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On a certain class of 1-thin distance-regular graphs

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Abstract

Let Γ denote a non-bipartite distance-regular graph with vertex set X, diameter $D \geq 3$, and valency $k \geq 3$. Fix $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. For any $z \in X$ and for $0 \leq i \leq D$, let $\Gamma_i(z) = \{w \in X : \partial(z, w) = i\}$. For $y \in \Gamma_1(x)$, abbreviate $D^i_j = D^i_j(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$ $(0 \leq i, j \leq D)$. For $1 \leq i \leq D$ and for a given y, we define maps $H_i \colon D^i_i \to \mathbb{Z}$ and $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ as follows:

$$H_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|, \qquad V_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|.$$

We assume that for every $y \in \Gamma_1(x)$ and for $2 \leq i \leq D$, the corresponding maps H_i and V_i are constant, and that these constants do not depend on the choice of y. We further assume that the constant value of H_i is nonzero for $2 \leq i \leq D$. We show that every irreducible T-module of endpoint 1 is thin. Furthermore, we show Γ has exactly three irreducible T-modules of endpoint 1, up to isomorphism, if and only if three certain combinatorial conditions hold. As examples, we show that the Johnson graphs $J(n, m)$ where $n \geq 7$, $3 \leq m < n/2$ satisfy all of these conditions.

Keywords: Distance-regular graph, Terwilliger algebra, subconstituent algebra.

Math. Subj. Class.: 05E30

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O nekaterih 1-tankih razdaljno-regularnih grafih

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Povzetek

Naj bo Γ ne-dvodelen razdaljno-regularen graf z množico vozlišč X, premerom $D \geq 3$, ter stopnjo $k \geq 3$. Izberimo si vozlišče $x \in X$ in naj bo $T = T(x)$ Terwilligerjeva algebra grafa Γ glede na x. Za vsako vozlišče $z \in X$ in za $0 \leq i \leq D$ naj bo $\Gamma_i(z) = \{w \in X : \partial(z,w) = i\}$. Za $y \in \Gamma_1(x)$ označimo $D^i_j = D^i_j(x,y) = \Gamma_i(x) \cap \Gamma_j(y)$ $(0 \leq i,j \leq D)$. Za $1 \leq i \leq D$ in za dano vozlišče y definirajmo preslikavi $H_i \colon D^i_i \to \mathbb{Z}$ in $V_i \colon D^i_{i-1} \cup D^{i-1}_i \to \mathbb{Z}$ takole:

$$H_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|, \qquad V_i(z) = |\Gamma_1(z) \cap D^{i-1}_{i-1}|.$$

Privzemimo, da sta za vsako vozlišče $y \in \Gamma_1(x)$ in za vsak $2 \leq i \leq D$ pripadajoči preslikavi H_i in V_i konstantni, ter da te konstante niso odvisne od izbire vozlišča y. Dalje tudi privzemimo, da so konstantne vrednosti preslikav H_i neničelne za $2 \leq i \leq D$. Pokažemo, da je vsak nerazcepen T-modul s krajiščem 1 tanek. Nadalje tudi pokažemo, da ima Γ do izomorfizma natančno natanko tri nerazcepne T-module s krajiščem 1 natanko takrat, ko veljajo trije kombinatorični pogoji (ki jih definiramo kasneje). Kot primer pokažemo, da ti trije kombinatorični pogoji veljajo za Johnsonove grafe $J(n,m)$, kjer je $n \geq 7$, $3 \leq m < n/2$.

Ključne besede: Razdaljno-regularen graf, Terwilligerjeva algebra, podkonstituentska algebra.

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