t_0 \le 1$, *there exists a unique representation with the index* $\mathcal{I}(\boldsymbol{c}) = m+1$ *(with* $m+1$ *terms) involving* t_0 *as a node.*

3.2. *Generalized Gaussian Quadratures for Exponentials*

Let us consider a family of real periodic functions

$$1, \cos(\pi t), \sin(\pi t), \ldots, \cos(\pi m t), \sin(\pi m t) \tag{3.5}$$

on the interval $[-1, 1]$. We treat the boundary points -1 and 1 as identical so that (3.5) is defined on the circle. The system of functions in (3.5) is a periodic Chebyshev system (T-system) in $[-1, 1]$ (see [12, 13] and Section 3.1 for a brief summary).

We also consider this family on a proper subinterval of $[-1, 1]$. On any subinterval $[a, b] \subset [-1, 1]$, the family in (3.5) is a T-system.

Let us consider the moments of the measure $\omega(\tau)\, d\tau$, where $\omega(\tau)$ is a weight function,

$$\alpha_k = \int_{-1}^{1} \omega(\tau) \cos(\pi k \tau)\, d\tau, \qquad k = 0, 1, \ldots, m, \tag{3.6}$$

and

$$\beta_k = \int_{-1}^{1} \omega(\tau) \sin(\pi k \tau)\, d\tau, \qquad k = 1, \ldots, m. \tag{3.7}$$

We also consider complex-valued moments,

$$\gamma_k = \alpha_k + \mathrm{i}\beta_k = \int_{-1}^{1} \omega(\tau) \mathrm{e}^{\mathrm{i}\pi k \tau}\, d\tau, \qquad k = 1, \ldots, m. \tag{3.8}$$

Let $\mathcal{M}_{2m+1}$ be the moment space and let $\mathrm{Int}(\mathcal{M}_{2m+1})$ be its interior, as defined in Section 3.1. We have

THEOREM 3.5 [12, VI, Sec. 4]. *For the periodic T-system* (3.5)*, a point* $\boldsymbol{c} = \{\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m\}$ *is a point of the moment space* $\mathcal{M}_{2m+1}$ *if and only if the Toeplitz matrix* $\{\gamma_{k'-k}\}_{k,k'=0,\dots,m}$ *is nonnegative definite.*

Furthermore, $\boldsymbol{c} \in \mathrm{Int}(\mathcal{M}_{2m+1})$ *if and only if the Toeplitz matrix* $\{\gamma_{k-k'}\}_{k,k'=0,\dots,m}$ *is positive definite.*

We also have

THEOREM 3.6 [12, VI, Sec. 2]. *A point* $\boldsymbol{c} = \{\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m\}$ *is a boundary point of the moment space* $\mathcal{M}_{2m+1}$ *if and only if there is a unique representation*

$$\gamma_k = \sum_{j=1}^{m'} \omega_j \mathrm{e}^{\mathrm{i}\pi\theta_j k}, \tag{3.9}$$

where $m' \leq m$ *and* $-1 \leq \theta_j \leq 1$.

If $\boldsymbol{c} \in \mathrm{Int}(\mathcal{M}_{2m+1})$*, then for each* $\tau_0 \in [-1, 1]$*, there exists a unique representation with* $m+1$ *nodes, including* τ_0 *as a node; that is,*

$$\gamma_k = \sum_{j=1}^{m} \omega_j \mathrm{e}^{\mathrm{i}\pi\theta_j k} + \omega_0 \mathrm{e}^{\mathrm{i}\pi\tau_0 k}, \tag{3.10}$$

where $-1 \leq \theta_j \leq 1$.

Let us consider weights ω supported in a subinterval of $[-1, 1]$. We then prove

THEOREM 3.7. *Let* ω *be a weight supported in some interval* $I = [a, b]$*,* $I \subseteq [-1, 1]$*. Then there exists a unique representation*

$$\int_{-1}^{1} \omega(t)\mathrm{e}^{\mathrm{i}\pi k t}\,dt = \sum_{j=1}^{m} \omega_j \mathrm{e}^{\mathrm{i}\pi\theta_j k} + \omega_0\,(-1)^k, \qquad \textit{for} \quad |k| \leq m, \tag{3.11}$$

where $a < \theta_j < b$ *and* $\omega_j > 0$ *for* $j = 0, \dots, m$.

Moreover, if $I = [-a, a]$*, where* $0 < a \leq 1/2$*, then*

$$\omega_0 \leq \frac{4\int_{-1/2}^{1/2} \omega(t)\,dt}{2 + (2+\sqrt{3})^m + (2-\sqrt{3})^m}. \tag{3.12}$$

Proof. Let us start by considering the periodic T-system (3.5) and $\boldsymbol{c} = \{\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m\}$ the point in the moment space obtained from the measure $d\mu(t) = \omega(t)\,dt$. It is easy to show that the Toeplitz matrix $\{\gamma_{j-k}\}$, obtained for the moments in (3.8), is positive definite (see (4.7)). Thus, Theorem 3.5 and (3.10) with $\tau_0 = 1$ imply a unique representation (3.11) with $-1 < \theta_j < 1$ and $k = 0, \dots, m$. Since the weights are real, (3.11) also holds for $k = -m, \dots, -1$. We want to show that, in fact, $a < \theta_j < b$.

Since the weight ω is supported in $[a, b] \subseteq [-1, 1]$, we can also consider $[a, 1]$ as its interval of definition. We note that the functions in (3.5) form a Chebyshev system on this interval (in fact, on any subinterval of $[-1, 1]$). Using Theorem 3.3 we construct a representation which includes the boundary point 1 of the interval $[a, 1]$ as a node. However, this representation also holds on $[-1, 1]$, where (3.11) guarantees uniqueness.

Thus, to avoid a contradiction, we conclude that $a < \theta_j \leq 1$ in (3.11). Similarly, let us consider the weight ω in $[-1, b]$. By the same argument we obtain that $-1 \leq \theta_j < b$ (the points -1 and 1 are identical on $[-1, 1]$). Therefore, we conclude that $a < \theta_j < b$ and (3.11) is established.

Let us now consider a periodic trigonometric polynomial

$$v_m(t) = T_m(1 - \cos(\pi t)), \tag{3.13}$$

where T_m is the Chebyshev polynomial of degree m. Since the degree of $v_m(t)$ does not exceed m, we have from (3.11)

$$\int_{-1/2}^{1/2} v_m(t)\omega(t)\,dt = \sum_{j=1}^{m} \omega_j v_m(\theta_j) + \omega_0 v_m(1), \tag{3.14}$$

where $-1/2 < \theta_j < 1/2$.

Let us compute $v_m(1) = T_m(2)$. Using the three-term recursion for the Chebyshev polynomials, we obtain

$$v_m(1) = ((2 + \sqrt{3})^m + (2 - \sqrt{3})^m)/2. \tag{3.15}$$

Since in the interval $[-1/2, 1/2]$ the absolute value of v_m does not exceed 1,

$$\omega_0 v_m(1) = \left| \int_{-1/2}^{1/2} v_m(t)\omega(t)\,dt - \sum_{j=1}^{m} \omega_j v_m(\theta_j) \right| \leq \int_{-1/2}^{1/2} \omega(t)\,dt + \sum_{j=1}^{m} \omega_j. \tag{3.16}$$

Setting $k = 0$ in (3.11), we obtain

$$\sum_{j=1}^{m} \omega_j = \int_{-1/2}^{1/2} \omega(t)\,dt - \omega_0 \tag{3.17}$$

and, combining with (3.16), we arrive at (3.12). ■

Remark 3.1. Since the numerator in (3.12) remains bounded and the denominator grows exponentially fast with m, the coefficient ω_0 is very small even for m of moderate size.

4. A NEW FAMILY OF GAUSSIAN-TYPE QUADRATURES

Let us consider the trigonometric moments of a weight $w(\tau)$,

$$t_k = \int_{-1}^{1} e^{i\pi\tau k} w(\tau)\,d\tau. \tag{4.1}$$

In our approach, it is essential to consider only weights supported inside $[-1/2, 1/2]$. Only then can the moments t_k be viewed as values of a properly sampled bandlimited function (see (6.2) and (6.3)).

In this section we start by using Carathéodory representation and Theorem 3.7 from the previous section, to construct two different Gaussian quadratures for integrals with weight w. These quadratures are exact for trigonometric polynomials of appropriate degree.

We then generalize these types of quadratures further and develop a new family of Gaussian-type quadratures. This family of quadrature formulas is parameterized by the eigenvalues of the Toeplitz matrix

$$\boldsymbol{T} = \{t_{l-k}\}_{0\le k,\, l\le N}. \tag{4.2}$$

Among these new quadrature formulas, only those corresponding to eigenvalues of small size are of practical interest. In fact, the size of the eigenvalue determines the error of the quadrature formula. To compute the weights and nodes of these quadratures, we develop a new algorithm which may be viewed as a (major) modification of Algorithm 2.1. The new algorithm is described in Section 5. The main results of this section are gathered in Theorem 4.1.

We start by using Theorem 3.7 to write

$$t_k = \sum_{j=1}^{N} \omega_j e^{i\pi\phi_j k} + \omega_0(-1)^k, \qquad \text{for } |k| \le N, \tag{4.3}$$

for unique positive weights ω_j and phases ϕ_j in $(-1, 1)$. Then, for any $A(z) = \sum_{|k|\le N} a_k z^k$ in Λ_N, the space of Laurent polynomials of degree at most N, we have

$$\int_{-1}^{1} A(e^{i\pi\tau}) w(\tau)\, d\tau = \sum_{|k|\le N} a_k t_k = \sum_{j=1}^{N} \omega_j A(e^{i\pi\phi_j}) + \omega_0 A(-1), \tag{4.4}$$

for unique positive weights ω_j and nodes $e^{i\pi\phi_j}$.

Alternatively, using Carathéodory representation (2.1) applied to the sequence $c_k = t_k$, $1 \le k \le N$,

$$\begin{aligned}\int_{-1}^{1} A(e^{i\pi\tau}) w(\tau)\, d\tau &= \sum_{j=1}^{M} \rho_j A(e^{i\pi\theta_j}) + (t_0 - c_0)\frac{1}{2}\int_{-1}^{1} A(e^{i\pi\tau})\, d\tau \\ &= \sum_{j=1}^{M} \rho_j A(e^{i\pi\theta_j}) + \lambda^{(N)}\frac{1}{2}\int_{-1}^{1} A(e^{i\pi\tau})\, d\tau,\end{aligned} \tag{4.5}$$

where $c_0 = \sum_{j=1}^{M} \rho_j$ and $\{e^{i\pi\theta_j}\}$ are the roots of the eigenpolynomial corresponding to the smallest eigenvalue $\lambda^{(N)}$ of $\boldsymbol{T}$.

Note that (4.5) is again valid for all $A(z)$ in Λ_N and that the positive weights ρ_j and phases θ_j in $(-1, 1]$ are unique.

Thus, we have two different quadratures that may not coincide. However, by considering $w(\tau)$ supported inside $(-1/2, 1/2)$, (3.12) implies that w_0 in (4.4) decreases exponentially fast with N and, since $\min w(\tau) = 0$ for $|\tau| \le 1$, we have

$$\lim_{N\to\infty} \lambda^{(N)} = 0, \tag{4.6}$$

as shown in [8, p. 65]. In consequence, for large N, the difference between these two quadratures can be made smaller than the accuracy sought.

A similar reasoning could be applied to other small eigenvalues of $\boldsymbol{T}$ provided we can generalize (4.5) to other eigenvalues and roots of the corresponding eigenpolynomials. For that purpose, we first describe some properties of these eigenpolynomials.

4.1. Toeplitz Matrices for Trigonometric Moments

We summarize in this section properties of eigenpolynomials of the Toeplitz matrix $\boldsymbol{T}$ with entries $\{t_{l-k}\}_{k,l=0,\dots,N}$. The matrix $\boldsymbol{T}$ is self-adjoint and positive definite since, for all $\boldsymbol{x} \in \mathbb{C}^{N+1}$,

$$\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{x}\rangle = \sum_{k,l=0,\dots,N} t_{l-k} x_l \overline{x_k} = \int_{-1}^{1} |P_{\boldsymbol{x}}(\mathrm{e}^{\mathrm{i}\pi\tau})|^2 w(\tau)\, d\tau, \tag{4.7}$$

where $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_k x_k \overline{y_k}$ is the usual inner product of two vectors and $P_{\boldsymbol{x}}(z) = \sum_k x_k z^k$.

More generally, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{N+1}$, $\boldsymbol{T}$ induces a weighted inner product for trigonometric polynomials,

$$\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y}\rangle = \int_{-1}^{1} P_{\boldsymbol{x}}(\mathrm{e}^{\mathrm{i}\pi\tau})\overline{P_{\boldsymbol{y}}(\mathrm{e}^{\mathrm{i}\pi\tau})} w(\tau)\, d\tau. \tag{4.8}$$

Since $\boldsymbol{T}$ is positive definite, there exists an orthonormal basis $\{\boldsymbol{v}^{(k)}\}_{k=0,\dots,N}$ of eigenvectors of $\boldsymbol{T}$ corresponding to eigenvalues $\lambda^{(0)} \geq \lambda^{(1)} \geq \cdots \geq \lambda^{(N)} > 0$. The corresponding eigenpolynomials $V^{(k)}(z) = \sum_j \boldsymbol{v}_j^{(k)} z^j$ satisfy

$$\int_{-1}^{1} V^{(k)}(\mathrm{e}^{\mathrm{i}\pi t})\overline{V^{(l)}(\mathrm{e}^{\mathrm{i}\pi t})}\, dt = 2\delta_{kl} \tag{4.9}$$

and, because of (4.8),

$$\int_{-1}^{1} V^{(k)}(\mathrm{e}^{\mathrm{i}\pi t})\overline{V^{(l)}(\mathrm{e}^{\mathrm{i}\pi t})} w(t)\, dt = \delta_{kl}\, \lambda^{(k)}. \tag{4.10}$$

For a vector $\boldsymbol{x} = (x_0, \dots, x_N)$, let us define the *reciprocal vector* of $\boldsymbol{x}$ as

$$\boldsymbol{x}_* = \left(\overline{x_N}, \dots, \overline{x_0}\right)$$

and, similarly, for a trigonometric polynomial $P(z)$, the *reciprocal polynomial* of P as

$$P_*(z) = \overline{P}(z^{-1}) = \overline{P(\overline{z}^{-1})}. \tag{4.11}$$

Since $\boldsymbol{T}$ is Toeplitz Hermitian, we have

$$\boldsymbol{T}\boldsymbol{x}_* = \lambda \boldsymbol{x}_*, \qquad \text{if } \boldsymbol{T}\boldsymbol{x} = \lambda \boldsymbol{x}.$$

In particular, if λ is a simple eigenvalue, its corresponding eigenspace can be generated by a self-reciprocal eigenvector $\boldsymbol{x}$, i.e., $\boldsymbol{x}_* = \boldsymbol{x}$, and the associated (self-reciprocal) eigenpolynomial will have roots in pairs $\{\gamma, \overline{\gamma}^{-1}\}$.

4.2. Gaussian-Type Quadratures on the Unit Circle

In this section we present the main results of the paper. We derive new Gaussian-type quadratures valid for any eigenvalue of the matrix $\boldsymbol{T}$ rather than just the smallest eigenvalue λ_N. These quadratures allow us to select the desired accuracy and thus, to construct accuracy-dependent families of quadratures.

The nodes of the quadrature in (4.5) are the roots of the eigenpolynomial corresponding to the least eigenvalue of $\boldsymbol{T}$ and, because of Carathéodory representation, we know that these roots are on the unit circle and that the weights are positive numbers. In our generalization, this standard property for the nodes and weights is no longer enforced. However, we will show that for nodes on the unit circle, the corresponding weights are real. Moreover, in all examples we have examined, for all small eigenvalues λ of $\boldsymbol{T}$, their negative weights are associated with the nodes outside the support of the weight and are comparable in size with λ. We believe this property to hold for a wide variety of weights.

We prove the following

THEOREM 4.1. *Assume that the eigenpolynomial $V^{(s)}(z)$ corresponding to the eigenvalue $\lambda^{(s)}$ of $\boldsymbol{T}$ has distinct, nonzero roots $\{\gamma_j\}_{j=1}^N$. Then there exist numbers $\{w_j\}_{j=1}^N$ such that*

(i) *For all Laurent polynomials $P(z)$ of degree at most N,*

$$\int_{-1}^{1} P(\mathrm{e}^{\mathrm{i}\pi t})w(t)\,dt = \sum_{j=1}^{N} w_j P(\gamma_j) + \lambda^{(s)}\frac{1}{2}\int_{-1}^{1} P(\mathrm{e}^{\mathrm{i}\pi t})\,dt. \tag{4.12}$$

(ii) *For each root γ_k with $|\gamma_k| = 1$, the corresponding weight w_k is a real number and*

$$w_k = \int_{-1}^{1} |L_k^s(\mathrm{e}^{\mathrm{i}\pi t})|^2 w(t)\,dt - \lambda^{(s)}\frac{1}{2}\int_{-1}^{1} |L_k^s(\mathrm{e}^{\mathrm{i}\pi t})|^2\,dt, \tag{4.13}$$

where

$$L_k^s(z) = \frac{V^{(s)}(z)}{(V^{(s)})'(\gamma_k)\,(z-\gamma_k)} \tag{4.14}$$

is the Lagrange polynomial associated with the root γ_k.

(iii) *If $\lambda^{(s)}$ is a simple eigenvalue, then for $k = 1, \ldots, N$, the weight w_k is nonzero and*

$$\frac{1}{w_k} = \sum_{\substack{0\le l\le N\\ l\ne s}} \frac{V^{(l)}(\gamma_k)\,V_*^{(l)}(\gamma_k)}{\lambda^{(l)} - \lambda^{(s)}}, \tag{4.15}$$

where $V_*^{(l)}(z) = \overline{V^{(l)}}(z^{-1})$ is the reciprocal polynomial of $V^{(l)}(z)$.

In particular, for each γ_k with $|\gamma_k| = 1$,

$$\frac{1}{w_k} = \sum_{\substack{0\le l\le N\\ l\ne s}} \frac{|V^{(l)}(\gamma_k)|^2}{\lambda^{(l)} - \lambda^{(s)}}. \tag{4.16}$$

(iv) *If $\lambda^{(s)}$ is a simple eigenvalue and all roots γ_k are on the unit circle, then the set $\{w_k\}_{k=1}^N$ contains exactly s positive numbers and $N - s$ negative numbers.*

In particular, if $s = 0$ or $s = N$, then all w_k are negative or positive, respectively.

Remark 4.1. Our approach to obtain Gaussian quadratures does not use Szegő polynomials and is therefore substantially different than the one in [11]. We briefly explain the approach in [11]. Note that (4.9) and (4.10) show that the polynomials $\{V^{(k)}(z)\}$ are orthogonal with respect to both the usual inner product for trigonometric polynomials and the weighted inner product with weight $w(t)$. We can also construct Szegő polynomials $\{p_k(z)\}$ orthogonal with respect to $w(t)$ and such that each $p_k(z)$ has precise degree k [26]. For any k, the roots of $p_k(z)$ are all in $|z| < 1$ [8].

Szegő polynomials and their reciprocals induce para-orthogonal polynomials [11],

$$B_n(z) = p_n(z) + \xi_n z^n (p_n)_*(z),$$

where ξ_n are complex constants, $|\xi_n| = 1$. The roots of $B_n(z)$ are on the unit circle and can be used as the nodes for Gaussian quadratures with respect to the weight $w(t)$.

Under appropriate assumptions to guarantee uniqueness, the quadratures in [11] should coincide with those obtained in Theorem 3.7.

In contrast to these exact quadratures, in Theorem 4.1 we derive a new family of Gaussian-type quadratures where, for each eigenvalue of the Toeplitz matrix (4.2), there is a corresponding quadrature formula. Even for the smallest eigenvalue $\lambda^{(N)}$, the quadratures in (4.12) and in Theorem 3.7 are different because of the extra integral term in (4.12). The size of this extra term is controlled by the size of the corresponding eigenvalue and, thus, it is never exactly zero. However, in applications, this extra term can be made as small as desired via oversampling (see (6.1)–(6.3) and note that (4.6) is valid for all small eigenvalues, not just the smallest [8, p. 65]).

Remark 4.2. For an eigenvalue $\lambda^{(s)}$ of small size, the integral term on the right-hand side of (4.12) can be neglected. This is the case of practical interest.

Remark 4.3. Even though the weights w_k could be complex valued, an important consequence of Theorem 4.1 is that in many important cases w_k are, in fact, real.

Remark 4.4. We have observed (see Table I) that the weights w_k corresponding to nodes outside the support of the weight $w(t)$ are small, negative, and roughly of the size of the eigenvalue $\lambda^{(s)}$. Although we now present a heuristic explanation of this behavior, we do not know if a proof can be obtained along this path.

Let us split the sum in (4.16),

$$\frac{1}{w_k} = \sum_{l:\, \lambda^{(l)} > \lambda^{(s)}} \frac{|V^{(l)}(\gamma_k)|^2}{\lambda^{(l)} - \lambda^{(s)}} - \sum_{l:\, \lambda^{(s)} > \lambda^{(l)}} \frac{|V^{(l)}(\gamma_k)|^2}{\lambda^{(s)} - \lambda^{(l)}}. \tag{4.17}$$

If $\lambda^{(s)}$ is in the range of the exponential decay of the eigenvalues, the first term in (4.17) turns out to be much smaller than the second term, which is approximately

$$\frac{1}{\lambda^{(s)}} \sum_{l:\, \lambda^{(s)} > \lambda^{(l)}} |V^{(l)}(\gamma_k)|^2.$$

For γ_k outside the support of the measure, we have observed (Figs. 2, 3, and 5–8) that

$$\sum_{l:\, \lambda^{(s)} > \lambda^{(l)}} |V^{(l)}(\gamma_k)|^2$$

is a constant of moderate size.

Thus, the second term in (4.17) is $O(1/\lambda^{(s)})$ and the weight is indeed negative and roughly of the size of the eigenvalue.

Remark 4.5. For the weight with value 1 in $(-1/2, 1/2)$ and 0 otherwise, the eigenpolynomials are the discrete PSWF. For these functions, we know that all eigenvalues are simple and that all eigenpolynomial roots are on the unit circle [23].

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1, *it follows that the Toeplitz matrix* $\boldsymbol{T}$ *in* (4.2) *has the following representation as a sum of rank-1 Toeplitz matrices,*

$$(\boldsymbol{T} - \lambda^{(s)} \boldsymbol{I})_{kl} = \sum_{j=1}^{N} w_j \gamma_j^{l-k},$$

where $\lambda^{(s)}$, w_j, *and* γ_j *are as in (4.12).*

This corollary should be compared with Remark 2.3 noting that, in the corollary, $\lambda^{(s)}$ is not necessarily the least eigenvalue of $\boldsymbol{T}$. For an alternative derivation see [4].

Proof of Theorem 4.1. (1) For $\boldsymbol{x} = (x_0, \ldots, x_N) \in \mathbb{C}^{N+1}$, let us define

$$A_{\boldsymbol{x}}(z) = \begin{cases} \displaystyle\sum_{l=-L}^{L} x_{l+L} z^l, & \text{if } N = 2L, \\ \displaystyle\sum_{l=-L+1}^{L} x_{l+L-1} z^l, & \text{if } N = 2L-1. \end{cases}$$

The values of $A_{\boldsymbol{x}}$ on the unit circle have a phase shift with respect to those of $P_{\boldsymbol{x}}$. In fact, depending on the parity of N, $A_{\boldsymbol{x}}(e^{i\pi t})$ is either $P_{\boldsymbol{x}}(e^{i\pi t})e^{-i\pi t L}$ or $P_{\boldsymbol{x}}(e^{i\pi t})e^{-i\pi t(L-1)}$.

Hence, (4.8) holds replacing $P_{\boldsymbol{x}}$ by $A_{\boldsymbol{x}}$, and then (4.9)–(4.10) also hold for the shifted eigenpolynomials.

We prove the theorem for the case $N = 2L$. (The case $N = 2L - 1$ is similar.) For this case, using the same notation $V^{(k)}$ for the shifted eigenpolynomials, we have

$$V^{(k)}(z) = \sum_{l=-L}^{L} \boldsymbol{v}_{l+L}^{(k)} z^l.$$

(2) Since $\{\gamma_j\}$ are distinct, we define $\{w_j\}_{j=1}^{N}$ as the unique solution of the Vandermonde system

$$\sum_{j=1}^{N} \gamma_j^{-k} w_j = \int_{-1}^{1} e^{-i\pi t k} w(t)\, dt, \qquad \text{for } k = 1, \ldots, N. \tag{4.18}$$

(3) Let $P \in \Lambda_N$; then $z^N P(z)$ is a polynomial of at most degree $2N$, and since $z^L V^{(s)}(z)$ is a polynomial of degree N, by Euclidean division, there exist polynomials $q(z)$ and $r(z)$ of degrees at most N and $N-1$ such that

$$z^N P(z) = z^L V^{(s)}(z) q(z) + r(z).$$

Thus,

$$P(z) = V^{(s)}(z) Q(z) + R(z), \tag{4.19}$$

where $Q(z) \in \Lambda_L$ and $R(z)$ has the form $R(z) = \sum_{k=1}^{N} r_k z^{-k}$ and hence

$$\int_{-1}^{1} R(\mathrm{e}^{\mathrm{i}\pi t})\, dt = 0.$$

Using the fact that $\{V^{(l)}\}_{l=0}^{N}$ is a basis of Λ_L, we write

$$\overline{Q(\mathrm{e}^{\mathrm{i}\pi t})} = \sum_{l=0}^{N} d_l\, V^{(l)}(\mathrm{e}^{\mathrm{i}\pi t}),$$

where d_l are some complex coefficients.

Using (4.10) and (4.18), we multiply both sides of (4.19) by $w(t)$ and integrate to obtain

$$\int_{-1}^{1} P(\mathrm{e}^{\mathrm{i}\pi t}) w(t)\, dt = \sum_{l=0}^{N} \overline{d_l} \int_{-1}^{1} V^{(s)}(\mathrm{e}^{\mathrm{i}\pi t}) \overline{V^{(l)}(\mathrm{e}^{\mathrm{i}\pi t})} w(t)\, dt + \int_{-1}^{1} R(\mathrm{e}^{\mathrm{i}\pi t}) w(t)\, dt$$
$$= \overline{d_s} \lambda^{(s)} + \sum_{j=1}^{N} w_j R(\gamma_j).$$

Now, (4.19) implies that the last sum equals $\sum_{j=1}^{N} w_j P(\gamma_j)$. To find the constant d_s, we integrate both sides of (4.19) and use (4.9) to obtain

$$\int_{-1}^{1} P(\mathrm{e}^{\mathrm{i}\pi t})\, dt = \sum_{l=0}^{N} \overline{d_l} \int_{-1}^{1} V^{(s)}(\mathrm{e}^{\mathrm{i}\pi t}) \overline{V^{(l)}(\mathrm{e}^{\mathrm{i}\pi t})}\, dt = 2\overline{d_s},$$

and thus (4.12).

(4) Let us assume that the node γ_k has unit norm, $|\gamma_k| = 1$, and let $P(z) = L_k^s(z)\, (L_k^s)_*(z)$. We have $P(\gamma_r) = \delta_{rk}$ and since $P \in \Lambda_{N-1}$, (4.12) implies

$$\int_{-1}^{1} |L_k^s(\mathrm{e}^{\mathrm{i}\pi t})|^2 w(t)\, dt = \lambda^{(s)} \frac{1}{2} \int_{-1}^{1} |L_k^s(\mathrm{e}^{\mathrm{i}\pi t})|^2\, dt + w_k.$$

Clearly w_k is real.

(5) We now show that, for $1 \le k,\ j \le N$,

$$w_k \sum_{\substack{0 \le l \le N \\ l \ne s}} \frac{V^{(l)}(\gamma_j)\, V_*^{(l)}(\gamma_k)}{\lambda^{(l)} - \lambda^{(s)}} = \delta_{kj}, \tag{4.20}$$

and thus, considering $k = j$, (4.15) follows. Note that we need $\lambda^{(s)}$ to be simple to guarantee $\lambda^{(l)} - \lambda^{(s)} \neq 0$, $l \neq s$ in (4.20).

If we view the left hand side of (4.20) as the entries A_{kj} of a matrix A and let B be the matrix of entries

$$B_{lk} = V^{(l)}(\gamma_k), \qquad \text{where } 0 \le l \le N,\ l \neq s, \text{and } 1 \le k \le N, \tag{4.21}$$

we can prove (4.20) by showing that $BA = B$ and that B is nonsingular.

For the latter claim, we simply check that the columns of B are linearly independent. Indeed, let $a_l,\ l \neq s$, be constants such that

$$\sum_{l \neq s} a_l V^{(l)}(\gamma_k) = 0, \qquad \text{for } k = 1, \ldots, N.$$

It follows that the polynomial $P(z) = \sum_{l \neq s} a_l V^{(l)}(z) \in \Lambda_L$ has the $N = 2L$ distinct roots γ_k. Since P and $V^{(s)}$ have the same degree and the same N distinct roots, $P(z) = cV^{(s)}(z)$, for some constant c. By (4.9), $V^{(s)}(z)$ is orthogonal to all the other eigenpolynomials and so $a_l = 0$.

To show that $BA = B$, we first substitute $P(z) = V^{(l)}(z)V_*^{(m)}(z)$ in (4.12) to obtain

$$\int_{-1}^{1} V^{(l)}(e^{i\pi t})\overline{V^{(m)}(e^{i\pi t})}w(t)\,dt = \lambda^{(s)}\frac{1}{2}\int_{-1}^{1} V^{(l)}(e^{i\pi t})\overline{V^{(m)}(e^{i\pi t})}\,dt + \sum_{j=1}^{N} w_j V^{(l)}(\gamma_j)V_*^{(m)}(\gamma_j).$$

Using (4.9)–(4.10), we rewrite the previous equation as

$$\delta_{lm}(\lambda^{(l)} - \lambda^{(s)}) = \sum_{j=1}^{N} w_j V^{(l)}(\gamma_j)V_*^{(m)}(\gamma_j) \tag{4.22}$$

and thus,

$$\begin{aligned}(BA)_{mn} &= \sum_j V^{(m)}(\gamma_j)w_j \sum_{l \neq s} \frac{V^{(l)}(\gamma_n)V_*^{(l)}(\gamma_j)}{\lambda^{(l)} - \lambda^{(s)}} \\ &= \sum_{l \neq s} \frac{V^{(l)}(\gamma_n)}{\lambda^{(l)} - \lambda^{(s)}} \sum_j w_j V^{(m)}(\gamma_j)V_*^{(l)}(\gamma_j) \\ &= V^{(m)}(\gamma_n) = B_{mn}.\end{aligned}$$

(6) To prove the last assertion of the theorem, we consider (4.20) when all γ_k have unit norm and thus all w_k are real. In this case,

$$V_*^{(l)}(\gamma_k) = \overline{V^{(l)}((\gamma_k)_*)} = \overline{V^{(l)}(\gamma_k)},$$

and we can rewrite (4.20) as a matrix identity

$$B^*\Gamma B = W, \tag{4.23}$$

where B is the invertible matrix defined in (4.21), B^* is its adjoint, and Γ and W are diagonal matrices with real entries $\{1/(\lambda^{(l)} - \lambda^{(s)})\}_{0 \le l \le N,\, l \ne s}$ and $\{1/(w_l)\}_{1 \le l \le N}$, respectively.

Using Sylvester's law of inertia [10, Theorem 4.5.8], (4.23) implies that Γ and W have the same inertia, that is, the same number of positive, negative, and zero eigenvalues. The result follows because we assumed $\lambda^{(s)}$ to be simple and then $\lambda^{(0)} \ge \cdots \ge \lambda^{(s-1)} > \lambda^{(s)} > \lambda^{(s+1)} \ge \cdots \ge \lambda^{(N)}$. ■

Techniques similar to those used in the proof of Theorem 4.1 allow us to derive several results for eigenpolynomials corresponding to multiple eigenvalues or for the case where their roots lie outside the unit circle. Here we limit our attention to the case of simple eigenvalues or eigenpolynomials with all roots on the unit circle.

Trench [27] has shown that both the multiplicity of the eigenvalues and the number of the eigenpolynomial zeros outside of the unit circle depend on the oscillations of the weight function $w(\tau)$. We state two of the results in [27] for $\boldsymbol{T}$ as in (4.2).

THEOREM 4.2 [27, Theorem 2.1]. *If λ is an eigenvalue of $\boldsymbol{T}$ with multiplicity m, then $w(\tau) - \lambda$ changes sign at least $2m - 1$ times in $(-1, 1)$.*

THEOREM 4.3 [27, Theorem 3.1]. *Let $u(z)$ be a self-reciprocal eigenpolynomial corresponding to the eigenvalue λ of $\boldsymbol{T}$. If $u(z)$ has $2m$ $(m \ge 1)$ zeros that are not on the unit circle, then $w(\tau) - \lambda$ changes sign at least $2m + 1$ times in $(-1, 1)$.*

4.3. Examples

We consider three examples with different weights and construct the appropriate quadratures. See Fig. 1.

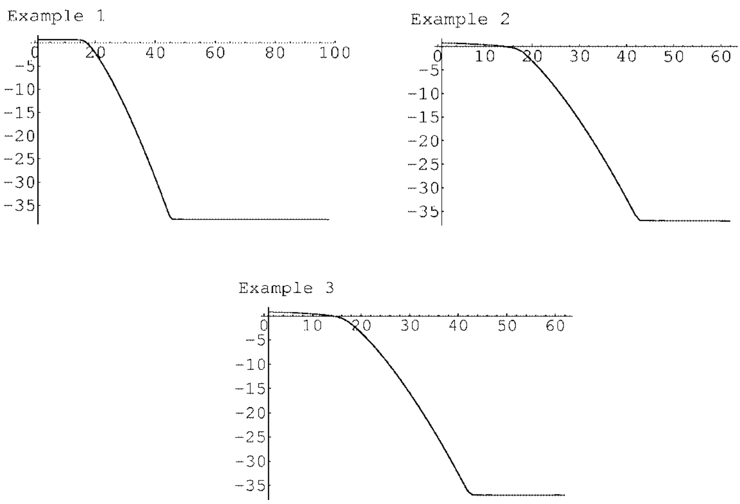

FIG. 1. Decay of the eigenvalues of the matrix $\boldsymbol{T}$ in Examples 1–3. The scale of the vertical axes is logarithmic ($\log_{10}$), whereas the horizontal axes display indices of eigenvalues. We note the exponential rate of decay. The flat portion of the graph for large indices is due to the limited precision of our computations. Thus, these graphs also illustrate the practical difficulty of finding the eigenpolynomial corresponding to the smallest eigenvalue.

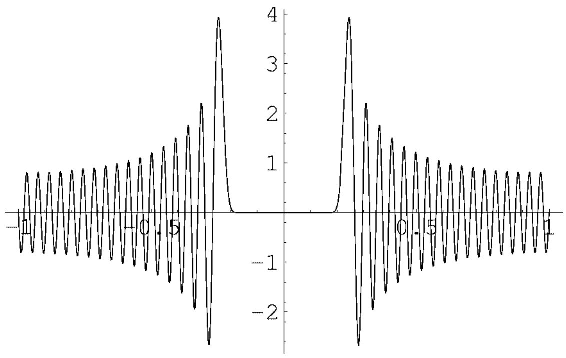

FIG. 2. Modified eigenpolynomial $e^{-i\pi t(N/2)}V^{(30)}(e^{i\pi t})$ on the interval $[-1, 1]$, where $N = 97$ and $V^{(30)}(e^{i\pi t})$ is the eigenpolynomial corresponding to the eigenvalue $\lambda^{(30)}$ in Example 1. The phase factor $e^{-i\pi tN/2}$ is introduced to make this function real.

EXAMPLE 1. First we consider the weight

$$w(t) = \begin{cases} 1, & t \in [-a, a],\ a \le 1/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.24}$$

For this weight, the eigenpolynomials $V^{(l)}(e^{i\pi t})$ of the $N + 1 \times N + 1$ Toeplitz matrix $\boldsymbol{T}$ are the discrete PSWF [23]. Thus the eigenpolynomial $V^{(l)}(e^{i\pi t})$ has all of its zeros on the unit circle. Moreover, it has exactly l zeros for t in the interval $(-a, a)$ and N zeros for t in $[-1, 1]$. In this example we have selected $N = 97$, $a = 1/6$, $c = 15\pi$. We then construct the matrix $\boldsymbol{T}$ and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(30)} = 9.77306136381891632828 \cdot 10^{-16}. \tag{4.25}$$

The eigenpolynomial $V^{(30)}(e^{i\pi t})$ is shown in Figs. 2 and 3. Locations of the zeros on the unit circle are displayed in Fig. 4. We then use the quadrature formula corresponding to this eigenvalue and tabulate the weights in Table I. Note that the weights for nodes inside the interval $[-1/6, 1/6]$ are positive and those for nodes outside this interval are negative and roughly of the size of $\lambda^{(30)}$.

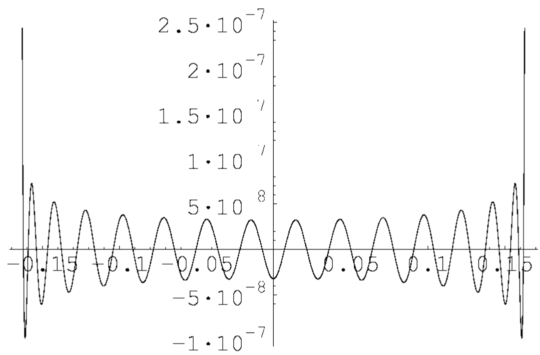

FIG. 3. The same function as in Fig. 2 on the interval $[-1/6, 1/6]$.

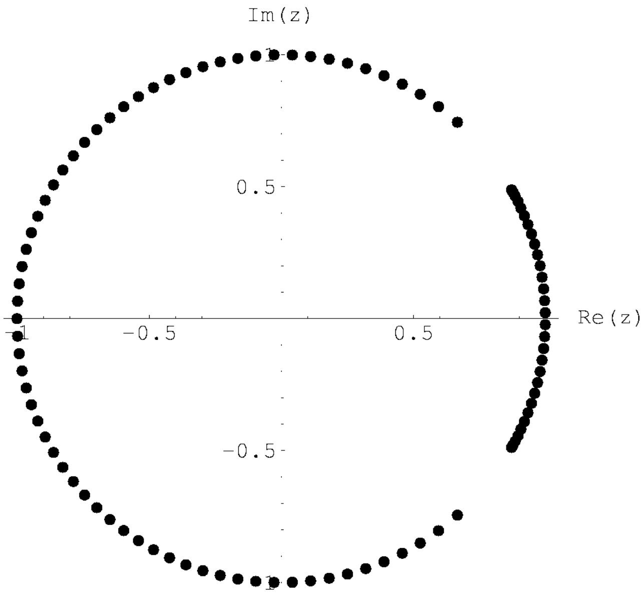

FIG. 4. Location of the zeros on the unit circle for the eigenpolynomial $V^{(30)}$ in Example 1.

TABLE I

Table of Weights for the Quadrature Formula with $\lambda^{(30)}$ in Example 1

#	Weights	#	Weights
1	$-1.0328 \cdot 10^{-17}$	50	0.04437549133235668283
2	$-1.0328 \cdot 10^{-17}$	51	0.04419611220330997984
3	$-1.0329 \cdot 10^{-17}$	52	0.04382960375644760677
⋮	⋮	53	0.04325984471286061543
		54	0.04246105337417774134
33	$-1.3518 \cdot 10^{-17}$	55	0.04139574827622469674
34	$-1.6030 \cdot 10^{-17}$	56	0.04001188663952018400
35	0.00580295532842819966	57	0.03823923547752508920
36	0.01310603337477264417	58	0.03598544514201341779
37	0.01959211245475268191	59	0.03313334531810570720
38	0.02506789313597245367	60	0.02954323947353217723
39	0.02954323947353217723	61	0.02506789313597245367
40	0.03313334531810570720	62	0.01959211245475268191
41	0.03598544514201341779	63	0.01310603337477264417
42	0.03823923547752508920	64	0.00580295532842819966
43	0.04001188663952018400	65	$-1.6030 \cdot 10^{-17}$
44	0.04139574827622469674	66	$-1.3518 \cdot 10^{-17}$
45	0.04246105337417774134	⋮	⋮
46	0.04325984471286061543		
47	0.04382960375644760677	96	$-1.0329 \cdot 10^{-17}$
48	0.04419611220330997984	97	$-1.0328 \cdot 10^{-17}$
49	0.04437549133235668283		

Note. The weight #1 corresponds to the node $\gamma_1 = -1$ (see Fig. 4).

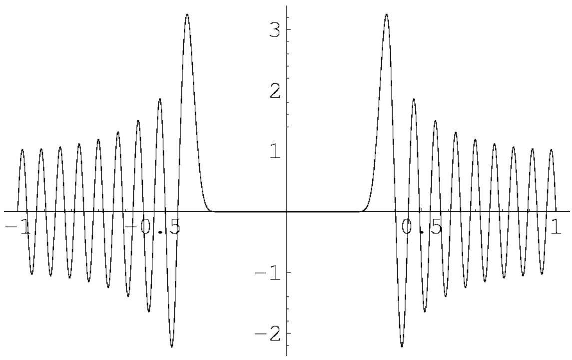

FIG. 5. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 2.

EXAMPLE 2. We consider the weight

$$w(t) = \begin{cases} |t|/a, & t \in [-a, a],\ a \leq 1/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.26}$$

In this example we have selected $N = 61$, $a = 1/4$, $c = 15\pi$. We then construct the matrix $\boldsymbol{T}$ and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 1.1159893168852370628 0 \cdot 10^{-14}. \tag{4.27}$$

The eigenpolynomial $V^{(28)}(e^{i\pi t})$ is shown in Figs. 5 and 6.

EXAMPLE 3. We consider a nonsymmetric weight

$$w(t) = \begin{cases} 1 + t/a, & t \in [-a, a],\ a \leq 1/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.28}$$

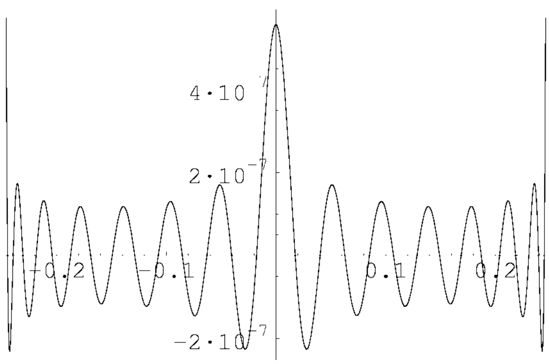

FIG. 6. The same function of Fig. 5 on the interval $[-1/4, 1/4]$.

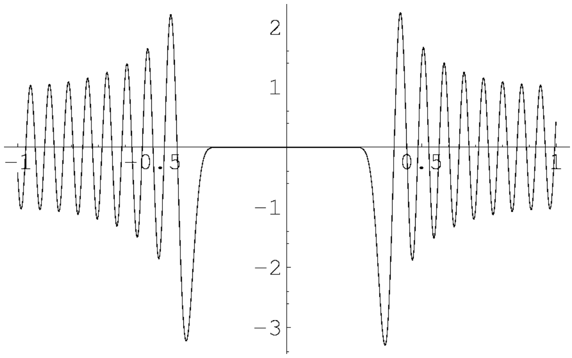

FIG. 7. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 3.

In this example we have selected $N = 61$, $a = 1/4$, $c = 15\pi$. We then construct the matrix $\boldsymbol{T}$ and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 4.68165338379692121389 \cdot 10^{-15}. \tag{4.29}$$

The eigenpolynomial $V^{(28)}(e^{i\pi t})$ is shown in Figs. 7 and 8. Although we do not have a proof at the moment, it appears that there is a class of weights for which eigenpolynomials corresponding to small eigenvalues mimic the behavior of the discrete PSWF with respect to locations of zeros. In Example 3 we know that all zeros are on the unit circle due to Theorems 4.2 and 4.3.

In Table II we illustrate the performance of quadratures for different bandlimits c. This table should be compared with [29, Table 1]. The performance of both sets of quadratures is very similar. Yet these quadratures are quite different as can be seen by comparing Table III with [29, Table 5]. Although the accuracy is almost identical, approximately 10^{-7}, the positions of nodes and weights differ by approximately 10^{-3}–10^{-4}.

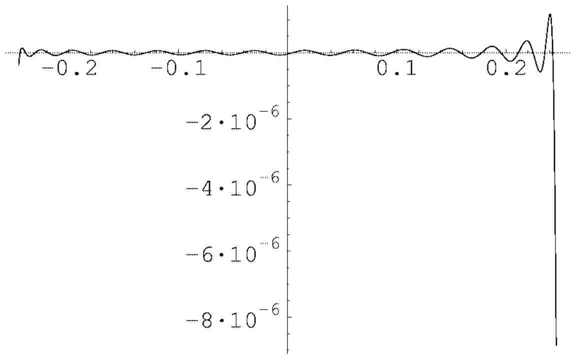

FIG. 8. The same function of Fig. 7 on the interval $[-1/4, 1/4]$.

TABLE II
Quadrature Performance for Varying Bandlimits

c	# of nodes	Maximum errors
20	13	$1.2 \cdot 10^{-7}$
50	24	$1.1 \cdot 10^{-7}$
100	41	$1.6 \cdot 10^{-7}$
200	74	$1.8 \cdot 10^{-7}$
500	171	$1.4 \cdot 10^{-7}$
1000	331	$2.4 \cdot 10^{-7}$
2000	651	$1.2 \cdot 10^{-7}$
4000	1288	$3.7 \cdot 10^{-7}$

5. A NEW ALGORITHM FOR CARATHÉODORY REPRESENTATION

5.1. Algorithm 2

We now describe an algorithm for computing quadratures via a Carathéodory-type approach based on Theorem 4.1. It is easy to see that, although there are similarities with

TABLE III
Quadrature Nodes for Exponentials with Maximum Bandlimit $c = 50$

Node	Weight
−0.99041609489889	2.42209284787E-02
−0.95238829377394	5.04152570050E-02
−0.89243677566550	6.82109308489E-02
−0.81807124037876	7.96841731718E-02
−0.73438712699465	8.71710040243E-02
−0.64454148960251	9.22000859355E-02
−0.55050369342444	9.56668891250E-02
−0.45355265507507	9.80920675810E-02
−0.35456254990620	9.97843340729E-02
−0.25416536256280	1.00930070892E-01
−0.15284664158549	1.01641529848E-01
−0.05100535080412	1.01982696564E-01
0.05100535080412	1.01982696564E-01
0.15284664158549	1.01641529848E-01
0.25416536256280	1.00930070892E-01
0.35456254990620	9.97843340729E-02
0.45355265507507	9.80920675810E-02
0.55050369342444	9.56668891250E-02
0.64454148960251	9.22000859355E-02
0.73438712699465	8.71710040243E-02
0.81807124037876	7.96841731718E-02
0.89243677566550	6.82109308489E-02
0.95238829377394	5.04152570050E-02
0.99041609489889	2.42209284787E-02

Note. The maximum error is $\approx 1.1 \cdot 10^{-7}$.

Pisarenko's method, the corresponding algorithms are substantially different. We plan to address implications for signal processing in a separate paper.

(1) Given t_k, the trigonometric moments of a measure, we construct the $(N+1) \times (N+1)$ Toeplitz matrix $\boldsymbol{T}_N$ with elements $(\boldsymbol{T}_N)_{kj} = t_{j-k}$. This matrix is positive definite and has a large number of small eigenvalues.

(2) For a given accuracy ϵ, we compute the inverse of the Toeplitz matrix $\boldsymbol{T}_N - \epsilon \boldsymbol{I}$. For a self-adjoint Toeplitz matrix, it is sufficient to solve $(\boldsymbol{T}_N - \epsilon \boldsymbol{I})\boldsymbol{x_0} = \boldsymbol{e_0}$, where $\boldsymbol{e_0} = (1, 0, \ldots, 0)^t$. After $\boldsymbol{x_0}$ is found, we use the Gohberg–Semencul representation of the inverse of the Toeplitz matrix [7] (see also [6] for a modern perspective) in order to apply it to a vector. If ϵ is too close to an eigenvalue of $\boldsymbol{T}_N$, it might be necessary to slightly modify the value of ϵ and repeat this step.

This step requires $\mathrm{O}(N^2)$ operations if we use the Levinson algorithm. However, we know how to build a stable $\mathrm{O}(N(\log N)^2)$ algorithm for this purpose, which we will present elsewhere.

(3) Using the power method for $(\boldsymbol{T}_N - \epsilon \boldsymbol{I})^{-1}$, we find an eigenvalue $\lambda^{(q)}$ close to ϵ and the corresponding eigenvector $\boldsymbol{q}$. This step requires $\mathrm{O}(N \log N)$ operations due to the Gohberg–Semencul representation of the inverse.

(4) Next, compute all zeros on the unit circle of the eigenpolynomial corresponding to the eigenvector $\boldsymbol{q}$.

This requires $\mathrm{O}(N \log N)$ operations since we use the unequally spaced fast Fourier transform [1, 5] to evaluate the trigonometric polynomial on the unit circle. We pick out the zeros within the support of the measure and denote their number by M.

(5) Using the algorithm described below, we find the weights $\boldsymbol{\rho}$ by solving the Vandermonde system for all nodes (including those outside the support). This algorithm takes $\mathrm{O}(N \log N)$ operations.

5.2. Solving Vandermonde Systems Using Polynomial Evaluation

To obtain the weights in step (5) of Algorithm 5.1, we need to solve an $M \times M$ Vandermonde system with nodes on the unit circle. In this section we discuss an algorithm to obtain the solution by evaluating certain polynomials on the Vandermonde matrix nodes. The algorithm can be derived from more general results [9, 16, 21]. Here we give a simpler presentation adapted to our particular application. Note that general algorithms to solve Vandermonde systems are unstable, unless there is a particular arrangement of the nodes.

Let $\{\gamma_1, \ldots, \gamma_M\}$ be distinct complex numbers and define

$$\boldsymbol{V} = \begin{bmatrix} 1 & \ldots & 1 \\ \gamma_1 & & \gamma_M \\ \vdots & & \vdots \\ \gamma_1^{M-1} & \ldots & \gamma_M^{M-1} \end{bmatrix} \in \mathbb{C}^{M \times M}.$$

Since the nodes $\{\gamma_r\}$ are distinct, $\det \boldsymbol{V} \neq 0$ and thus, given $\boldsymbol{b} = (b_0, \ldots, b_{M-1})^t$, there is a unique $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_M)^t$ such that

$$\boldsymbol{V}\boldsymbol{\rho} = \boldsymbol{b}. \tag{5.1}$$

If we define

$$Q(z) = \prod_{k=1}^{M}(z-\gamma_k) = \sum_{k=0}^{M} q_k z^k, \tag{5.2}$$

then, for any polynomial P of degree at most $M-1$,

$$\frac{P(z)}{Q(z)} = \sum_{r=1}^{M} \frac{P(\gamma_r)}{Q'(\gamma_r)(z-\gamma_r)}.$$

Thus, for $|z| < \min|\gamma_r|^{-1}$,

$$\frac{z^{M-1}}{z^M}\frac{P(z^{-1})}{Q(z^{-1})} = \sum_{r=1}^{M}\frac{P(\gamma_r)}{Q'(\gamma_r)}\sum_{k=0}^{+\infty}\gamma_r^k z^k = \sum_{k=0}^{+\infty}\left(\sum_{r=1}^{M}\frac{P(\gamma_r)}{Q'(\gamma_r)}\gamma_r^k\right)z^k. \tag{5.3}$$

Now choose P to be the unique polynomial with $P(\gamma_r) = \rho_r\, Q'(\gamma_r)$ for $1 \le r \le M$, and let $B(z) = \sum_{k=0}^{M-1} b_k z^k$. Substituting in (5.3),

$$\frac{z^{M-1}P(z^{-1})}{z^M Q(z^{-1})} = \sum_{k=0}^{M-1}\underbrace{\sum_{r=1}^{M}\rho(r)\gamma_r^k}_{b_k}\, z^k + \text{higher powers} = B(z) + z^M(\cdots).$$

If we denote

$$\tilde{P}(z) = z^{M-1}P(z^{-1}), \qquad \tilde{Q}(z) = z^M Q(z^{-1}),$$

then

$$\tilde{P}(z) = \tilde{Q}(z)B(z) + z^M\tilde{Q}(z)(\cdots), \tag{5.4}$$

and thus the coefficients of $\tilde{P}$ correspond to the first M coefficients of $\tilde{Q}(z)B(z)$.

As a result, we have the following algorithm.

5.3. Algorithm to Solve Vandermonde Systems

(1) Given $\{\gamma_k\}$, compute $\boldsymbol{q} = (q_0, \ldots, q_M)$.

(2) Given $\boldsymbol{b}$ and $\boldsymbol{q}$, compute

$$\tilde{p}(s) = \sum_{l=0}^{s}\tilde{q}(s-l)b(l)$$

for $0 \le s \le M-1$.

(3) Compute $\boldsymbol{\rho}$ as

$$\rho_r = \frac{P(\gamma_r)}{Q'(\gamma_r)} = -\frac{\gamma_r\tilde{P}(1/\gamma_r)}{\tilde{Q}'(1/\gamma_r)}$$

for $1 \le r \le M$.

This algorithm is equivalent to the following factorization of the inverse of the Vandermonde matrix in terms of a diagonal matrix, its transpose $\boldsymbol{V}^t$, and a triangular Hankel matrix,

$$\boldsymbol{V}^{-1} = \begin{bmatrix} \frac{1}{Q'(\gamma_1)} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{1}{Q'(\gamma_M)} \end{bmatrix} \boldsymbol{V}^t \begin{bmatrix} q_1 & q_2 & \cdots & q_M \\ q_2 & \cdots & q_M & 0 \\ \vdots & & & \vdots \\ & & \cdots & 0 \\ q_M & \cdots & 0 & 0 \end{bmatrix}. \tag{5.5}$$

This description is a particular case of the inversion formulae for Löwner–Vandermonde [21] or close to Vandermonde matrices [9, Corollary 2.1, p. 157]. We can state those results as (see [21, p. 548])

$$\boldsymbol{V}^{-1} = \begin{bmatrix} x_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & x_M \end{bmatrix} \boldsymbol{V}^t \begin{bmatrix} -y_2 & -y_3 & \cdots & 1 \\ -y_3 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ & & \cdots & 0 \\ 1 & \cdots & 0 & 0 \end{bmatrix},$$

where the vectors $\boldsymbol{x} = (x_1, \ldots, x_M)^t$ and $\boldsymbol{y} = (y_1, \ldots, y_M)^t$ are solutions of

$$\boldsymbol{V}\boldsymbol{x} = (0, \ldots, 1)^t \qquad \text{and} \qquad \boldsymbol{V}^t \boldsymbol{y} = [\gamma_r^M]_{r=1}^M.$$

Since γ_r are the roots of $Q(z)$, we can take $\boldsymbol{y} = -(q_0, \ldots, q_{M-1})^t$, and if $B(z) = z^M$ in (5.4), then $P(z) = 1$ and $\boldsymbol{x} = (1/Q'(\gamma_1), \ldots, 1/Q'(\gamma_M))^t$.

Remark 5.1. For Algorithm 5.1, we first obtained the eigenvector $\boldsymbol{q}$ corresponding to an eigenvalue close to ϵ. Thus, step (1) of the Vandermonde algorithm is already accomplished and step (2) can be performed using the FFT. Furthermore, the nodes γ_k belong to the unit circle and, via the unequally spaced fast Fourier transform, we have a fast algorithm to obtain the weights.

Remark 5.2. As an example, we use this approach to derive the solution of the Vandermonde system with nodes at $\gamma_r = \mathrm{e}^{\mathrm{i}2\pi(r-1)/M}$, $1 \le r \le M$. In this case, $Q(z) = 1 - z^M$ and $\tilde{P}(z) = z^{M-1}P(z^{-1}) = \tilde{Q}(z)B(z) + z^M(\cdots) = -B(z) + z^M(\cdots)$. We conclude $P = -\tilde{B}(z)$ and thus

$$\rho_r = \frac{P(\gamma_r)}{Q'(\gamma_r)} = \frac{-\gamma_r^{M-1}B(\overline{\gamma_r})}{-M\gamma_r^{M-1}} = \frac{1}{M}\sum_{k=0}^{M-1} b_k e^{-\mathrm{i}2\pi rk/M}.$$

As expected, we obtained the inverse of the discrete Fourier transform matrix.

6. APPROXIMATION OF BANDLIMITED FUNCTIONS

Let us consider the problem stated in (1.3)–(1.5); that is, given

$$u(x) = \int_{-1}^{1} w(\tau)\mathrm{e}^{\mathrm{i}c\tau x}\, d\tau, \tag{6.1}$$

construct the function $\tilde{u}(x)$ in (1.4) such that (1.5) holds. We show in this section that the approximation obtained with exponential sums holds in any subinterval of $(-1, 1)$. In fact, in Theorem 6.1, we prove the existence of such an approximation even though our proof does not provide a practical method to obtain the nodes and weights in the exponential sum. In practice, we obtain them using Algorithm 5.1.

We assume that we have access to values of $u(x)$ for x uniformly sampled. We select the sampling rate to be at least twice the Nyquist sampling rate for $u(x)$. We have observed that this is the minimal rate for which our method works properly.

In fact, let us discretize $u(x)$ at nodes $x_k = k/N$, for $|k| \leq N$, and pick N such that $N \geq 2c/\pi$. The value of N determines our sampling rate. The resulting values are

$$u_k = u(x_k) = \int_{-1}^{1} w(\tau)\mathrm{e}^{\mathrm{i}c\tau(k/N)} d\tau. \tag{6.2}$$

Defining $\nu = c/\pi N$, then $\nu \leq 1/2$, and by changing variables $t = \nu\tau$,

$$u_k = \int_{-\nu}^{\nu} \sigma(t)\mathrm{e}^{\mathrm{i}\pi tk}\, dt, \qquad \text{for } |k| \leq N, \tag{6.3}$$

are the trigonometric moments of a new weight $\sigma(t) = \frac{1}{\nu} w(t/\nu)$ supported in $(-\nu, \nu) \subseteq [-1/2, 1/2]$.

Now, assume we can approximate $\int_{-1}^{1} \sigma(t)\mathrm{e}^{\mathrm{i}\pi ty}\, dt$ by $\sum_{j=1}^{N} w_j \mathrm{e}^{\mathrm{i}\pi\theta_j y}$ for $|y| \leq N$. Then, since

$$u(x) = \int_{-1}^{1} w(\tau)\mathrm{e}^{\mathrm{i}c\tau x}\, d\tau = \int_{-\nu}^{\nu} \sigma(t)\mathrm{e}^{\mathrm{i}\pi tNx}\, dt,$$

we can approximate $u(x)$ for $|x| \leq 1$.

Indeed, we now show that, for any d, $0 < d < 1$, we can approximate $u(x)$ for $|x| \leq d$.

THEOREM 6.1. *Let σ be a weight supported in $[-\nu, \nu]$, $0 < \nu \leq 1/2$, and let ϵ and d be positive numbers with $d < 1$. Then, for N sufficiently large, there exist real constants $\{w_1, \ldots, w_N\}$ and $\{\theta_1, \ldots, \theta_N\}$, with $w_j > 0$ and $|\theta_j| < \nu$, such that*

$$\left| \int_{-1}^{1} \sigma(t)\mathrm{e}^{\mathrm{i}\pi ty}\, dt - \sum_{j=1}^{N} w_j \mathrm{e}^{\mathrm{i}\pi\theta_j y} \right| < \epsilon, \qquad \textit{for } |y| \leq dN + 1. \tag{6.4}$$

For the proof, we will use the fact that exponential functions can be well approximated by splines interpolating them at integer values.

For fixed $t \in [-\pi, \pi]$ and positive integer m, consider the exponential Euler spline of order $2m - 1$,

$$S_{2m-1}(x, \mathrm{e}^{\mathrm{i}t}) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}tk} L_{2m-1}(x - k), \tag{6.5}$$

where $L_n(x)$ is the fundamental cardinal spline of order n, $L_n(r) = \delta_{r0}$, for all $r \in \mathbb{Z}$.

We will use the following properties (see [22, pp. 29, 30, and 35]) valid for all $x, t \in \mathbb{R}$,

$$\left|S_{2m-1}(x, \mathrm{e}^{\mathrm{i}t})\right| \leq 1, \tag{6.6}$$

$$\left|\mathrm{e}^{\mathrm{i}\pi tx} - S_{2m-1}(x, \mathrm{e}^{\mathrm{i}\pi t})\right| < 3|t|^{2m}, \tag{6.7}$$

$$|L_{2m-1}(x)| \leq d_m\, \mathrm{e}^{-\alpha_m|x|}, \tag{6.8}$$

for positive constants d_m and α_m.

Proof of Theorem 6.1. Let

$$u(y) = \int_{-1}^{1} \sigma(t) e^{i\pi t y} \, dt,$$

and, for each m, define the spline of order $2m - 1$ interpolating $u(y)$ at the integers,

$$a(y) = \sum_{k} u(k) L_{2m-1}(y-k) = \int_{-1}^{1} \sigma(t) S_{2m-1}(y, e^{i\pi t}) \, dt.$$

By (6.7),

$$|u(y) - a(y)| \leq 3 \int_{-\nu}^{\nu} \sigma(t) |t|^{2m} \, dt \leq 3\nu^{2m} \|\sigma\|_1,$$

where $\|\sigma\|_1 = \int_{-1}^{1} \sigma(t)\, dt$. We choose m such that $3\nu^{2m}\|\sigma\|_1 < \epsilon/4$.

On the other hand, for each N, Theorem 3.7 allows us to represent the moments $u(k)$, $|k| \leq N$,

$$u(k) = \int_{-1}^{1} \sigma(t) e^{i\pi k t} \, dt = \sum_{j=1}^{N} w_j e^{i\pi \theta_j k} + w_0 (-1)^k, \tag{6.9}$$

where

$$w_0 \leq \frac{4\|\sigma\|_1}{2 + (2+\sqrt{3})^N + (2-\sqrt{3})^N}. \tag{6.10}$$

Let

$$\tilde{u}(y) = \sum_{j=1}^{N} w_j e^{i\pi \theta_j y};$$

then $u(k) = \tilde{u}(k) + w_0(-1)^k$ for $|k| \leq N$, and defining

$$\tilde{a}(y) = \sum_{k} \tilde{u}(k) L_{2m-1}(y-k) = \sum_{j=1}^{N} w_j S_{2m-1}(y, e^{i\pi\theta_j}),$$

(6.7) gives the estimate

$$|\tilde{u}(y) - \tilde{a}(y)| \leq 3 \sum_{j=1}^{N} w_j |\theta_j|^{2m} \leq 3\nu^{2m}(u(0) - w_0) \leq 3\nu^{2m} \|\sigma\|_1 < \frac{\epsilon}{4}.$$

We have shown that $u(y)$ is close to $a(y)$ and $\tilde{u}(y)$ is close to $\tilde{a}(y)$. To finish the proof, we need to show that $|a(y) - \tilde{a}(y)| < \epsilon/2$, for $|y| \leq dN + 1$. Now,

$$\begin{aligned} a(y) - \tilde{a}(y) &= \sum_{|k|\leq N} w_0(-1)^k L_{2m-1}(y-k) + \sum_{|k|>N} (u(k) - \tilde{u}(k)) L_{2m-1}(y-k) \\ &= w_0 S_{2m-1}(y, e^{i\pi}) + \sum_{|k|>N} (u(k) - \tilde{u}(k) - w_0(-1)^k) L_{2m-1}(y-k) \end{aligned}$$

and

$$\begin{aligned} |u(k) - \tilde{u}(k) - w_0(-1)^k| &\leq |u(k)| + |\tilde{u}(k)| + w_0 \leq \sum_{j=0}^{N} w_j + \sum_{j=1}^{N} w_j + w_0 \\ &\leq 2u(0) = 2\|\sigma\|_1, \end{aligned}$$

where we used (6.9).

Using (6.6) and (6.8),

$$|a(y) - \tilde{a}(y)| \leq w_0 + 2\|\sigma\|_1\, d_m \sum_{|k|>N} \mathrm{e}^{-\alpha_m |y-k|}.$$

Hence, for large N, we can estimate w_0 using (6.10), and for the last sum, when $|y| \leq dN + 1$,

$$\sum_{|k|>N} \mathrm{e}^{-\alpha_m|y-k|} = \sum_{k=N+1}^{\infty} (\mathrm{e}^{-\alpha_m(k-y)} + \mathrm{e}^{-\alpha_m(k+y)}) = \frac{\mathrm{e}^{\alpha_m y} + \mathrm{e}^{-\alpha_m y}}{1-\mathrm{e}^{-\alpha_m}} \mathrm{e}^{-\alpha_m(N+1)}$$
$$\leq \frac{2}{1-\mathrm{e}^{-\alpha_m}} \mathrm{e}^{-\alpha_m(1-d)N}. \blacksquare$$

Remark 6.1. In the proof of Theorem 6.1, we used Theorem 3.7 to represent the moments (6.3) as

$$\int_{-\nu}^{\nu} \sigma(t)\mathrm{e}^{\mathrm{i}\pi tk}\, dt = \sum_{j=1}^{N} w_j \mathrm{e}^{\mathrm{i}\pi\theta_j k} + w_0(-1)^k,$$

where w_0 decreases exponentially with N. The approximation in (6.4) is obtained using these w_j and θ_j. Nevertheless, in practice, we use instead the weights and nodes from the quadratures in (4.12). We also note that, as the proof of the theorem indicates, for (1.5) to hold on most of the interval $(-1, 1)$, we should appropriately oversample the function u in (1.3).

7. APPROXIMATION OF INTEGRALS BY LINEAR COMBINATIONS OF EXPONENTIALS

In this section we show that by solving the problem (1.3)–(1.5) for $x \in [-1, 1]$, we also find quadratures for functions that can be considered as a linear combination of exponentials with bandlimit c. We provide two examples with Bessel functions and with PSWF.

PROPOSITION 7.1. *For $x \in [-1, 1]$, let us consider*

$$v(x) = \int_0^1 w(\tau) J_{2n}(cx\tau)\, d\tau, \tag{7.1}$$

where $w \geq 0$ is a weight, J_{2n} is the Bessel function of order $2n$, $n \geq 0$, and c is a positive real constant. Then we have

$$|v(x) - \tilde{v}(x)| \leq \frac{2}{\pi}\epsilon, \tag{7.2}$$

where

$$\tilde{v}(x) = \sum_{k=1}^{M} w_k J_{2n}(cx\theta_k), \tag{7.3}$$

and the nodes θ_k and the weights w_k are as in (1.4) *but for the weight $\tilde{w}$ defined as $\tilde{w}(\tau) = w(\tau)/2$ for $0 \leq \tau \leq 1$ and $\tilde{w}(\tau) = w(-\tau)/2$, $-1 \leq \tau \leq 0$.*

Since J_{2n} is an even function, we have

$$v(x) = \int_{-1}^{1} \tilde{w}(\tau) J_{2n}(cx\tau)\, d\tau. \tag{7.4}$$

Using

$$J_{2n}(\xi) = \frac{(-1)^n}{\pi} \int_{-1}^{1} \frac{T_{2n}(y)}{\sqrt{1-y^2}} \mathrm{e}^{\mathrm{i}y\xi}\, dy, \tag{7.5}$$

we obtain

$$v(x) - \tilde{v}(x) = \frac{(-1)^n}{\pi} \int_{-1}^{1} \frac{T_{2n}(y)}{\sqrt{1-y^2}} \left[\int_{-1}^{1} \tilde{w}(\tau) \mathrm{e}^{\mathrm{i}cxy\tau}\, d\tau - \sum_{k=1}^{M} w_k \mathrm{e}^{\mathrm{i}cxy\theta_k} \right] dy. \tag{7.6}$$

Since $|y| \leq 1$, we have (by selecting nodes and weights as in (1.4))

$$\left| \int_{-1}^{1} \tilde{w}(\tau) \mathrm{e}^{\mathrm{i}cxy\tau}\, d\tau - \sum_{k=1}^{M} w_k \mathrm{e}^{\mathrm{i}cxy\theta_k} \right| \leq \epsilon \tag{7.7}$$

for $x \in [-1, 1]$. Thus, we obtain

$$\begin{aligned} |v(x) - \tilde{v}(x)| &\leq \frac{\epsilon}{\pi} \int_{-1}^{1} \frac{|T_{2n}(y)|}{\sqrt{1-y^2}}\, dy \\ &= \frac{\epsilon}{\pi} \int_{0}^{\pi} |\cos(2nx)|\, dx = \begin{cases} \frac{2}{\pi}\epsilon, & \text{if } n \neq 0, \\ \epsilon, & \text{if } n = 0. \end{cases} \end{aligned} \tag{7.8}$$

A similar result holds for the PSWF, which are defined as the eigenfunctions of the operator

$$F_c(\phi)(x) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}cxt} \phi(t)\, dt, \tag{7.9}$$

where c is a positive real constant (bandlimit) and F_c: $L^2[-1, 1] \to L^2[-1, 1]$. These bandlimited functions satisfy

$$\lambda_j \psi_j(x) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}cxt} \psi_j(t)\, dt, \tag{7.10}$$

where the eigenvalues λ_j, $j = 0, 1, \ldots,$ are all nonzero and simple, and are arranged so that $|\lambda_{j-1}| > |\lambda_j|$, $j = 1, 2, \ldots.$

PROPOSITION 7.2. *For all nonnegative integers j, let us consider integrals*

$$v_j = \int_{-1}^{1} w(\tau) \psi_j(\tau)\, d\tau, \tag{7.11}$$

where $w \geq 0$ is a weight and ψ_j is the PSWF corresponding to the bandlimit $c > 0$. Then

$$|v_j - \tilde{v}_j| \leq \frac{\sqrt{2}}{|\lambda_j|} \epsilon, \tag{7.12}$$

where

$$\tilde{v}_j = \sum_{k=1}^{M} w_k \psi_j(\theta_k), \tag{7.13}$$

and the nodes θ_k *and the weights* w_k *are the same as in* (1.4).

For large c, the spectrum of F_c can be divided into three groups. The first group contains approximately $2c/\pi$ eigenvalues with absolute value very close to 1. They are followed by order $\log c$ eigenvalues whose absolute values make an exponentially fast transition from 1 to 0. The third group consists of exponentially decaying eigenvalues that are very close to zero. For precise statements see [14, 24, 25, 29].

Therefore, it follows from (7.12) that, for the first $\approx 2c/\pi$ eigenfunctions, the integrals in (7.11) are well approximated by the quadratures in (7.13). To prove (7.12), use (7.10), to write

$$v_j - \tilde{v}_j = \frac{1}{\lambda_j} \int_{-1}^{1} \left(\int_{-1}^{1} w(\tau) e^{ic\tau t}\, d\tau - \sum_{k=1}^{M} w_k e^{ic\theta_k t} \right) \psi_j(t)\, dt. \tag{7.14}$$

Since $|t| \leq 1$, we have

$$\left| \int_{-1}^{1} w(\tau) e^{ic\tau t}\, d\tau - \sum_{k=1}^{M} w_k e^{ic\theta_k t} \right| \leq \epsilon, \tag{7.15}$$

and $\|\psi_j\|_2 = 1$ implies $\int_{-1}^{1} |\psi_j(t)|\, dt \leq \sqrt{2}$. ∎

Remark 7.1. If the weight function w is chosen as in (8.39), then the eigenpolynomials are the discrete PSWF [23] (see Example 1 in Section 4.3). In this case, the nodes $\{\theta_k\}$ are zeros of a discrete PSWF corresponding to a small eigenvalue. Therefore, these nodes are Gaussian nodes for PSWF (appropriately scaled to the interval $[-1, 1]$) as stated in Proposition 7.2.

8. INTERPOLATING BASES FOR BANDLIMITED FUNCTIONS

In this section we construct bases for bandlimited functions with bandlimit $c > 0$ using exponentials $\{e^{ict_l x}\}_{l=1}^{M}$, where $\{t_l\}$ are some quadrature nodes with $|t_l| < 1$. In particular, we derive interpolating bases as linear combinations of such exponentials.

We start by constructing, for some $\epsilon > 0$ and bandlimit $2c > 0$, the nodes $|t_l| < 1$ and the weights $w_l > 0$, $l = 1, \ldots, M$, where $M = M(c, \epsilon)$, such that for all $x \in [-1, 1]$,

$$\left| \int_{-1}^{1} e^{2icxt}\, dt - \sum_{l=1}^{M} w_l e^{2icxt_l} \right| < \epsilon^2, \tag{8.1}$$

where, for each l, there exists l' such that $t_{l'} = -t_l$ and $w_{l'} = w_l$.

Since $\int_{-1}^{1} e^{2icxt}\, dt = \sin(2cx)/cx$, we have from (8.1)

$$\left| \frac{\sin c(x-t)}{c(x-t)} - \frac{1}{2} \sum_{l=1}^{M} w_l e^{ic(x-t)t_l} \right| < \frac{\epsilon^2}{2}, \tag{8.2}$$

where $|x|, |t| \leq 1$.

In considering bandlimited functions we will use the PSWF (see [15, 24], and a more recent paper [29]). The PSWF are real eigenfunctions of the operator F_c in (7.9) with eigenvalues λ_j, $j = 0, 1, \ldots$, such that $|\lambda_0| > |\lambda_1| > \cdots > 0$. They are also eigenfunctions of the operator $Q_c = (c/2\pi) F_c^* F_c$, namely,

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sin c(x-t)}{(x-t)} \psi_j(t)\, dt = \mu_j \psi_j(x), \tag{8.3}$$

with eigenvalues

$$\mu_j = \frac{c}{2\pi} |\lambda_j|^2, \qquad j = 0, 1, \ldots. \tag{8.4}$$

We prove

THEOREM 8.1. *For x in $[-1, 1]$ and for any $|b| \le c$ and $\epsilon > 0$, there exist coefficients $\{\alpha_l\}_{l=1}^M$ and constants A_1, A_2 such that*

$$\left\| e^{ibx} - \sum_{l=1}^{M} \alpha_l e^{ict_l x} \right\|_\infty \le A_1 \epsilon \tag{8.5}$$

and

$$\left\| e^{ibx} - \sum_{l=1}^{M} \alpha_l e^{ict_l x} \right\|_2 \le A_2 \epsilon, \tag{8.6}$$

where the nodes $|t_l| < 1$, $l = 1, \ldots, M$, are the same as in (8.1) *and do not depend on b.*

In other words, for a given precision ϵ, the functions $\{e^{ict_l x}\}_{l=1}^M$ form an approximate basis for bandlimited functions.

The proof of this theorem uses the fact that the PSWF are uniformly bounded

$$\|\psi_j\|_\infty \le C_\infty, \qquad j = 0, 1, \ldots, \tag{8.7}$$

where C_∞ does not depend on j. Through direct numerical examination, it is not difficult to verify that C_∞ is a fairly small constant that weakly depends on the bandlimit c. However, we are not aware of a proof that provides a tight bound in (8.7). A proof that C_∞ exists can be constructed using the fact that, for $j \gg c$, the functions ψ_j approach the Legendre polynomials. In order to obtain appropriate estimates, one can use the recurrence relations derived in [29].

Proof of Theorem 8.1. Let us start by expanding e^{ibx} into the basis $\{\psi_j\}_{j=0}^\infty$ corresponding to the bandlimit c. We have

$$e^{ibx} = e^{ic(b/c)x} = \sum_{j=0}^{\infty} \lambda_j \psi_j(b/c) \psi_j(x), \tag{8.8}$$

where $|b/c| \leq 1$, and by (8.7)

$$\left| e^{ibx} - \sum_{j=0}^{M-1} \lambda_j \psi_j(b/c) \psi_j(x) \right| \leq \sum_{j=M}^{\infty} |\lambda_j| \, |\psi_j(b/c)| \, |\psi_j(x)| \leq C_\infty \sum_{j=M}^{\infty} |\lambda_j| \, |\psi_j(x)|. \tag{8.9}$$

Thus we obtain

$$\left\| e^{ibx} - \sum_{j=0}^{M-1} \lambda_j \psi_j(b/c) \psi_j(x) \right\|_\infty \leq C_\infty^2 \sum_{j=M}^{\infty} |\lambda_j|, \tag{8.10}$$

and, using (8.8) and the orthonormality of ψ_j on $[-1, 1]$,

$$\left\| e^{ibx} - \sum_{j=0}^{M-1} \lambda_j \psi_j(b/c) \psi_j(x) \right\|_2 = \sqrt{\sum_{j=M}^{\infty} |\lambda_j|^2 \, |\psi_j(b/c)|^2} \leq C_\infty \sqrt{\sum_{j=M}^{\infty} |\lambda_j|^2}. \tag{8.11}$$

From (7.10), we have

$$\int_{-1}^{1} e^{-ictt_l} \psi_j(t) \, dt = \bar{\lambda}_j \psi_j(t_l), \tag{8.12}$$

and, using (8.2)–(8.3) and the relationship between μ_j and λ_j in (8.4), we obtain

$$\left| \sum_{l=1}^{M} w_l e^{icxt_l} \bar{\lambda}_j \, \psi_j(t_l) - |\lambda_j|^2 \psi_j(x) \right| \leq \epsilon^2 \int_{-1}^{1} |\psi_j(t)| \, dt \leq \sqrt{2} \epsilon^2. \tag{8.13}$$

Thus, we arrive at

$$\left\| \sum_{l=1}^{M} w_l e^{icxt_l} \, \psi_j(t_l) - \lambda_j \, \psi_j(x) \right\|_\infty \leq \sqrt{2} \frac{\epsilon^2}{|\lambda_j|} \tag{8.14}$$

and

$$\left\| \sum_{l=1}^{M} w_l e^{icxt_l} \psi_j(t_l) - \lambda_j \psi_j(x) \right\|_2 \leq 2 \frac{\epsilon^2}{|\lambda_j|}. \tag{8.15}$$

Combining (8.10) and (8.14), we have

$$\left\| e^{ibx} - \sum_{l=1}^{M} \sum_{j=0}^{M-1} w_l e^{icxt_l} \psi_j(b/c) \psi_j(t_l) \right\|_\infty \leq C_\infty^2 \sum_{j=M}^{\infty} |\lambda_j| + \sqrt{2} \, C_\infty \sum_{j=0}^{M-1} \frac{\epsilon^2}{|\lambda_j|}. \tag{8.16}$$

Similarly, combining (8.11) and (8.15), we obtain

$$\left\| e^{ibx} - \sum_{l=1}^{M} \sum_{j=0}^{M-1} w_l e^{icxt_l} \psi_j(b/c) \psi_j(t_l) \right\|_2 \leq C_\infty \sqrt{\sum_{j=M}^{\infty} |\lambda_j|^2} + 2 \, C_\infty \sum_{j=0}^{M-1} \frac{\epsilon^2}{|\lambda_j|}. \tag{8.17}$$

By setting

$$\alpha_l = w_l \sum_{j=0}^{M-1} \psi_j(b/c)\,\psi_j(t_l), \tag{8.18}$$

and observing that $|\lambda_M| \approx \epsilon$ and that $|\lambda_j| \ll |\lambda_M|$ for $j > M$, we obtain (8.5) and (8.6). ■

We now construct two useful bases as linear combinations of the functions $\{e^{ict_l x}\}_{l=1}^{M}$. First, let us consider the following algebraic eigenvalue problem,

$$\sum_{l=1}^{M} w_l e^{ict_m t_l} \Psi_j(t_l) = \eta_j \Psi_j(t_m), \tag{8.19}$$

where t_l and w_l are the same as in (8.1). By solving (8.19), we find η_j and $\Psi_j(t_l)$. We then consider functions Ψ_j, $j = 1, \ldots, M$, defined for any x as

$$\Psi_j(x) = \frac{1}{\eta_j} \sum_{l=1}^{M} w_l e^{icxt_l} \Psi_j(t_l). \tag{8.20}$$

The functions Ψ_j in (8.20) are linear combinations of the exponentials $\{e^{icxt_l}\}_{l=1}^{n}$. We will show that the functions Ψ_j are nearly orthonormal and we will use them as an approximate basis for weighted bandlimited functions with bandlimit c.

Remark 8.1. The functions Ψ_j mimic the PSWF. However, one has to be careful relating ψ_j and Ψ_j. Since a large number of eigenvalues λ_j satisfy both, $|\lambda_j|$ is close to $\sqrt{2\pi/c}$ and $\eta_j - \lambda_j = O(\epsilon^2)$, then, in solving (8.19), we also obtain a large number of eigenvalues η_j with absolute value close to $\sqrt{2\pi/c}$. Therefore, in spite of the fact that all λ_j are distinct, we may obtain a group of eigenvalues that are identical within the precision of the computation. In this case the functions Ψ_j correspond to linear combinations of the PSWF. In other words, we need to impose additional conditions in (8.19) to maintain a proper correspondence with the PSWF. However, in many applications there is no apparent need to make such an identification since in all cases the resulting functions span the same subspace.

PROPOSITION 8.1. *The functions Ψ_j, $j = 1, \ldots, M$, are nearly orthogonal and satisfy*

$$\left| \int_{-1}^{1} \Psi_j(t)\Psi_{j'}(t)\,dt - \delta_{jj'} \right| < \frac{\epsilon^2 \sum_{k=1}^{M} w_k}{|\eta_j|\,|\eta_{j'}|}. \tag{8.21}$$

Proof. We start by defining $q_l^j = \sqrt{w_l}\,\Psi_j(t_l)$. We substitute in (8.19) to obtain

$$\sum_{l=1}^{M} \sqrt{w_m} e^{ict_m t_l} \sqrt{w_l}\, q_l^j = \eta_j q_m^j. \tag{8.22}$$

The vectors $\{\boldsymbol{q}^j\}$ are eigenvectors of the matrix S, $S_{ml} = \sqrt{w_m} e^{ict_m t_l} \sqrt{w_l}$. If we take into account the symmetry of nodes t_l and weights w_l in (8.1), we obtain that the matrix S is normal, $SS^* = S^*S$, and, in addition, $\bar{S} = S^*$. In Proposition 8.2 we show that, for such matrices, there exists an orthonormal basis of real eigenvectors. Thus, computed via (8.22), we assume q_l^j to be a real orthogonal matrix and then

$$\sum_{j=1}^{M} \sqrt{w_l}\, \Psi_j(t_l)\, \Psi_j(t_m) \sqrt{w_m} = \delta_{lm} \tag{8.23}$$

and

$$\sum_{l=1}^{M} \Psi_j(t_l)\, w_l\, \Psi_{j'}(t_l) = \delta_{jj'}. \tag{8.24}$$

We have

$$\int_{-1}^{1} \Psi_j(t)\Psi_{j'}(t)\, dt = \frac{1}{\eta_j \eta_{j'}} \sum_{l,l'=1}^{M} w_l w_{l'} \Psi_j(t_l)\Psi_{j'}(t_{l'}) \int_{-1}^{1} \mathrm{e}^{ict(t_l+t_{l'})}\, dt \tag{8.25}$$

and, from (8.1), we obtain

$$\begin{aligned} &\left| \int_{-1}^{1} \Psi_j(t)\Psi_{j'}(t)\, dt - \frac{1}{\eta_j \eta_{j'}} \sum_{l,l'=1}^{M} w_l w_{l'} \Psi_j(t_l)\Psi_{j'}(t_{l'}) \sum_{k=1}^{M} w_k \mathrm{e}^{ict_k(t_l+t_{l'})} \right| \\ &\quad \le \frac{\epsilon^2 \sum_{k=1}^{M} w_k}{|\eta_j|\,|\eta_{j'}|}. \end{aligned} \tag{8.26}$$

For the last inequality we also used Schwarz inequality and (8.24) to estimate

$$\begin{aligned} \left| \sum_{l,l'=1}^{M} w_l\, w_{l'}\, \Psi_j(t_l)\, \Psi_{j'}(t_{l'}) \right| &\le \sum_l w_l\, |\Psi_j(t_l)| \sum_l w_l\, |\Psi_{j'}(t_l)| \le \left(\sum_{l=1}^{M} w_l |\Psi_j(t_l)| \right)^2 \\ &\le \left(\sum_{l=1}^{M} w_l \right) \left(\sum_{l=1}^{M} w_l |\Psi_j(t_l)|^2 \right) = \sum_{l=1}^{M} w_l. \end{aligned}$$

To finish, using (8.19) and (8.24), we simplify (8.26) and arrive at (8.21). ■

We still need to prove

PROPOSITION 8.2. *Let S be a normal matrix such that $\bar{S} = S^*$. Then there exists an orthonormal basis of real eigenvectors of S.*

Proof. First, since S is normal, eigenspaces corresponding to different eigenvalues are orthogonal. Thus, it is sufficient to prove the proposition for any eigenspace $E(\lambda) = \{\boldsymbol{x}\colon\ S\boldsymbol{x} = \lambda\boldsymbol{x}\}$. Also, normality of S implies that, for any $\boldsymbol{x} \in E(\lambda)$, we have $\bar{\boldsymbol{x}} \in E(\lambda)$. Indeed, from $S^*\boldsymbol{x} = \bar{\lambda}\boldsymbol{x}$ and $\bar{S} = S^*$, it follows that $S\bar{\boldsymbol{x}} = \lambda\bar{\boldsymbol{x}}$. Consequently, if $\{\boldsymbol{v_k}\}_{k=1}^m$ is a basis of $E(\lambda)$, then $E(\lambda)$ can be spanned by the real and imaginary parts of $\boldsymbol{v_k}$,

$$\boldsymbol{\mathcal{A}} = \{\mathrm{Re}(\boldsymbol{v_k}), \mathrm{Im}(\boldsymbol{v_k})\}_{k=1}^m,$$

where, for any vector $\boldsymbol{x} = (x_1, \ldots, x_M)$, $\mathrm{Re}(\boldsymbol{x}) = (\mathrm{Re}(x_1), \ldots, \mathrm{Re}(x_m))$, and similarly for $\mathrm{Im}(\boldsymbol{x})$. By Gram–Schmidt orthonormalization of the $2m$ (linearly dependent) vectors $\boldsymbol{\mathcal{A}}$, we obtain the desired result. See another proof in [10, Theorem 4.4.7]. ■

Let us now construct interpolating bases as linear combinations of the exponentials $\{e^{icxt_l}\}_{l=1}^n$. We define functions R_k, $k = 1, \ldots, M$, as

$$R_k(x) = \sum_{l=1}^{M} r_{kl} e^{icxt_l}, \tag{8.27}$$

where

$$r_{kl} = \sum_{j=1}^{M} w_k \Psi_j(t_k) \frac{1}{\eta_j} \Psi_j(t_l) w_l = \sum_{j=1}^{M} \sqrt{w_k} q_k^j \frac{1}{\eta_j} q_l^j \sqrt{w_l}. \tag{8.28}$$

By direct evaluation in (8.19) and (8.23), we verify that functions R_k are interpolating,

$$R_k(t_m) = \delta_{km}. \tag{8.29}$$

Let us show that the integration of $R_k(t)e^{iat}$, where $|a| \leq c$, yields a one-point quadrature rule of accuracy $O(\epsilon)$.

PROPOSITION 8.3. *For $|a| \leq c$, let*

$$\Delta_k = \int_{-1}^{1} R_k(t) e^{iat}\, dt - w_k e^{iat_k}. \tag{8.30}$$

Then we have

$$|\Delta_k| \leq \|\Delta\|_2 \leq \sqrt{M} \frac{\max_{k=1,\ldots,M} |w_k|}{\min_{k=1,\ldots,M} |\eta_k|} \epsilon^2, \tag{8.31}$$

where $\|\Delta\|_2 = \sqrt{\sum_{k=1}^{M} |\Delta_k|^2}$.

Proof. Using (8.27) and (8.29),

$$\sum_{l=1}^{M} r_{kl} \sum_{m=1}^{M} w_m e^{ict_m(t_l + a/c)} = \sum_{m=1}^{M} w_m R_k(t_m) e^{iat_m} = w_k e^{iat_k}, \tag{8.32}$$

and, therefore, Δ_k in (8.30) can be written as a matrix-vector multiplication $\Delta_k = \sum_{l=1}^{M} r_{kl} s_l$, where

$$s_l = \int_{-1}^{1} e^{ict(t_l + a/c)}\, dt - \sum_{m=1}^{M} w_m e^{ict_m(t_l + a/c)}. \tag{8.33}$$

The inequality (8.31) is then obtained via the usual l^2-norm estimates, taking into account that the matrices q_k^j and q_l^j in (8.28) are orthogonal and that, for functions e^{iax}, where $|a| \leq c$, (8.1) implies $|s_l| \leq \epsilon^2$. ■

We have observed (via computation) that $\max_{k=1,\ldots,M} |w_k| = O(1)$ and $\min_{k=1,\ldots,M} |\eta_k| = O(\epsilon)$ in (8.31), thus resulting in $\|\Delta\|_2 = O(\epsilon)$. Next we derive a weak estimate showing that the functions R_k are close to being an interpolating basis for band-limited exponentials.

PROPOSITION 8.4. *For every b, $|b| \leq c$, let us consider the function*

$$\Omega_b(t) = \mathrm{e}^{\mathrm{i}bt} - \sum_{k=1}^{M} \mathrm{e}^{\mathrm{i}bt_k} R_k(t). \tag{8.34}$$

Then, for every $|a| \leq c$, we have

$$\left| \int_{-1}^{1} \Omega_b(t) \mathrm{e}^{\mathrm{i}at}\, dt \right| \leq \left(1 + M \frac{\max_{k=1,\dots,M} |w_k|}{\min_{k=1,\dots,M} |\eta_k|} \right) \epsilon^2. \tag{8.35}$$

Proof. Using (8.30), we have

$$\int_{-1}^{1} \Omega_b(t) \mathrm{e}^{\mathrm{i}at}\, dt = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}(b+a)t}\, dt - \sum_{k=1}^{M} w_k\, \mathrm{e}^{\mathrm{i}(b+a)t_k} - \sum_{k=1}^{M} \mathrm{e}^{\mathrm{i}bt_k} \Delta_k, \tag{8.36}$$

where

$$\Delta_k = \int_{-1}^{1} R_k(t) \mathrm{e}^{\mathrm{i}at}\, dt - w_k \mathrm{e}^{\mathrm{i}at_k}. \tag{8.37}$$

Applying (8.1), we obtain

$$\left| \int_{-1}^{1} \Omega_b(t) \mathrm{e}^{\mathrm{i}at}\, dt \right| \leq \epsilon^2 + \sqrt{M}\, \|\Delta\|_2. \tag{8.38}$$

The estimate (8.35) then follows from Proposition 8.3. ■

Remark 8.2. Using the functions R_k, $k = 1, \dots, M$, on a hierarchy of intervals, it is possible to construct a multiresolution basis (for a finite number of scales) similar to multiwavelet bases. We will consider such construction and its applications elsewhere.

8.1. Examples

For the weight

$$\omega(t) = \begin{cases} 1, & t \in [-a, a],\ a \leq 1/2, \\ 0, & \text{otherwise,} \end{cases} \tag{8.39}$$

we construct a 30-node quadrature formula so that (8.1) is satisfied with $\epsilon^2 \approx 10^{-15}$. We display the error in Fig. 9. For bandlimit $c = 7.5\pi$, we construct an interpolating basis $\{R_k\}_{k=1}^{30}$ and display one of the functions in Fig. 10. We then demonstrate the performance of the interpolation using this basis for three examples,

$$g_1(t) = \cos(ct), \tag{8.40}$$

$$g_2(t) = t, \tag{8.41}$$

$$g_3(t) = P_9(t), \tag{8.42}$$

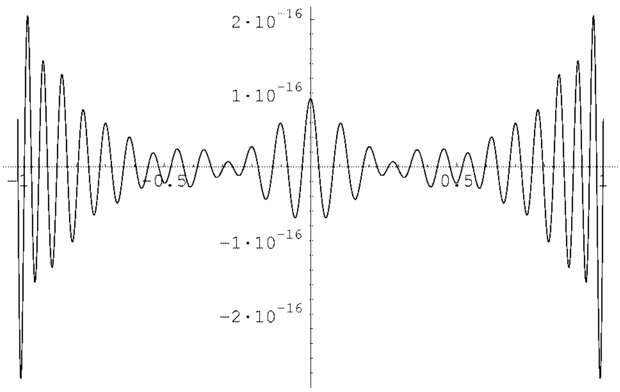

FIG. 9. Error in (8.1) for Example 1.

where P_9 is the Legendre polynomial of degree 9. These three functions are not periodic and we use

$$\tilde{g}_1(t) = \sum_{l=1}^{30} \cos(ct_l) R_l(t), \tag{8.43}$$

$$\tilde{g}_2(t) = \sum_{l=1}^{30} t_l R_l(t), \tag{8.44}$$

$$\tilde{g}_3(t) = \sum_{l=1}^{30} P_9(t_l) R_l(t), \tag{8.45}$$

as approximations. We display the function g_1 in Fig. 11 and the error of approximation by $\tilde{g}_1$ in Fig. 12. Similarly, we display the function g_2 in Fig. 13 and the error of approximation by $\tilde{g}_2$ in Fig. 14, the function g_3 in Fig. 15, and the error of approximation by $\tilde{g}_3$ in Fig. 16.

9. CONCLUSIONS

In this paper we have introduced a new family of Gaussian-type quadratures for weighted integrals of exponentials. These quadratures are parameterized by the eigenvalues of the positive definite Toeplitz matrix constructed from the trigonometric moments of the weight. The eigenvalues of this matrix accumulate toward zero and appear to have

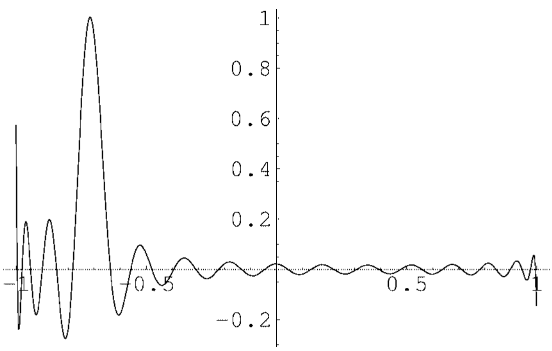

FIG. 10. Interpolating function $R_7(t)$ on the interval $[-1, 1]$.

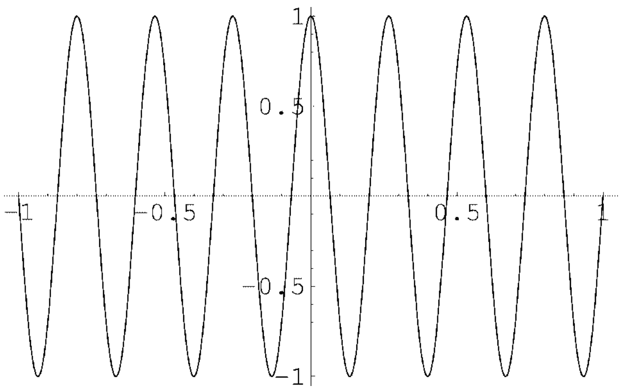

FIG. 11. Function $g_1(t)$ on the interval $[-1, 1]$.

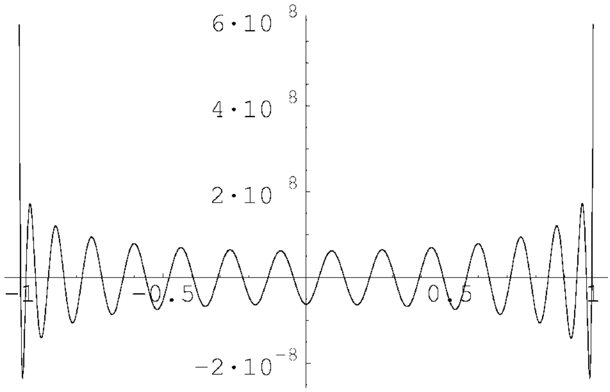

FIG. 12. Difference $g_1(t) - \tilde{g}_1(t)$ on the interval $[-1, 1]$.

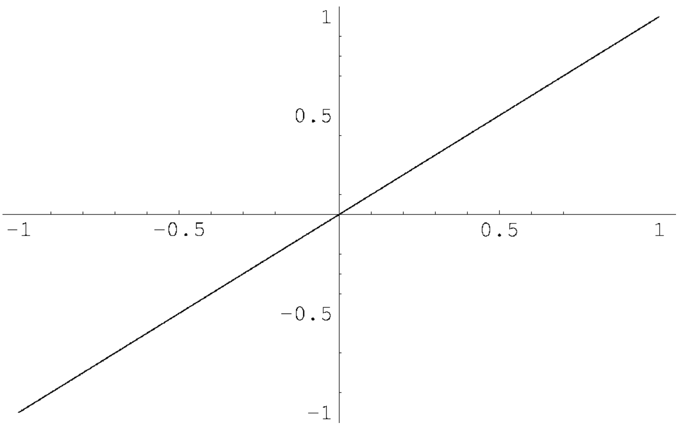

FIG. 13. Function $g_2(t)$ on the interval $[-1, 1]$.

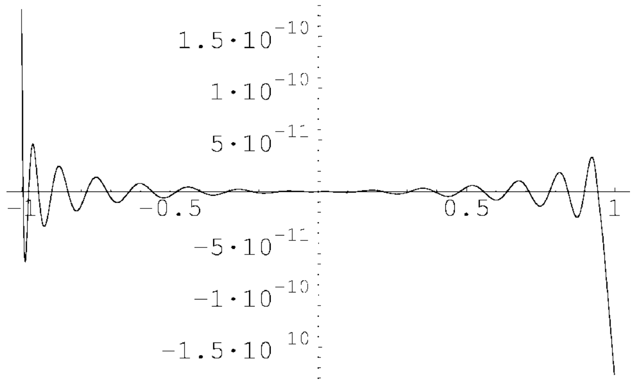

FIG. 14. Difference $g_2(t) - \tilde{g}_2(t)$ on the interval $[-1, 1]$.

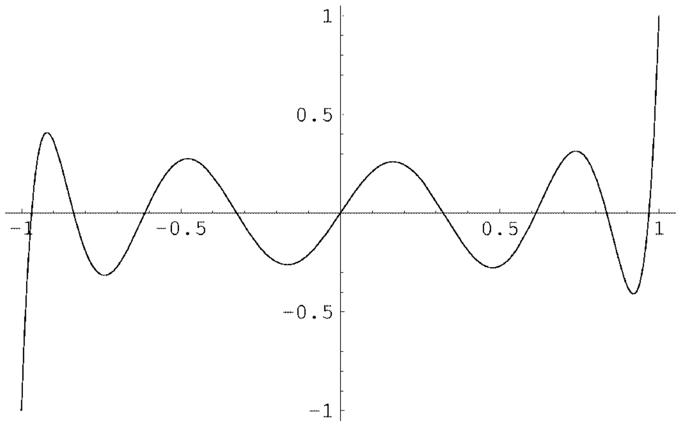

FIG. 15. Function $g_3(t)$ on the interval $[-1, 1]$.

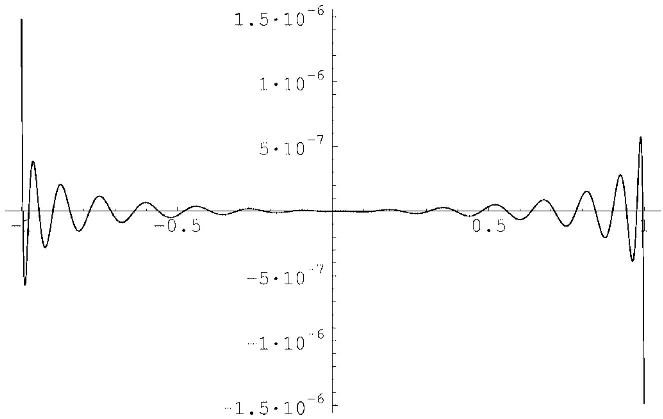

FIG. 16. Difference $g_3(t) - \tilde{g}_3(t)$ on the interval $[-1, 1]$.

exponential decay (see Fig. 1). For small eigenvalues, these quadratures are of practical interest.

The remarkable feature of these quadratures is that they have nodes outside the support of the measure and, as it turns out, the corresponding weights are negative and small, roughly of the size of the eigenvalue. The case corresponding to the smallest eigenvalue is equivalent to the classical Carathéodory representation.

As an application of the new quadratures, we show how to approximate and integrate several (essentially) bandlimited functions. We also have constructed, using quadrature nodes and for a given precision, an interpolating basis for bandlimited functions on an interval.

In the paper we made a number of observations for which we do not have proofs. Let us finish by stating two unresolved issues. First, it is desirable to have tight uniform estimates for the L_∞-norm of the PSWF (with a fixed bandlimiting constant) or, ideally, for the eigenfunctions associated with more general weights. Second, we conjecture that in Theorem 4.1, it is not necessary to require distinct roots for the eigenpolynomial since it might be a consequence of the eigenvalue being simple. We have neither a proof nor a counterexample at this time.

APPENDIX: PROOF OF THEOREM 2.2

We use a technique that goes back to [2] (see [28, Theorem 7.3] and [19, Chapter 5] for more details) which involves the Fejér kernel,

$$F_L(x) = \sum_{|k|\le L} \left(1 - \frac{|k|}{L+1}\right) e^{i\pi kx} = \frac{\sin^2((L+1)\frac{\pi x}{2})}{(L+1)\sin^2\frac{\pi x}{2}}, \tag{A.1}$$

for real x.

We need the following result.

THEOREM A.1 [19, Theorem 8, Chapter 5]. *For $|k| \le N$, let*

$$c_k = \sum_{j=1}^{M} \rho_j z_j^k,$$

where $\rho_j \ge 0$ and $|z_j| = 1$. Then, for all L, $0 \le L \le N$,

$$(L+1)\,\|\boldsymbol{\rho}\|_2^2 \le c_0^2 + 2\sum_{k=1}^{L} |c_k|^2.$$

Proof. Let $a_k = 1 - |k|/L + 1$ be the coefficients of the Fejér kernel F_L and write $z_j = e^{i\pi\theta_j}$. Since $\rho_j \ge 0$ and $F_L(\theta) \ge 0$ for all θ,

$$\begin{aligned}\sum_{|k|\le L} a_k |c_k|^2 &= \sum_{|k|\le L} a_k \sum_{j,l} \rho_j \rho_l \left(\frac{z_j}{z_l}\right)^k \\ &= \sum_{j,l} \rho_j \rho_l F_L(\theta_j - \theta_l) \ge F_L(0) \sum_{j=1}^{M} \rho_j^2 = (L+1) \sum_{j=1}^{M} \rho_j^2.\end{aligned}$$

The theorem follows because $a_0 = 1$ and $a_k \le 1$. ∎

Proof of Theorem 2.2. We first use (2.1) to extend the definition of c_k as $c_{-k} = \overline{c_k}$ for $k = 1, \ldots, N$ and $c_0 = \sum_{j=1}^{M} \rho_j$. We then define the Toeplitz matrix $\boldsymbol{T}_N$, $(\boldsymbol{T}_N)_{kj} = (c_{j-k})_{0\le k,j\le N}$, and the polynomial

$$Q(z) = \prod_{j=1}^{M} (z - e^{i\pi\theta_j}) = \sum_{k=0}^{M} q_k z^k.$$

Then $\boldsymbol{q} = (q_0, \ldots, q_M, 0, \ldots, 0)^t$ belongs to the null subspace of $\boldsymbol{T}_N$ because, for all l,

$$\sum_{k=0}^{M}\sum_{j=1}^{M} \rho_j e^{i\pi\theta_j(k-l)} q_k = \sum_{j=1}^{M} \rho_j e^{-i\pi\theta_j l} Q(e^{i\pi\theta_j}) = 0,$$

since, for all j, $Q(e^{i\pi\theta_j}) = 0$.

The matrix $\boldsymbol{A} = \boldsymbol{T}_N - c_0\boldsymbol{I}$ has the eigenvalue $-c_0$ because $\boldsymbol{T}_N$ is singular. We can bound c_0^2 by the Frobenius norm of $\boldsymbol{A}$ to obtain

$$c_0^2 \le 2\sum_{j=1}^{N} j|c_{N+1-j}|^2 \le 2N\|\boldsymbol{c}\|_2^2.$$

To finish the proof, we use Theorem A.1 with $L = N$. ■

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