The Mercator Redemption

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Abstract

The Mercator projection is arguably the most famous projection in cartography and it has a long and useful history, especially as a navigation tool. In the last century, the Mercator projection's reputation has suffered a setback. This paper hopes to restore some of the projection's former glory by applying the Mercator projection to spherical content and the related *complex logarithm* function to planar content. We take advantage of its cylindrical and conformal properties and we showcase the projection's ability to zoom through many orders of scale. Finally, we show some related image manipulations such as the Droste effect, and a new *Conformal Spherical Stretching* operation by the first author that allows for a new degree of freedom when composing spherical panoramas.

Introduction

In the 16th century, Gerardus Mercator invented the Mercator projection (see Figure 1) to map a sphere (typically our Earth) to a plane (a map). It was designed such that any line on the map would trace the path of a ship keeping a constant compass bearing. One of the most notable features about Mercator world maps is that the features close to the poles are significantly scaled up compared to those at the equator. It is the one that shows Greenland the size of Africa.

The Mercator projection is famous. For most of its history, the Mercator map was widely used for general reference and is a household name. It was so popular that it is still sometimes mistakenly used generically as a term for any projection. Many of us who grew up in the 20th century remember it on our wall.

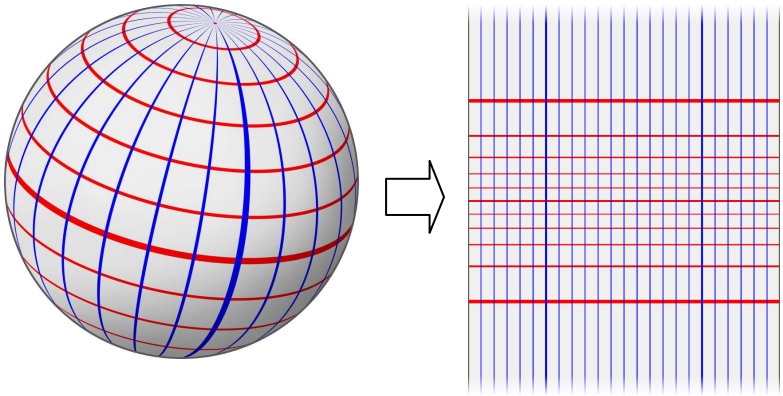

Figure 1: *Mapping the sphere to the plane with the Mercator projection*

But the Mercator projection is infamous too. It is beyond debate that the Mercator plot overstates the areas of countries far from the equator. In the 1960s, Arno Peters led a campaign against the Mercator projection and its politically incorrect scaling. While there are some serious drawbacks with Peters' proposed alternative, the controversy served to gradually dethrone the Mercator projection as the go-to projection for wall maps. To cement the Mercator projection's diminished status, cartographers adopted a resolution "strongly urging [mapmakers] to cease using rectangular world maps for general purposes or artistic displays" [1]. The Mercator projection's low standing is further underscored by the xkcd comic in Figure 2.

Figure 2: *xkcd's view of the Mercator projection*

Now, if mapping a globe was all the Mercator projection ever had to do, that might be the end of the discussion. However, as projections do, it will map *any* spherical imagery to a plane, not just the continents of the Earth. In this paper we hope to give the Mercator projection a new lease on life by using it to map spherical art and photography. We will also adapt it to map *planar* imagery for artistic purposes.

In the next section, we will discuss some of the mathematical properties of the Mercator projection. Following that, we show some artistic examples that use the Mercator projection, paying special attention to its ability to zoom in through many orders of magnitude. Then, we show how this projection is suitable for mapping visible spheres, specifically high resolution digital panoramas. Finally we will explore two other related image transformations: the Droste effect, and a new Conformal Spherical Stretching effect.

Properties of the Mercator Projection

As a map projection, the Mercator is known for being the projection with a lot of distortion, but in two important ways it is in fact *without* distortion. By design, it is *cylindrical* and it is *conformal*. A cylindrical projection is one that preserves orientation: North and South remain vertical and East and West are horizontal. A conformal projection is one that locally preserves angles. That is, if you look at small enough features on the map (i.e., small enough to ignore any change in scale), they will appear correctly, without any shearing or non-uniform stretching. These conditions are ideal for mapping photography and art as the details of the mapped imagery will never be crooked, upside down, or squashed/stretched into an unrecognizable streak of colours.

The Mercator projection is the *only* cylindrical and conformal map. We can sketch an argument of why this is here. For all cylindrical projections, points at longitude λ are mapped directly to their horizontal position x on the plane untouched. And a point's vertical position y is a function of only its latitude ϕ. Thus, cylindrical projections are distinguished from one another by this vertical function. Now, if we require a cylindrical projection to be conformal, then any given spherical curve that crosses all meridians at a constant angle (i.e., a *rhumb-line*) must cross vertical lines in the plane at this same angle (only a straight line does this). Determining the unique vertical stretch function that gives straight rhumb-lines in the plane yields the Mercator projection: $(x, y) = f(\lambda,\phi) = (\lambda, \ln(\tan(\pi/4 + \phi/2)))$ with its inverse being $(\lambda,\phi) = f^{-1}(x, y) = (2\tan^{-1}(e^{y}) - \pi/2, x)$.

A little analysis will show that the magnitude of y grows asymptotically as ϕ approaches the poles. This explains why a Mercator map is an infinitely long strip that is always necessarily truncated at the top and the bottom when on display.

We have defined the Mercator projection above as going directly from the sphere to an infinite strip on the plane. An alternative but equivalent construction takes two steps. First we project the sphere to an intermediate plane using the standard stereographic projection, which is conformal. In this paper, we'll refer to this plane as the *stereographic plane*. Second, by equating the stereographic plane to the complex plane, we can calculate the Mercator projection as the image of this plane under the *complex logarithm*

function, which is everywhere conformal except at the origin (with a branch cut); see Figure 3. With the complex logarithm $f(z) = \ln(z)$, we now have a function to map *planar* imagery with many of the qualities of the Mercator projection. Effectively, a complex point z with polar coordinates (ρ,θ) is mapped to $f(z) = \ln(\rho) + i\theta$. This yields a *horizontal* strip, but we can get the standard *vertical* Mercator mapping by rotating 90° afterwards, or equivalently $f(z)$ can be defined as $f(z) = i\ln(z)$.

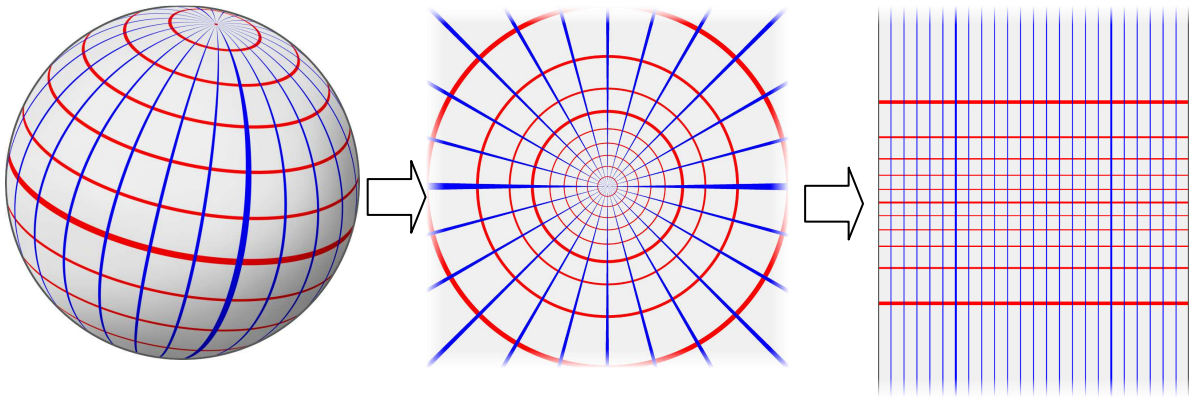

Figure 3: *The stereographic projection composed with the complex logarithm function results in the Mercator projection*

Zooming

In the physical world, we don't see Mercator maps that extend infinitely far up or down. They are cropped. It is easy to miss this point as Antarctica is shown as a strip of land along the bottom edge in the same way that finite cylindrical projections show it. But we can ask, "What would the map look like if we didn't crop it so soon?" Figure 4(a) shows such a map and we encourage the reader to print a copy on stiff paper to use as a bookmark. As we look downwards towards the South Pole, we zoom into smaller and smaller details, past the science station, until we see a snowflake that is centered at the South Pole with all six arms pointing northward. But even in this case, we have cropped the map before seeing molecules and beyond.

This zooming and scaling quality of the Mercator projection, which is the bane of some cartographers, can be a boon for photographers and artists. Here we have an interesting way to display an image that contains an increasing amount of detail while zeroing in on a selected point of interest. While the Mercator projection zooms into the poles, for the complex logarithm, the point of interest is at the origin where details are blown up exponentially.

In 2004, and again in 2007, two collaborative art projects called Zoomquilt [2] and Zoomquilt II [3] were created consisting of an interactive series of image patches, with each patch zooming in on details of the previous. Figure 4(b) and (c) show the unwrapped versions in one continuous strip. What this Zoomquilt example highlights is that the Mercator projection is an interesting way of saving multi-scale zoom images as contrasted with the 88 separate images that were needed to represent the Zoomquilt II in its standard zoom setup.

Of course when one thinks of math, art, and varying levels of zoom, fractals inevitably come to mind. Figure 4(d) zooms in on one particular detail of the Mandelbrot set. You can see the repeated horizontal motifs which correspond to some interesting rotationally symmetric patterns that surround the point of interest. The Mandelbrot shape at the top is the "standard" one while the one at the bottom is one of its copies at a miniscule scale.

Our last example is taken from the short film "Powers of Ten" by Ray and Charles Eames in 1977. The film zooms both out to a cosmic scale, and in to a microscopic scale. Using stills from this film we

have created Figure 4(e) which shows the Milky Way at the top and a sea of quarks along the bottom, all in one image.

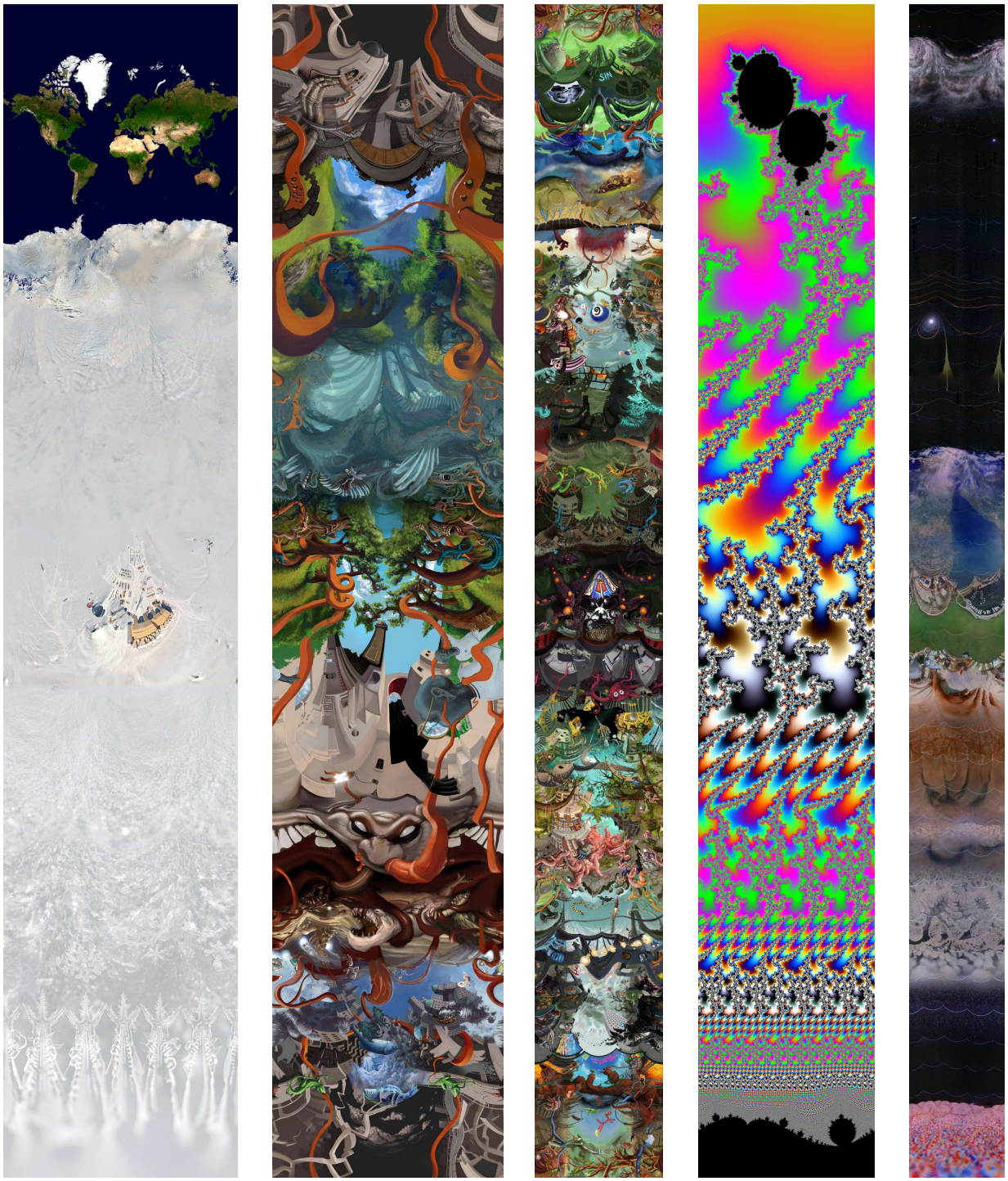

Figure 4: *a) The Mercator world map; and the complex logarithm function applied to b) Zoomquilt, c) Zoomquilt II, d) Mandelzoom, and e) Powers of 10 short film*

An interesting property of rectangular images mapped with Mercator projections or complex logarithms is that we can easily calculate the amount of zooming from their aspect ratios alone. We define

the *zoom-depth* of such images as the ratio between the sizes of the innermost and outermost circles when the image is projected back on the stereographic plane. The zoom-depth corresponds to the amount of zooming in that one would require to travel through the Mercator image from top to bottom. If the image is H units high and W units wide, then it has an aspect ratio of $A=W/H$ and the zoom-depth equals $f(A) = e^{2\pi/A}$. Table 1 shows the zoom-depth for the images in Figure 4.

Table 1: *The approximate zoom-depths of the various images in Figure 4*

	Aspect Ratio	Zoom-depth
World map	0.200	4.4×10^{13}
Zoomquilt	0.196	5.1×10^{13}
Zoomquilt II	0.107	1.4×10^{25}
Mandelzoom	0.160	1.1×10^{17}
Powers of 10	0.078	6.4×10^{34}

Mapping the Visible Sphere

Most of our examples so far have used planar imagery. Where can we get interesting *spherical* imagery to map? Let's consider one class of spherical imagery: the visible sphere. The *visible sphere* is a virtual sphere of imagery matching the solid-angular space around an eye-point such as star charts, Google Street View, and digital panoramas stitched together from photos taken in every direction. Several papers [4] [7] discuss artistic projections of visible spheres. Just as digital cameras have enabled visible sphere panoramas to become a common phenomenon, robotic camera mounts such as GigaPan are making high resolution gigapixel panoramas feasible. We have the perfect opportunity to showcase the high resolution of these panoramas by applying the Mercator projection to them.

Before looking at our next example, let us quickly introduce the *transverse* Mercator projection, which is just the Mercator projection applied sideways (i.e., rather than the poles being stretched out, two opposite points on the equator will be); see Figure 5.

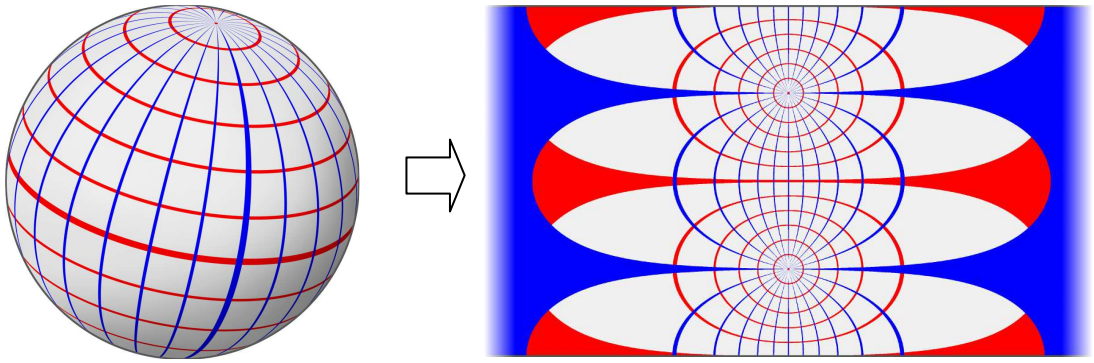

Figure 5: *The transverse Mercator projection*

The transverse Mercator image shown in Figure 6(d) started out as a 97,736 pixel wide panorama consisting of a narrow band close to the horizon captured by Larry Paul at a marina in Long Beach**.** Figure 6 shows how the Mercator projection allows us to see both the broad scope of an entire panorama, and the high level of detail at two opposite points (the trees and building top) at the same time.

Variations

Since the Mercator "strip" is in fact a cylinder (the left and right sides of the projection could be glued together), it is possible to transform any such cylindrical image back to either the stereographic plane or to the sphere (and then from the sphere on to any other projection). This basic concept is the source of

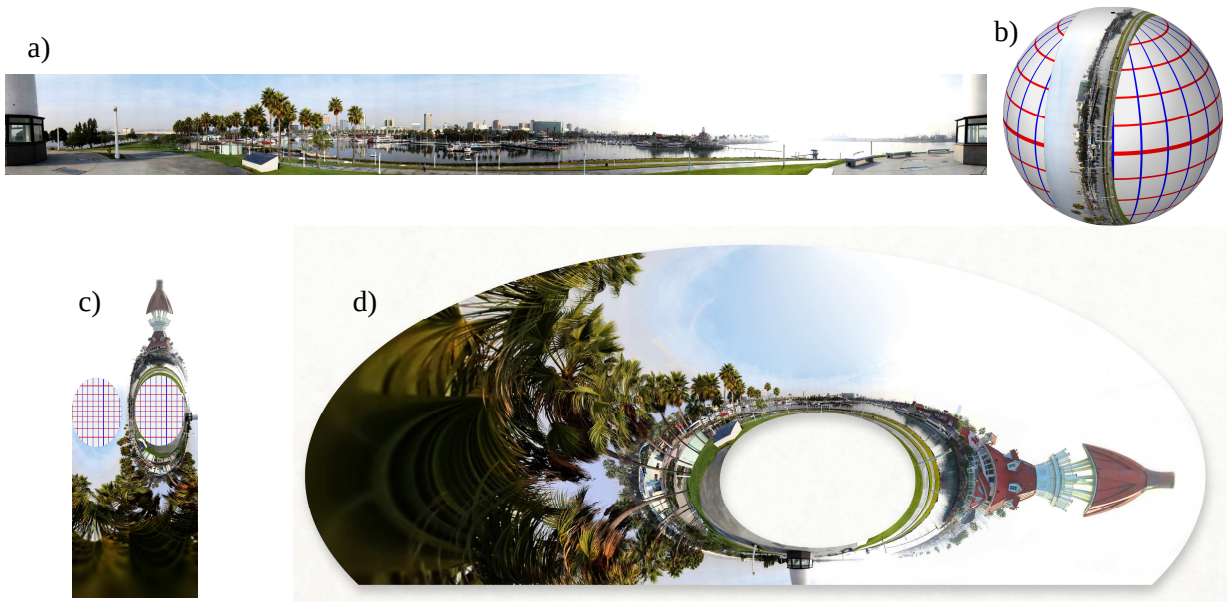

Figure 6: *a) A high resolution 360° panorama; b) points of interest are rotated to the poles; c) the Mercator projection which zooms in on the points of interest; d) artistically filled and cropped.*

two variations on the Mercator map; the first variation is accomplished by translating the Mercator map vertically, the second by combining a rotation and a scale change.

Conformal Spherical Stretching. This transformation first maps a sphere to a plane with a Mercator projection, it shifts the Mercator image vertically, and then it re-projects back to the sphere. This transformation is conformal, since all three steps are conformal. The result is that the equator is shifted downwards (say), increasing the relative size of the northern hemisphere, while still preserving angles. See Figure 7(a). Even more interesting is to apply this transformation to achieve the stretching of a different, non-polar point of the viewable sphere. For example, by rotating the sphere so that the point of interest is at the pole, stretching the pole as described above, and then undoing the first rotation, it is possible to conformally distort an image in quite drastic ways as shown in Figure 7(b).

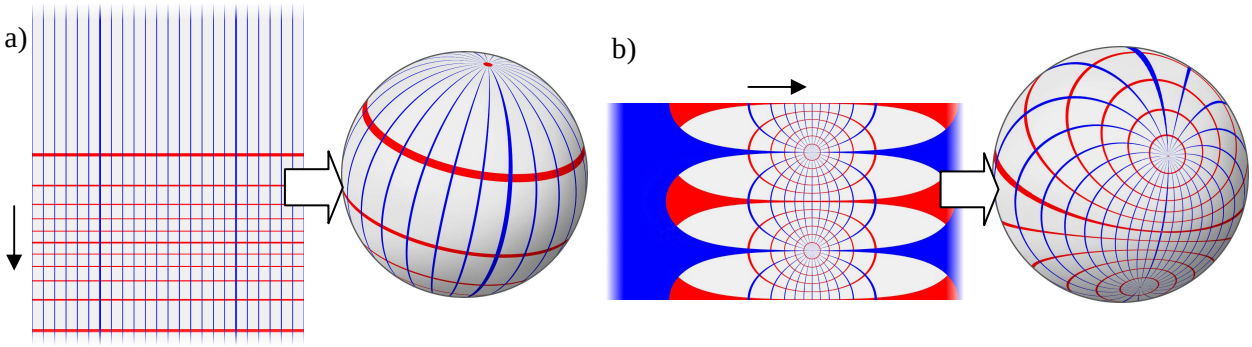

Figure 7: *Conformal Spherical Stretching: a) shift a Mercator projection and then apply the inverse to get back to the sphere; b) the transverse equivalent*

The effect can be thought of as a Möbius transformation (a class of functions on the complex plane) under the inverse image of the Mercator projection. In this paper, we would like to emphasize its status as a spherical transformation – especially for its use on spherical imagery which can then undergo further transformations such as equirectangular projections or immersive views.

To see this conformal stretching in action, Figure 8 shows examples which emphasize some elements and deemphasize others. Since these transformations are conformal, close inspection of the resulting images will reveal little, while the overall effect in an immersive viewer with a large field of view will be delightfully destabilizing [5].

a)

b)

Figure 8: *Two examples of Conformal Spherical Stretching. Original equirectangular image on the left, Conformally stretched equirectangular images on the right. a) Cathédrale de Reims, conformal stretching of the zenith, highlighting the vault. b) Picking blueberries, lateral conformal stretching*

The Droste effect. The *Droste effect* [6], is a fascinating conformal transformation that turns a planar image into a spiral around the origin and was pioneered by M. C. Escher in his *Print Gallery*. We mention it here because the Droste effect, which can be accomplished by relatively simple operations in Mercator space, highlights the Mercator projection's utility.

Consider now, not a Mercator strip, but a tileable Mercator rectangle forming a grid in the plane, so that in addition to the two vertical sides, the top and the bottom are tileable as well. If we rotate and stretch this grid such that the horizontal periodicity of the image is preserved then transforming this back to the stereographic plane will achieve the spiral effect. Figure 9 demonstrates these operations: The original image with a hole in the middle (d) is projected to (a), and copied vertically so that we have a tileable rectangle (b). We can rotate this tile as shown in (c) and project it back to get the final result in (f).

Conclusion

The Mercator projection and its close relative, the complex logarithmic function, are far from new and the Mercator world map is (justifiably) in decline. However, in a world with new technologies such as digital photography, immersive panorama viewers and gigapixel images, and new mathematical ideas such as conformal stretching and Droste spirals, there are certainly exciting and fascinating artistic uses for these functions yet to be discovered. We feel this will not be the last time you will hear of the Mercator projection.

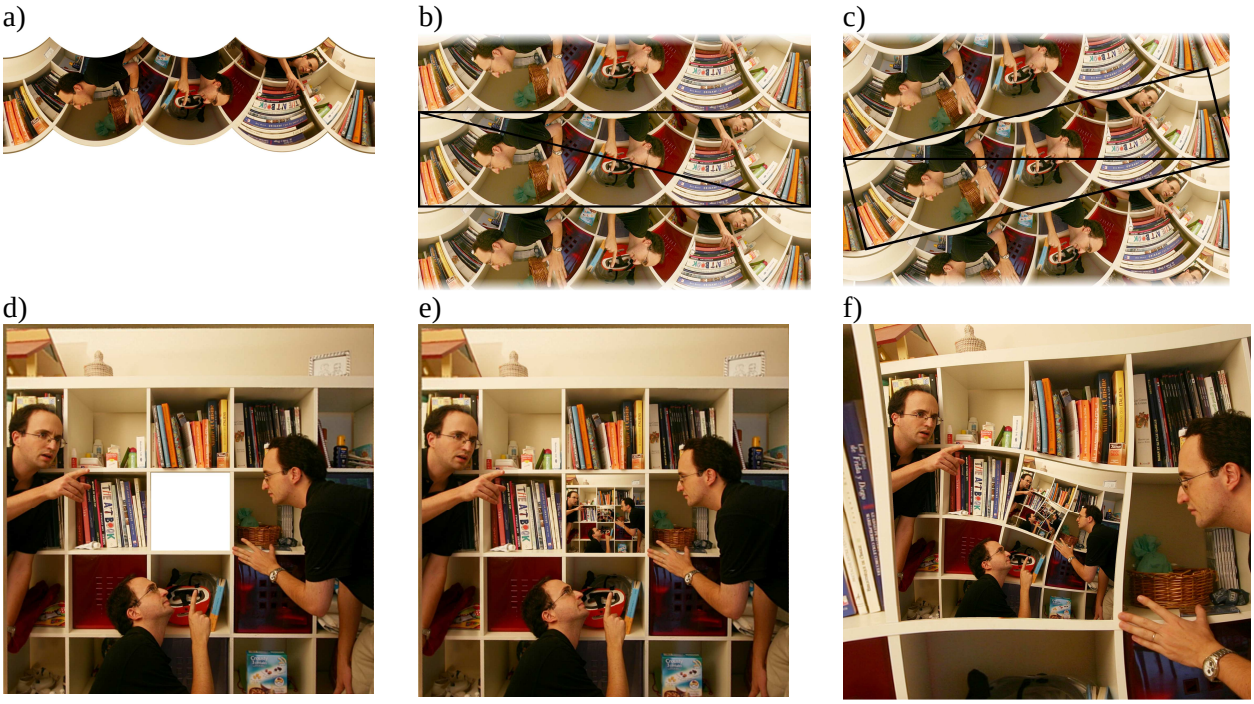

Figure 9 *The Droste effect is accomplished by simple operations in Mercator space (a, b, c);, with their corresponding effects shown in (d, e, f)*

Acknowledgements

The source imagery for the following figures are used with permission courtesy of the following people: Figure 2 is taken from xkcd.com/977 (Creative Commons); Figure 4(a), NASA; Figure 6, Larry Paul; Figure 4(b) and (c), ZoomQuilt; Figure 4(d), Figure 8, and Figure 9, Sebastien Perez-Duarte; Figure 4(e), from "Power of Ten" by Charles and Ray Eames; All other figures by David Swart.

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