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COALITION AND PARTY FORMATION IN A LEGISLATIVE VOTING GAME

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Abstract

We examine a legislative voting game where decisions are being made over both ideological and distributive dimensions, and legislators' preferences are separable over the two dimensions. In equilibrium legislators prefer to make proposals for the two dimensions together, rather than offering sequential proposals on the two dimensions separately. The equilibria exhibit interaction between the ideological and distributive dimensions and in any equilibrium there is a positive probability that a proposal is made and approved which excludes the median legislator (as defined over the ideological dimension), in contrast with a game where no distributive decision is being made. Moreover, in any stationary equilibrium there is more than one ideological decision that has a positive probability of being proposed and approved.

We show that legislators can gain from forming political parties, and consider examples where predictions can be made about the composition of parties. We discuss the impact of political parties on the outcome.

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Coalition and Party Formation in a Legislative Voting Game[*]

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1 Introduction

In this paper we examine the equilibrium patterns of proposed and approved decisions, as well as the winning coalition structure in a simple legislative game. The understanding of such issues is fundamental to the understanding of the operation of a legislature, committee, or the formation of a parliamentary government. The main focus of our work is on the importance of relative ideological positions of in a legislative decision making game. We begin by analyzing the equilibrium outcomes of the game without any external party influence, and then illustrate the usefulness of the model by considering the issue of party formation and how external party influence can alter the outcomes.

Our approach is to model the legislative procedure as a non-cooperative game, building on the seminal bargaining approach of Baron and Ferejohn (1989).[1] They considered a non-cooperative legislative bargaining game and explicitly modeled the process by which legislators are recognized, make proposals and vote on proposals. Despite the fact that Baron and Ferejohn considered a pure bargaining setting where the decision was entirely distributive, their non-cooperative approach and explicit focus on process allowed for predictions in situations where voting cycles exist (and the core is empty), and thus produced new insights relative to the existing spatial voting literature. The predictions of the Baron and Ferejohn model are simple, intuitive, and provide insight into the give and take present in a legislature and how the decision making depends on the specifics of the procedure.

Of course, the main limitation of considering such a pure bargaining model is that it offers little predictive power concerning the specifics of coalition formation (other than

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[1]See also Baron (1989) and Harrington (198).

confirming Riker's (1962) minimal winning coalition ideas), and offers no insight into the relationship between legislative behavior and the ideological positions of the legislators. In order to provide insight into these issues we consider a legislature that must make a decision about both an ideological (or public good) dimension, over which legislators have single peaked preferences, and a purely distributive (or private good) dimension for which each legislator prefers to have a larger amount allocated to his or her constituency. We examine a random recognition rule where a legislator is randomly selected to make a proposal. The legislator may make a proposal over either dimension or both dimensions, and the proposal is then put to a vote. If the proposal fails to receive a majority of the vote, the process is repeated. If the proposal receives a majority of the vote and it involves both dimensions, then the game ends. If the proposal passes and only involves one dimension, then the process is repeated with the restriction that new proposals can only consider the remaining dimension.

In the context of this legislative game, we begin by showing that even though the ideological and distributive issues may be considered separately, the equilibria will involve a proposal and approval of both dimensions simultaneously. The ideological issues cannot be divorced from distributive issues because of the usefulness of the distributive dimension as an instrument for compromise. For example, in a legislature which is deciding on both the level of gun control and a division of government spending across states or provinces, it is useful to tie the consideration of these decisions together since bargaining over the distribution of spending can be used for compromise on the decisions concerning gun control. This becomes especially important when legislators' preferences vary in intensity over the ideological dimension, as then there are significant possibilities for compromise and tradeoff. The outcome will generally *not* be a median decision on the level of gun control and separate bargaining over distribution of spending.

Once this interaction between the dimensions is explored, we provide results regarding the structure of the equilibria of the legislative game. First, we show that every stationary equilibrium results in some level of randomization over approved decisions. Most importantly, in any equilibrium there is more than one ideological position which has a chance of being approved. The intuition for this result is fairly straightforward: Given that only a majority is needed for approval, the choice of the proposal that a legislator makes will depend on that legislator's ideological position. The set of other legislators whose approval the proposer attempts to win depends on the legislator's ideological position, and the intensity of other legislators' preferences and their willingness to trade off ideology for the distributive dimension. Given this heterogeneity, different proposers will find different groups of legislators to be attractive as potential allies in forming a majority. This is in contrast with the homogeneous setting of Baron and Ferejohn (1989).

Second, we examine the structure of the winning coalitions in a class of stationary equilibria. In particular, we show that every legislator has a chance of being excluded from the winning proposal, as well as being included in the winning proposal. This means that no legislator is indifferent among all the outcomes which have a probability of approval in an equilibrium, and so there are some that they would like to vote yes on, and others that they would like to vote no on. The surprising aspect of this is that there are necessarily some proposals which are made and approved which exclude the median legislator. Thus, there are proposals which win approval of legislators whose ideological positions are not adjacent, and this is true regardless of the relative locations and strength of the ideological positions. There are two parts to the intuition behind this. One part comes from the bargaining and is related to the intuition one obtains from the work of Baron and Ferejohn (1989): including a legislator in too many proposals strengthens their bargaining position and makes it relatively expensive to obtain their vote. The other part of the intuition is that the expectation of what will happen if the proposal is voted down is the important benchmark for what is needed to get agreement. Thus, the attractiveness of an ideological proposal is measured relative to this expectation rather than on an absolute scale. For instance, a legislator with an ideological position at one extreme needs only worry about how the proposal he or she makes compares to the expected continuation in order to win approval of a legislator with an opposite ideological consideration, and may not have move dramatically away from that expectation in order to win approval of the other legislator.

Along these lines, the results exhibit some intuitive comparative statics. The set of proposals that are approved in an equilibrium generally exhibit some variation around their expectation (which is the relevant continuation expectation in a stationary equilibrium). So, there are winning proposals with ideological positions both to the left and right of the expected proposal. As distributive considerations are relatively less important to legislators and ideology is relatively more important, both the ability to compromise and the variation of the ideological dimension of the winning proposals decrease. In the limit, the winning proposals converge to the median position. At the other extreme, as distributive considerations are relatively more important to legislators, there is more room for compromise on ideology and correspondingly a larger variation along the ideological dimension of the winning proposals.

Finally, the structure and variation in the equilibria lead to a natural role for political parties. Given that legislators are not indifferent among the possible outcomes in an equilibrium, they may gain by forming a binding alliance with other legislators in the form of a political party. We discuss how this view relates to and differs from related analyses of political parties in legislative settings. We consider examples where sharp predictions can be made concerning a stable political party, and examine the proposal that would emerge in the presence of the party. We also show that there are examples where there may be several political parties which could form and be stable (so that no members would choose to defect and ally themselves with others to form a new party), and examples where there are stable parties consisting of legislators with opposite extremes in the ideological spectrum.

Before proceeding to the model, let us discuss the relationship of this work to two other models which are closely related to the Baron and Ferejohn (1989) approach. Baron (1991) extends the Baron-Ferejohn model to the case of two dimensional decisions where agents have circular preferences over outcomes. This produces interesting insight into situations where there are three bargainers, which can be of significant use in understanding government formation in a parlimentary system. However, the model turns out not to be tractable with larger numbers of players or with more general preferences. Thus, little can be said analytically about the general behavior of the equilibria and so is difficult to use to analyze coalition formation and legislative behavior.

Calvert and Dietz (1996) note this difficulty and take a different approach, still keeping with the original Baron-Ferejohn one dimensional pure bargaining model, but allowing legislators to care not only about their own share but also about the shares of others. Their model is tractable, as there is a natural tendency to form coalitions with other legislators about whose allocation you care most. This also allows for an analysis of party formation (which we will come back to discuss later). An important difference of the Calvert and Dietz approach and the one we take here is in terms of the motivation for forming coalitions. In their model this motivation comes from externalities in preferences. In our approach it comes from relative ideological positions and convictions. These different approaches offer complementary views of coalition and ultimately party formation and, as will become evident, different insights into coalition and party formation. We also take a different point of view on what a party is and how it works, treating it as an stable organization external to the game, rather than as a (non-stationary) equilibrium phenomenon of the game. We offer a detailed discussion of this view.

Let us mention one final, but central, motivation for including an ideological dimension in a legislative model. Ultimately, it is important to marry a model of the internal workings of a legislature with models of elections of legsilators, as well as the interactions of the legislature with other branches of government. As ideological considerations are critical to understanding these relationships (especially the election process), it is important to understand their role in the legislative setting.

2 The Legislative Game

Legislators

There are n *legislators*, where $n \geq 3$ is an odd number.

Decisions

A *decision* is a vector $(y, x_1, \ldots, x_n)$ consisting of an *ideological decision* y and a *distributive decision* $x_1, \ldots x_n$. The set of feasible public decisions is $[0, Y]$ where $Y \in [0, 1]$ and the set of private decisions are those such that $\sum x_i \leq X$ where $X \geq 0$. The set of possible decisions is denoted D with generic element $d \in D$.

In the case where $Y = 0$, the model simplifies to that of Baron and Ferejohn (1989), and so the X dimension captures decisions that are purely distributive with no particular ideological component to them. In the other extreme case, where $X = 0$, the model is one of a pure ideological decision as in a median voting model, and so the Y dimension captures decisions that are public.

Preferences

Each legislator i has preferences over decisions that depend only on the public decision and his or her own component of the private decision. So, preferences of legislator i are represented by a utility function $u_i : [0,1] \times I\!R_+ \to [0,1]$ that depends on y and x_i. The utility function, $u_i(y, x_i)$, is nonnegative, continuous, and is strictly increasing in x_i for every $y \in Y$. Also, u_i is single peaked in y for every x_i.[2] We denote the peak of u_i by $\widehat{y}_i$. Legislators evaluate randomizations over decisions through expected utility calculations.

The preference ranking of each legislator i over ideological decisions is *separable* from the distributive decision. More formally, for any $(y, x_1, \ldots, x_n)$ and $(y', x'_1, \ldots, x'_n)$, $u_i(y, x_i) > u_i(y', x_i)$ if and only if $u_i(y, x'_i) > u_i(y', x'_i)$. This restriction actually provides for stronger results since we will show that equilibrium behavior exhibits a strong interaction between the dimensions, despite the separability of preferences.

Without loss of generality, order legislators so that $\widehat{y}_i \leq \widehat{y}_j$ if $i \leq j$. In any case where $Y > 0$, assume that $\widehat{y}_1 < \widehat{y}_n$. Let $\widehat{y}_{\text{med}}$ be the median of $\widehat{y}_1, \ldots \widehat{y}_n$.

Legislators discount time at a rate δ where $0 < \delta \leq 1$. So, their utility for reaching an agreement $(y, x_1, \ldots, x_N)$ at time t is $\delta^t u_i(y, x_i)$.

The Legislative Game

The *legislative game*[3] consists of a potentially infinite number of *sessions*. Time is indexed by sessions $t \in \{1, 2, \ldots\}$. At the beginning of each session a legislator is recognized at random to make a proposal. Legislator i is recognized with probability $p_i > 0$, where $\sum_i p_i = 1$ and the recognition probabilities are the same in each session. Next, the recognized legislator proposes a decision $(y, x_1, \ldots, x_n)$. (Shortly, we will consider a more general version where the legislator may choose to make a proposal in only one dimension.) Then, in a fixed order[4] (the same in each session) the legislators are sequentially called on to vote 'yes' or 'no'. If a majority of legislators (at least $\frac{n+1}{2}$) vote 'yes', then the game ends and the decision $(y, x_1, \ldots, x_n)$ is taken. Otherwise the game proceeds to the next session, where the process is repeated.[5]

[2] That is, there exists $\widehat{y}_i$ such that $u_i(\widehat{y}_i, x_i) > u_i(y, x_i)$ for every $y \neq \widehat{y}_i$, and $y' < y < \widehat{y}_i$ or $\widehat{y}_i > y > y'$ implies $u_i(y, x_i) > u_i(y', x_i)$.

[3] This game is consistent with the 'closed rule' version of Baron-Ferejohn (1989).

[4] The order is not important to the results. A random or simultaneous order will support the same equilibria. One difference, however, is that a simultaneous vote will result in additional equilibria where all legislators vote yes for an arbitrary proposal expecting that this is the case and thus having no chance of being pivotal. Similarly a simultaneous vote can introduce equilibria where all legislators vote no for the same reason. Such degenerate equilibria are not possible in the case in a subgame perfect equilibrium of an ordered (roll call) vote.

[5] In some legislative settings there are restrictions on whether (or when) some issues can be reconsidered once they are voted down. To the extent that different but related proposals can still be submitted, the game above is still a good approximation. Also, there are many decisions that legislative rules

For the case that the game never ends, assign a default decision denoted $(y^0, x_1^0, \ldots, x_n^0)$. For the case where $\delta < 1$ this is irrelevant. For the case where $\delta = 1$ it is conceivable that the default (viewed as a status quo) would matter, but we will prove that this is not the case.

Each legislator observes all the actions that precede any action he or she decides upon, so that the game is one of perfect information and the definitions of strategies and subgame perfection are standard.

Although, we will use the term legislative game in what follows, it should be clear that the game above is also a useful model for committee interactions in a variety of different settings (for instance, a faculty committee), as well as the formation of a government. For more discussion for how such a model might fit with the formation of a government, see Baron (1989) and Baron and Ferejohn (1989).

Stationary and Simple Equilibria

Generally, the set of equilibria can be quite large and complicated in a game such as the one described above. Indeed, a 'folk-theorem' type of result along the lines of Proposition 2 of Baron and Ferejohn (1989) holds here as well, where a large set of equilibrium outcomes can be supported.[6] However, the types of strategies needed to support arbitrary sorts of outcomes are quite complicated and can be criticized on several grounds. Baron and Ferejohn argue for limiting attention to equilibria involving stationary strategies, based on focalness of such equilibria and the on the finite horizon of individual legislators. Rather than repeat those arguments here, we refer the interested reader to their discussion. (See also Hart and Mas-Colell (1996) for further discussion in a game theoretic bargaining context.)

A strategy is *stationary* if each legislator's continuation strategy is the same at the beginning of any session, regardless of history. An equilibrium is *stationary* if it is a subgame perfect equilibrium and each legislator's strategy is stationary.[7]

In some situations strategies turn out to satisfy further restrictions in terms of the number of proposals that a legislator randomizes over when called upon. A legislator's proposal must win the approval at least $\frac{n-1}{2}$ other legislators to become an equilibrium outcome. For any given legislator there exist $M = (n-1)!/[((n-1)/2)!]^2$ sets of exactly $\frac{n-1}{2}$ other legislators.

generally allow to be reconsidered, even if previously voted down. Take, for example, the ideological dimension to be the overall size of a budget and the distributive dimension to be the percent of spending allocated to different constituencies. As mentioned below the model may also be interpreted to be a general model of committee behavior in a variety of settings, or a model of government formation where such restrictions rarely apply.

[6] Here there are restrictions on the ideological outcomes that can be supported as depending on the size of (or relative preference for) X. But, with large enough X and δ, any outcome can be supported in subgame perfect equilibrium.

[7] Of course, non-stationary deviations are allowed when applying the definition of equilibrium.

A *simple equilibrium* is a stationary equilibrium in which (1) each legislator when called on to propose randomizes over at most M proposals, and (2) each such proposal can be identified with a distinct C such that $i \notin C$, $\#C = \frac{n-1}{2}$ and the legislators in C (and perhaps others) vote 'yes' on the proposal.

Note that a simple equilibrium does not require that the sets of legislators who approve two different proposals by i be distinct as i may make proposals that are approved by supermajorities.

3 Benchmarks

Let us begin with a result for the case where $X = 0$.

Benchmark *If $X = 0$ and $\delta = 1$ for each i, then in any stationary equilibrium $\widehat{y}_{\text{med}}$ is proposed and eventually approved with probability one. Furthermore, there exists a simple equilibrium in which any recognized legislator in any session proposes $\widehat{y}_{\text{med}}$ and it is approved by all legislators. Also, for any $\epsilon > 0$ there exists $\overline{\delta} < 1$ such that if $\delta \geq \overline{\delta}$, then the (possibly random) outcome of any stationary equilibrium is within ϵ of $\widehat{y}_{\text{med}}$ (with probability one).*

The above benchmark shows that in the purely ideological case, any stationary equilibrium outcome[8] is close to the median legislator's ideological ideal point. The intuition is that a proposal too far away from the median legislator's ideal point should not win approval, given that the median and legislators to the other side can wait and do better. A detailed proof is offered in the appendix.

The other extreme benchmark is the purely private case examined by Baron and Ferejohn (1989).

Benchmark (Baron and Ferejohn (1989)). If $Y = 0$, there are equal probabilities of recognition, and $u_i(0, x_i) = x_i$ for all i, then in any stationary equilibrium each legislator has an expected distributive allocation of $\frac{X}{n}$. Furthermore, there exists a simple equilibrium in which any recognized legislator proposes a share $X(1 - \delta\frac{n-1}{2n})$ for him or herself, and $\delta\frac{X}{n}$ to each of a randomly selected $\frac{n-1}{2}$ other legislators, and this is approved by these randomly selected legislators.

In this benchmark we see the pure bargaining aspect where agents are offered something which makes them indifferent between voting yes now and waiting for the continuation, with the randomly chosen proposer keeping the excess. The result extends to situations where the probabilities of recognition are not quite equal (with some adjustments necessary in the probabilities of who to propose to). This follows from the

[8] In this case, it is reasonable to conjecture that this is true of any subgame perfect equilibrium, although this conjecture is not relevant for comparison in this work.

balancing that goes on in the purely distributive game: a legislator always wants to make an offer to the cheapest (in terms of expectations) other legislators. This keeps any single legislator from having too high an expectation since in that case other proposers would not want to offer that legislator anything. Clearly, if a single legislator has an overwhelming probability of being recognized then this reasoning breaks down.

4 Agenda Setting with both Ideological and Distributive Decisions

We now move to the general case of both ideological and distributive decisions. First, we show that it is without loss of generality that we restrict the game to one where proposals are made over both ideological and distributive decisions simultaneously. That is, we show that in a game where the proposer has a choice of whether to propose on just one dimension at a time or on both simultaneously, he or she chooses to propose on both dimensions simultaneously (except in certain degenerate situations where the outcome is equivalent in any case).

A More General Legislative Game

Consider the following legislative game. The structure of the game is the same as the one described previously, except that the proposer may choose to propose a decision in both dimensions, or may choose to propose a decision in either of the single dimensions. In the case where a proposal is made and approved which involves just one dimension, then that decision is fixed and the game is continued with a new random recognition of proposer to decide on the remaining dimension. The definition of stationary strategy is extended so that an agent's strategy can depend on the previously approved proposal of one dimension if there is one.

Proposition 1 *Consider any stationary equilibrium of the general legislative game with concave utility functions. If $\delta < 1$, then the game ends in the first session with an approved proposal that involves both dimensions. If $\delta = 1$, then for any stationary equilibrium, there exists a stationary equilibrium with exactly the same probability over eventually approved decisions which ends in the first session. Moreover, if $\delta = 1$, then some proposal is approved (with probability 1) in the first session, and any proposal which is approved and does not involve both dimensions has the distributive dimension proposed and approved in the first session and then the median proposal approved in some subsequent session.*

Note that the last case is non-generic as it not only depends on $\delta = 1$, but also requires special configurations of preferences. Generally, there is compromise to be made and we should expect the decisions to be taken together. This is detailed in Proposition 5.

For the case of $\delta < 1$, the fact that a decision on both dimensions is approved in the first period is not surprising, as there is a loss to waiting. The more interesting case is when $\delta = 1$. Once a proposer has the floor, it is in his or her interest to propose a distributive decision that will be approved, as otherwise they may be excluded in what follows. The fact that they will choose to also propose an ideological decision is where the importance of compromise comes in. If only the distributive decision is made and cannot be changed, then in what follows Benchmark 1 will apply and we should expect the median ideological decision to be taken. Thus, any compromise that is to be made must be made simultaneously with the distributive decision.

For the remainder of the paper, given the equivalence between the outcomes of the two games established by Proposition 1,[9] we restrict our attention to the game where legislators propose both dimensions simultaneously.

5 Winning Proposals and Coalitions

We now provide a sequence of results which describe properties of simple and/or stationary equilibria.

The proposition below establishes the existence of simple equilibria. This is important since otherwise the analysis which follows could be vacuous. It is also interesting in that it demonstrates that despite the potential complexity of the game, there always exists a set of simple strategies that legislators can follow that are optimal with respect to each other.

Proposition 2 *If u_i is concave for each i, then there exists a simple equilibrium. Moreover, if each u_i is strictly concave then all stationary equilibria are simple.*

Standard game theoretic results concerning the existence of equilibrium do not apply here given the continuum of actions and the stochastic nature of the game,[10] and moreover because we are establishing existence of simple equilibrium. Thus we offer a direct proof of Proposition 2 in the appendix.[11] We remark that the proof of Proposition 2 does not rely on the separability of preferences.

Next we establish some basic characteristics of the equilibria.

Proposition 3 *In any stationary equilibrium:*

[9] Although not stated in the proposition, it is clear that any equilibrium of the restricted game is also an equilibrium of the more general game.

[10] The game may be viewed as stochastic by coding the random choice of proposer into the state, and also by having different states depending on whether or not a proposal has been accepted in the past.

[11] Independent work by Banks and Duggan (1998) offers a similar result, as well as a result like Lemma 1.

- *if utility functions are concave, then the legislative game ends in the first session (even if $\delta = 1$)*

- *any approved decision distributes X among an exact majority,*

- *if the utility functions are concave, then the equilibrium is independent of the default decision (even if $\delta = 1$).*

The fact that the legislative game ends in the first session is fairly clear for the case that $\delta < 1$, but for the case of $\delta = 1$ the argument is a bit more subtle as one can imagine a legislator being indifferent between the approved proposals and thus willing to make a realistic proposal with probability less than 1, with the expectation that sooner or later it will be made and approved. The key to the proof comes from the following propositions which establish a certain heterogeneity in equilibrium proposals, and show that any proposer has a chance of being excluded from some proposal. This last fact breaks indifference so that a recognized legislator strictly cares to propose a decision that will be approved.

The fact that X is distributed among an exact majority, reaffirms the logic of Riker (1962) and Baron and Ferejohn (1989).[12] Note that this depends on the closed rule nature of the legislative game, and would not necessarily hold in an open rule version where ammendments can be proposed. Such ideas are explored in Baron and Ferejohn (1989) and not reconsidered here.

Finally, the fact that the equilibrium is independent of the default decision when $\delta = 1$, is again related to some of the reasoning which is presented below. Each legislator will have some chance of being excluded from a winning proposal, which induces a form of impatience when they have a chance to propose. This (induced) impatience makes the default outcome irrelevant.

Legislators have well-defined ex ante expected utilities for a given strategy profile. Given the stationarity, at any point in the legislative game these expected utilities also represent the expectated utility of the continuation, conditional on the current proposal not being approved. We generally denote these by v_i. The following definitions identify how a legislator ranks a proposal relative to the continuation.

A proposal $(y, x_1, \dots x_n)$ *excludes* legislator i (relative to v_i) if $u_i(y, x_i) < v_i$.

A proposal $(y, x_1, \dots x_n)$ *includes* legislator i (relative to v_i) if $u_i(y, x_i) \geq v_i$.

Note that the definition of 'exclude' and 'include' is made relative to the legislator's preferences and not their voting behavior. It is possible in equilibrium for a legislator to vote 'yes' on a proposal when they prefer the continuation, provided they are in a

[12]In fact, this idea appears in von Neumann and Morgenstern (1944).

situation where they are certain that the proposal will be approved regardless of their vote. That is, if a proposal has received or will receive a majority independent of a given legislator's vote, then he or she will be indifferent between voting 'yes' or 'no.' Nevertheless, we still have information about the legislator's preferences and thus know how they would vote if they were to cast the decisive vote, which is the idea behind the definitions of being 'excluded' or 'included.'

Proposition 4 *Suppose that legislators' utility functions are concave. There exists $\overline{\delta} < 1$ such that for all $\delta \geq \overline{\delta}$ in any simple equilibrium for every i there is a positive probability that a proposal is made and approved which excludes i. Furthermore, if $\delta = 1$ then this is true of every stationary equilibrium.*

One implication of the above proposition is that the median is excluded from some proposal. This means that in any stationary equilibrium there is a positive probability that a proposal wins the approval of (and includes the members of) a disjoint coalition.

In Proposition 4, it is not clear that the exclusion of a legislator involves anything more than the distributive dimension, which would simply follow the logic of Baron and Ferejohn (1989). In fact, there are interesting tradeoffs occuring in the ideological dimension as well. The following proposition makes this point clear.

In what follows, let us restrict attention to quasi-linear utility functions. A legislator's preferences are said to be quasilinear if there exists a single peaked $\overline{u}_i : [0,1] \rightarrow [0,1]$ such that $u_i(y, x_i) = \overline{u}_i(y) + x_i$. The quas-linear preferences permit a more transparent presentation of the following proposition. They are not necessary, but without this assumption the extension of the following definition is more complicated.

We say that *preferences admit local compromise* if there exists some exact majority $C \subset N$ such that $\widehat{y}_{\text{med}}$ is not a local maximum of $\sum_{i \in C} \overline{u}_i(y)$.[13]

The above condition is a very weak one. It states that there exists *some* majority coalition that could improve its overall utility by moving the ideological decision slightly away from the median decision.

Proposition 5 *Suppose that there are equal probabilities of recognition, and legislators' preferences are quasi-linear and admit local compromise. Then in any stationary equilibrium there is more than one y (and thus more than one proposal) that has positive probability of being an equilibrium decision.*

One of the important implications of Proposition 5, when compared with Benchmark 3 is that the possibility of joint consideration of the two dimensions impacts the outcome on each dimension. Note that this true even though agents have additively separable

[13]For any $\epsilon > 0$ there exists y such that $|y - \widehat{y}_{\text{med}}| < \epsilon$ and $\sum_{i \in C} \overline{u}_i(y) > \sum_{i \in C} \overline{u}_i(\widehat{y}_{\text{med}})$.

preferences, and so even when the dimensions are completely independent. Thus, this differs from results in the spatial model where agents have circular or elliptical preferences, as in Enelow and Hinich (1984). There, the cardinal impact of changes in one dimension depends on the choice in the other dimension, and so there is some interaction between dimensions. In particular, how much a legislator cares about one dimension versus another can depend on the location in the space.[14]

Proposition 5 does not indicate what equilibrium implications are for the distributive dimension. As illustrated in the examples that follow, the distribution may be quite asymmetric depending on the relative ideological intensities of the legislators.

6 Comparative Statics

So far we have established a number of general properties of simple (and in many cases stationary) equilibria, which under the assumption of concave utility functions are loosely summarized as follows.

- Simple equilibria always exist and in such an equilibrium:
- Both dimensions will be considered together and a decision will be approved in the first session.
- Each legislator is excluded from some decision that has a chance of being approved.
- If preferences admit local compromise, then there are at least two different ideological decisions that have a chance of being approved.

In order to take our understanding further, we now examine some comparative statics. First, we develop a result that allows one to examine limiting behavior, and to draw conclusions from the Benchmarks we presented earlier. Second, we consider a specific parametric example of the model with three legislators, and examine changes in the equilibrium as the intensity of ideological preferences and locations of ideal points vary.

The following lemma shows that the set of simple equilibria are well behaved as one varies the set of parameters of the legislative game. The lemma is also an integral part of the proof of Propositions 2 and 4.

Lemma 6 *Let* $(\delta^k, Y^k, X^k, u_1^k, \ldots, u_n^k) \to (\overline{\delta}, \overline{Y}, \overline{X}, \overline{u}_1, \ldots, \overline{u}_n)$*, be a converging sequence of discount factors, ideological intervals, distributive intervals, and preference profiles,*

[14] For instance, if preferences are circular, then it is almost always true that $u_i(y, x_i) - u_i(y, \widetilde{x}_i) \neq u_i(\widehat{y}, x_i) - u_i(\widehat{y}, \widetilde{x}_i)$.

with corresponding simple equilibria ℓ^k.[15] *Choose any convergent subsequence of the equilibria and let its limit be ℓ.*[16] *Then ℓ is a simple equilibrium of the legislative game for $(\overline{\delta}, \overline{Y}, \overline{X}, \overline{u}_1, \ldots, \overline{u}_n)$,*

Although the lemma is technical in nature, establishing upper-hemicontinuity of the simple equilibrium correspondence, its has important implications for limiting comparative statics. For example, Benchmarks 1 and 2 can be used to understand what happens when X becomes relatively large or small in consideration compared to Y. Consider the case where $\delta = 1$. As the size of X goes to zero, then simple equilibria are close to one where all proposals involve $\widehat{y}_{\rm med}$, as in Benchmark 1. As the size of Y goes to zero, then simple equilibria are close to pure bargaining ones analyzed by Baron and Ferejohn (1989), as in Benchmark 2.

To gain a better understanding for the intermediate cases, where both X and Y are important considerations and there is a possibility for compromise, let us consider a few examples.

There are three legislators. Normalize by setting $Y = 1$, and for simplicity restrict attention to the case where $\delta = 1$ and legislators have an equal probablity of recognition, $p_1 = p_2 = p_3$. Label leglislators in order of their peaks so that $\widehat{y}_1 = 0$ $\widehat{y}_2 = \widehat{y}_{\rm med}$, and $\widehat{y}_3 = 1$. The preferences of legislator i are represented by $-b_i|y - \widehat{y}_i| + x_i$. Thus, legislators care about the distance of y from $\widehat{y}_i$ in a linear fashion.

Say that i *proposes to* j if the proposal by i includes j and excludes the remaining legislator. Let y_{ij}, x_{ij} denote the decision proposed by i when proposing to j, and a_ij denote the probability that i proposes to j.

There is a unique stationary equilibrium (and thus simple equilibrium) which is described in the appendix. We illustrate it here for specific cases.

Example 1: $b_1 = 1$, $b_2 = 3$, and $b_3 = 6$.

In this case there is a cycle where legislator 1 proposes to legislator 2, legislator 2 proposes to legislator 3, and legislator 3 proposes to legislator 1.

The specifics of this equilibrium are: [17]

[15] Measure distance between u_i and $\widetilde{u}_i$ by $sup_{y,x_i \in [0,1]\times I\!R_+}|u_i(y,x_i) - \widetilde{u}_i(y,x_i)|$, and assume that $(\overline{u}_1^n, \ldots, \overline{u}_N^n)$ are admissible. Simple equilibrium strategies may be set in a finite dimensional Euclidean space, as outlined in the proof of Proposition 2.

[16] Since there may be multiple simple equilibria, the sequence may not converge. However, any cluster point of the sequence will be a simple equilibrium.

[17] In each of the examples that follow, it must be that X is small enough so that $0 \leq y_{ij} \leq 1$. A bound on X then follows directly from the given expressions.

$$y_{12} = \widehat{y}_{\text{med}} - \tfrac{X}{6}, \qquad x_{12} = (X, 0, 0)$$

$$y_{23} = \widehat{y}_{\text{med}} + \tfrac{X}{6}, \qquad x_{23} = (0, X, 0)$$

$$y_{31} = \widehat{y}_{\text{med}} + \tfrac{X}{2}, \qquad x_{31} = (X, 0, 0).$$

Note that the equilibrium exhibits the properties that each legislator is excluded from some decision and there are several ideological decisions that are possible. Also, the decisions are efficient for the two legislators in question in that X goes to the legislator who cares less about ideology, and Y lies between the peaks of the two legislators.

There are some other interesting things to note.

First, the ideological decisions are all described relative to $\widehat{y}_{\text{med}}$ (and for instance, shifting y_1 or y_3 while preserving the ordering of ideal points actually has no effect). Keeping all else constant, if we shift $\widehat{y}_{\text{med}}$ then the equilibrium shifts completely as is, without changes in the relative positions of each decision. This illustrates an anchoring effect of the median benchmark. In the absence of any X (or as we can see by letting $X \to 0$ in the equilibrium), the decision would be the median yhm. Thus all compromise occurs relative to that anchoring position, regardless of whether it is closer to the left or the right.

Second, the range of the ideological decisions (y_{ij}'s) increases as X increases. Larger X permits greater tradeoff and thus results in a larger dispersion of ideological choices. In the extreme, as X becomes very large, proposed y's would always be either $\widehat{y}_2$ or $\widehat{y}_3$, depending on the coalition.

Third, there is a 1/3 chance that the decision is made by the coalition comprising legislators 1 and 3, who are not ideologically adjacent. This particular coalition has a nice intuition: it is the legislator with the most intense ideological preferences proposing to the legislator with the least intense ideological preferences, who thus offers the best compromise, even though their ideological positions are the most extreme. An important thing to note here is that it is the expected continuation proposals that are the anchor for the bargaining. From 3's perspective, both legislators 1 and 2 are to the left of where 3 would like the outcome to be, and 3 will end up offering the full X to whomever he proposes to in order to get the most favorable Y position.[18]

Some aspects of the above equilibrium generalize to other cases and some do not. The full description of all cases is outlined in the appendix. In each case where $b_1 < b_2 < b_3$, 1 always proposes to 2. As 1 will always get the full X in such a proposal, 2 has less intense ideological preferences and offers a better y position for 1. More generally, however, 2 and sometimes 3 may mix over who they propose to. For 2 it is a choice of being the

[18] Here 3 would get exactly same outcome by proposing to 2 instead of 1, *given the current expectations*. However, if 3 proposed to 2 then 2's expectations would be higher and 3 would have to move y closer to $\widehat{y}_{\text{med}}$ to get 2's approval. So the only equilibrium has 3 proposing to 1.

relatively less ideological legislator in a proposal to 3, or the relatively more ideological legislator in a proposal to 1 - and in many cases in equilibrium 2 can be indifferent. Also, as mentioned above, 3 would like to propose to the cheaper of the two other legislators. It turns out that as b_2 increases relative to b_1, then 3 strictly prefers to propose to 1. However, as b_2 decreases, then 3 becomes indifferent and mixes in equilibrium.

Let us examine other cases to get a feel for the types of behavior that are possible in equilibrium.

Example 2: $b_1 = 1$, $b_2 = 5$, $b_3 = 6$.

Legislator 1 proposes to legislator 2, legislator 2 mixes over proposing to the other two, and legislator 3 proposes to legislator 1.

The specifics of this equilibrium are:

$$y_{12} = \widehat{y}_{\text{med}} - X\tfrac{36}{215}, \quad x_{12} = (X, 0, 0)$$

$$y_{21} = \widehat{y}_{\text{med}}, \quad x_{21} = (X\tfrac{25}{43}, X\tfrac{18}{43}, 0),\ a_{21} = \tfrac{4}{25}$$

$$y_{23} = \widehat{y}_{\text{med}} + X\tfrac{5}{43}, \quad x_{23} = (0, X, 0), \quad a_{23} = \tfrac{21}{25}$$

$$y_{31} = \widehat{y}_{\text{med}} + X\tfrac{18}{43}, \quad x_{31} = (X, 0, 0).$$

In this case, 2 mixes in equilibrium between proposing to 3 and proposing to 1. Note that 2's proposal to 1 involves a split of the X. This happens because 2 is able to set $y = \widehat{y}_{\text{med}}$ and so cannot improve on this dimension.

Note also that compared to equilibrium 1, having more intense ideological preferences has helped agent 2: the proposals are all better for 2 than the corresponding proposals in Example 1. The intuition for this is that the ideological dimension has become more important, moving the relative position of the y proposals closer to the median. The distribution of the X dimension is determined by the ordering of the ideological intensities, but not the exact intensity.

Example 3: $b_1 = 4$, $b_2 = 5$, $b_3 = 6$.

Legislator 1 always proposes to legislator 2, legislator 2 mixes over proposing to the other two, and legislator 3 mixes over proposing to the other two.

The specifics of this equilibrium are:

$$y_{12} = \widehat{y}_{\text{med}}, \quad x_{12} = (X\tfrac{70}{81}, X\tfrac{11}{81}, 0)$$

$$y_{21} = \widehat{y}_{\text{med}}, \quad x_{21} = (X(\tfrac{25}{81}), X\tfrac{56}{81}, 0), \quad a_{21} = \tfrac{4}{5}$$

$$y_{23} = \widehat{y}_{\text{med}} + X\tfrac{5}{81}, \quad x_{23} = (0, X, 0), \quad a_{23} = \tfrac{1}{5}$$

$$y_{31} = \widehat{y}_{\text{med}} + X\tfrac{11}{81}, \quad x_{31} = (X, 0, 0), \quad a_{31} = \tfrac{5}{9}$$

$$y_{32} = \widehat{y}_{\text{med}} + X\tfrac{11}{81}, \quad x_{32} = (0, X, 0), \quad a_{32} = \tfrac{4}{9}.$$

Note that in this case, both legislators 2 and 3 mix. Compared to Example 2, legislator 3 sees less advantage to proposing to legislator 1 as the ideological intensities of leglislators 1 and 2 are now closer, and is thus willing to mix over proposing to 1 or 2.

Example 4: $b_1 = 1$, $b_2 = 1.25$, $b_3 = 6$.

This turns out to be exactly the same as Example 3. This illustrates that it is only the relative intensity of 1 and 2's ideological preference that matters in equilibrium. This is true since 3 will never get any X in the proposals (given the intensity of his preferences) and so it is only the relative willingness to compromise of the other legislators that matters.

Examples where the ordering over b_1, b_2, and b_3 changes, offer similar insight.

Example 5: $b_1 = 5$, $b_2 = 4$, $b_3 = 6$.

Legislator 1 mixes over proposing to the other two, legislator 2 proposes to legislator 3 and legislator 3 mixes over proposing to the other two.

The specifics of this equilibrium are:

$$y_{12} = \widehat{y}_{\text{med}} - X\tfrac{7}{8}, \quad x_{12} = (0, X, 0), \quad a_{12} = \tfrac{3}{4}$$

$$y_{13} = \widehat{y}_{\text{med}} + X\tfrac{1}{8}, \quad x_{12} = (X, 0, 0), \quad a_{13} = \tfrac{1}{4}$$

$$y_{23} = \widehat{y}_{\text{med}} + X\tfrac{1}{8}, \quad x_{23} = (0, X, 0),$$

$$y_{31} = \widehat{y}_{\text{med}} + X\tfrac{7}{8}, \quad x_{31} = (X, 0, 0), \quad a_{31} = \tfrac{1}{2}$$

$$y_{32} = \widehat{y}_{\text{med}} + X\tfrac{7}{8}, \quad x_{32} = (0, X, 0), \quad a_{32} = \tfrac{1}{2}.$$

Note that this example is similar to Example 3 except that the intensity of legislators 1 and 2's preferences are reversed. Correspondingly the identity of which of these two mixes is reversed.

7 Political Parties

In the above examples, if two legislators were to get together before the legislative game and bind themselves to cooperate with each other, then they could strictly improve over what they expect in the equilibrium. Thus, the legislative game offers an explicit reason for legislators to try to form such binding agreements, and in turn provides an opportunity for political parties to offer improvement. This motivation for party formation was developed in a legislative bargaining context by Baron (1989 and 1991).[19] Let us explore what insight our model has to offer to the specifics of party formation.

Krehbiel (1993) provides an important set of issues to be analyzed in a model of party formation. His central point is that similarities in preferences or other motivating factors may naturally result in voting patterns that appear consistent with party behavior, independent of the party's existence. Thus, it is important to distinguish between "party-like" behavior and significant changes in behavior due to parties. The idea being that one might observe something which appears to be party behavior, but really matches the natural underlying equilibrium behavior of the individuals in any case. In such a situation, the fact that there exists something labelled a 'party' is a natural result of the incentives of individuals to vote similarly, and not something that externally affects the outcome.

Calvert and Dietz (1996) point out that while the Baron (1991) model is in principle well suited for an analysis of party and party-like behavior, it is close to intractable. Thus, Calvert and Dietz (1996) propose a variation on the Baron and Ferejohn (1989) model where they allow legislators to have preferences over the distributive allocations going to others as well as themselves. This turns out to be more tractable and in particular, Calvert and Dietz demonstrate a natural affinity for legislators who care about each other's allocations to vote and propose closely together. Thus, they provide a benchmark of party-like behavior, that consistent with Krehbiel's thesis, appears consistent with a party organization even though there are no binding forces.

Calvert and Dietz (1996) do not model parties, and thus do not contrast party behavior with party-like behavior, but they outline a method for doing so in future research. Their suggestion is to examine non-stationary equilibria in the context of their game, and use that as the model of a party organization. To the contrary, we argue that this would not be a model of party behavior, but again a model of party-like behavior. A non-stationary equilibrium, just like a stationary equilibrium, is entirely self-enforcing. There is no outside force necessary to induce players to follow the prescribed behavior. If one believes that non-stationary equilibria are plausible, then no party is needed—

[19]The underlying ideas have a rich history and are in part the motivation for the early literature on cooperative game theory which was tied to what cooperating players could earn in non-cooperative games, again dating to von Neumann and Morgenstern. In the political science literature, there are a variety of motivations for party formation of which some representative references are the classics by Duverger (1954) and Riker (1962), and recent analyses of the role of parties in legislative contexts such as Cox and McCubbins (1993) and Kiewiet and McCubbins (1991). Most relevant for our analysis are papers by Krehbiel (1993) and Calvert and Dietz (1996).

simply knowledge of the equilibrium strategies themselves. Moreover, the non-stationary behavior needed to establish folk-theorem like results is often recursive. Thus, we argue against the idea that non-stationary equilibrium should be interpreted as party behavior, and stationary equilibrium play should be interpreted as party-like behavior.

Instead, our approach to modeling the party as an influence on the legislative game is to treat the party as an organization that is external to the game and through rewards and punishments can enforce behavior that would otherwise not be observed in the game. Thus, we stick with stationary equilibrium as the benchmark for the behavior in the game in the absence of any external force, and compare it to a situation where an external force is present. As an example, by controlling things such as committee appointments and campaign funding (or intangeables like brand-recognition and reputation as suggested in Kiewiet and McCubbins (1991)) a political party can serve as a device that binds legislators to following certain strategies, which they would otherwise not choose to. This can in turn improve their legislative performance.

One might reply to this view by saying that if one enlarges the scope of the model to be a much larger game including committee formation, campaigning, and elections, etc., then the party is again simply an equilibrium phenomenon and does not differ from what players naturally would like to do. With this we would have to agree. However, this is a version of Krehbiel's point taken absurdum. It is useful to distinguish the behavior of political players inside institutions such as a legislature or an election where there are well defined rules which lend naturally to equilibrium modeling techniques, from the general political arena. We argue that viewing a political party as an organization that operates in the general political arena, and has an ability to influence play inside a legislative body, is a reasonable and productive modeling choice.

Let us now reexamine the examples in the last section in the context of party formation.

A party is modeled as a binding agreement among the members of the party to act as one player in the legislative game. This generally poses difficult questions about how to model the decision of the party regarding how to act in the legislative game. As a first step in this direction, restricting our attention to the setting of the examples in the last section, however, considerably simplifies things and results in natural ways to model party behavior.

Consider two legislators who are forming a party. The party will generate benefits for them relative to the legislative game without any parties. They must split these benefits in some way. For instance, they could choose a platform that would give all the benefits to the first legislator and leave the second indifferent between being in the party or playing the straight legislative game. Similarly, they could reverse this, or they could end up somewhere in between. How these relative benefits are split implicitly determines the behavior of the party (and vice versa). So let us analyze this splitting.

The feasible pairs of gains in utility relative to the disagreement outcome (of the game without any parties) form a nice closed and convex set, as in the classical bargaining analysis of Nash. We use the Nash bargaining solution as a prediction for the surplus split among the party members.[20] In the case where the possible Pareto optimal utility combinations for the party form a linear frontier, this ends up giving each of the two party members half of their maximal (potential) surplus relative to threat point of no party.

Now, given an idea of how a given party will act, and what utilities it will generate for its members, we can step back and ask which parties are most likely to form. In particular, we can look for a (core) stable party structure such that neither legislator in the resulting party would rather be in another party.

In particular, say that a party is *stable* if there is no other majority party for which all of its members are strictly better off. This definition applies for the case of 3 legislators, but can be extended to more in the natural way, allowing for the possibility of multiple (and minority) parties.

Let us illustrate this approach in some detail by looking back at the examples of the previous section.

Example 1 Revisited: $b_1 = 1$, $b_2 = 3$, and $b_3 = 6$.

Recall that legislator 1 proposes to legislator 2, legislator 2 proposes to legislator 3, and legislator 3 proposes to legislator 1, and:

$$y_{12} = \widehat{y}_{\text{med}} - \tfrac{X}{6}, \qquad x_{12} = (X, 0, 0)$$

$$y_{23} = \widehat{y}_{\text{med}} + \tfrac{X}{6}, \qquad x_{23} = (0, X, 0)$$

$$y_{31} = \widehat{y}_{\text{med}} + \tfrac{X}{2}, \qquad x_{31} = (X, 0, 0).$$

The expected utilities from the equilibrium are:

$$v_1 = -\widehat{y}_{\text{med}} + X\tfrac{1}{2}$$

$$v_2 = -X\tfrac{1}{2}$$

$$v_3 = -6 + 6\widehat{y}_{\text{med}} + X$$

The following observation is helpful in mapping out the range of possible utility combinations that a party can generate for its members, and holds more generally.

[20] We do not model the explicit bargaining that goes on within a party. In the case where the frontier is linear, most extensive form bargaining procedures and bargaining solutions coincide (as $\delta \to 1$) with the Nash solution in any case.

Observation 1: If in the simple equilibrium of the game without parties i makes a proposal to j, then the maximum (potential) surplus that i can get from a party with j subject to j weakly preferring the party to the equilibrium outcome, is the same as the surplus that i gets from this proposal over the expected continutation.

Observation 1 is based on the fact that in an equilibrium, when a legislator gets to propose, he will choose a proposal that maximizes his surplus, subject to making another legislator indifferent between voting yes or no. This is the most that the legislator could get from a party, subject to both members being above the disagreement point and helps to define the possibility set for the party.

Let us now return to analyzing the example with Observation 1 in hand.

Let us first analyze the potential party of legislators 2 and 3.

Given observation 1, to identify the maximum potential surplus that could go to legislator 2, we can examine the utility from the proposal y_{23}, x_{23} compared to the expected equilibrium continuation. Here $u_2(y_{23}, x_{23}) = X\frac{1}{2}$, so $u_2 - v_2 = X$. So if the party acted purely in legislator 2's interest, subject to legislator 3 being as well off as without the party, then it would improve legislator 2's payoff by X. To find out the most that party $\{2,3\}$ could do for legislator 3 (subject to being above disagreement for legislator 2), we find the best proposal for 3 subject to $u_2 \geq v_2$. In this case, a simple calculation shows that $y = \widehat{y}_{\mathrm{med}} + \frac{X}{2}$, $x = (0, X, 0)$ is the best. This results in $u_3 = -6 + 6\widehat{y}_{\mathrm{med}} + 3X$. So $u_3 - v_3 = 2X$.

Thus, the range of possible utility combinations that the party $\{2,3\}$ can generate relative to the disagreement point of the stationary equilibrium without parties are pictured in Figure 1. Applying, the Nash bargaining solution to this set results in a surplus of $\frac{X}{2}$ going to legislator 2, and X to legislator 3. This corresponds to having the party $\{2,3\}$ make the proposal[21] $y = \widehat{y}_{\mathrm{med}} + \frac{X}{3}$, $x = (0, X, 0)$, and approving it.

Next we can analyze the potential party $\{1,3\}$. Very similar calculations result in $\frac{X}{6}$ going to legislator 1, and X to legislator 3. The party $\{1,3\}$ would make the proposal $y = \widehat{y}_{\mathrm{med}} + \frac{X}{3}$, $x = (X, 0, 0)$, and approve it in equilibrium. Note that the proposal is the same (except for X going to 1 instead of 2) as that in the party $\{2,3\}$, because 3's position is the constraining one in each of the party.

Finally, let us analyze the potential of party $\{1,2\}$. The calculations of the frontier are a bit more complicated in this case. First, using Observation 1, we can calculate the maximal potential surplus for legislator 1, which is $\frac{2}{3}X$, obtained when $y = y_{12}$, $x = x_{12}$. For player 2, the best that the party could do subject to 1 being as well off as in disagreement is to propose, $y = \widehat{y}_{\mathrm{med}}$, $x = (X\frac{1}{2}, X\frac{1}{2}, 0)$, which results in a maximal potential surplus for legislator 2 of X. However, in this case the possibility frontier is not

[21] It is also possible to let the party randomize, with not changes in expected action or utility.

linear. The frontier of utility combinations that the party $\{1,2\}$ can generate is found by moving efficiently between $y = y_{12} = \widehat{y}_{\text{med}} - \frac{1}{6}X$, $x = x_{12} = (X, 0, 0)$ and $y = \widehat{y}_{\text{med}}$, $x = (\frac{1}{2}X, \frac{1}{2}X, 0)$. There is a kink at the point $\widehat{y}_{\text{med}}$, $(X, 0, 0)$, as player 3 moving y down from that point results in incremental losses to u_2 at the rate $b_3 = 3$, while moving x_2 up from that point increases u_2 only at a rate of 1. This is pictured in Figure 2. Nash bargaining in this case leads to the surplus combination of $\frac{1}{2}X$ to legislator 1, and $\frac{1}{2}X$ to legislator 2, which corresponds to the proposal $\widehat{y}_{\text{med}}$, $(X, 0, 0)$.[22]

From these calculations, it follows that 1 strictly prefers party $\{1,2\}$ to $\{1,3\}$, 2 is indifferent between parties $\{1,2\}$ and $\{2,3\}$, and 3 is indifferent between parties $\{1,3\}$ and $\{2,3\}$.

Relative to these preferences there are two stable parties: $\{1,2\}$ and $\{2,3\}$. The party $\{1,3\}$ is not stable as legislator 1 would prefer to change and form party $\{1,2\}$.

Note that party behavior is distinguished from the non-party (or party-like) equilibrium behavior in terms of both actions and resulting utilities.

Next, let us reexamine Example 2 to see that the stable party structure depends on the particulars of the situation.

Example 2 Revisited:

Based on a similar set of calculations as those described above, it follows that 1 strictly prefers party $\{1,2\}$ to $\{1,3\}$, 2 strictly prefers party $\{1,2\}$ to $\{2,3\}$, and 3 is strictly prefers party $\{1,3\}$ to $\{2,3\}$.

Relative to these preferences there is a unique stable party: $\{1,2\}$. Both legislators 2 and 3 have relatively strong ideological positions and would rather form a party with legislator 1 than with each other. Legislator 1 gets more out of forming a party with legislator 2 that is the unique party outcome.

Example 5 Revisited:

In this example simple calculations show that 1 is indifferent between parties $\{1,2\}$ and $\{1,3\}$, 2 strictly prefers party $\{2,3\}$ to $\{1,2\}$, and 3 is indifferent between parties $\{1,3\}$ and $\{2,3\}$,

Relative to these preferences both parties $\{1,3\}$ and $\{2,3\}$ are possible stable party structures.

[22] In this case, no mixing is possible for the party given the kink at this point in the frontier, and so this behavior is unique.

Most importantly, this example points out that it is possible for a party with disjoint ideological positions, such as $\{1,3\}$ to be stable. Moreover, this is not simply in a situation where all parties are stable as party $\{1,2\}$ is not stable.

Note, however, that the proposal that party $\{1,3\}$ would approve would be $y = \widehat{y}_{\text{med}} + \frac{1}{2}X$, $x = (X,0,0)$, which differs from the simple equilibrium play in the absence of a party, but is not an extreme position.

Generally, although parties have an impact and change predicted behavior relative to an equilibrium without parties, that change is constrained by the comparisons to the disagreement simple equilibrium outcome implicit in the formation of the party platform.

8 Concluding Remarks

In this paper we have proposed a model of legislative bargaining that extends the Baron and Ferejohn (1989) model to include ideological considerations in a natural and tractable way. These considerations are essential to developing a working of legislative decision making, as well as understanding the workings of a legislature in a broader political context. With this in mind, we have provided a series of results characterizing equilibrium behavior.

We have also used the model for some simple examinations of the roles and formation of political parties. As seen in the examples in the last section, sharp predictions concerning the constituency and position of stable political parties can be made in such a model, and these depend on the particular ideological positions and intensities of the legislators. This suggests that the methodology we have outlined here will be useful in further analysis of the roles of political parties in legislative contexts.

Figure 1

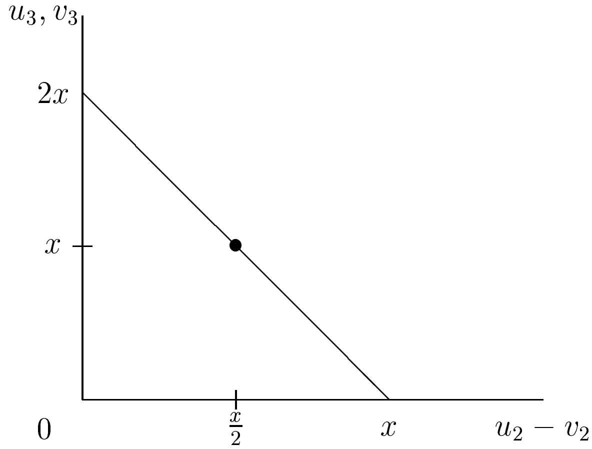
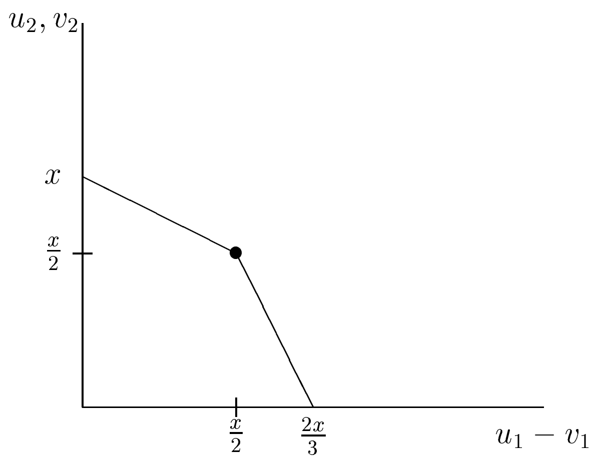


Figure 2



Appendix

Proof: **Proof of Benchmark 3:**
The first part of the proposition is straightforward.

Consider the second part. Note that in a stationary equilibrium, either with probability 1 the game never terminates, or else with probability 1 the game terminates at some finite date. To see this let a be the probability that the game terminates in the first session. If $a = 0$, then given stationarity the game never terminates. If $a > 0$, then given stationarity the probability that the game lasts at least t sessions is $(1-a)^t$ which converges to 0 as t becomes large.

Consider the possibility that in equilibrium no proposal is ever approved and the default y^0 is the outcome with probability 1. If $y^0 = \widehat{y}_{\text{med}}$, then the conclusion of the proposition is true. If $y^0 < \widehat{y}_{\text{med}}$ (the case $y^0 > \widehat{y}_{\text{med}}$ is analagous), then when the median legislator is recognized and proposes $\widehat{y}_{\text{med}}$ it will be approved by a majority (given the continuation expectation of y^0 which must be worse for everyone to the right of (and including) the median). Thus, the median should deviate and propose $\widehat{y}_{\text{med}}$ which is a contradiction.

Next, consider the possibility that some proposal is made and approved with positive probability. Let y_{min} be the lowest y and y_{max} be the highest y which is approved with positive probability in equilibrium.[23]

First, let us examine the case where $y_{min} \neq y_{max}$. Note that $y_1 \leq y_{min}$ (and similarly $y_{max} \leq y_n$), since otherwise y_1 could be proposed in place of y_{min} and strictly preferred by all, which contradicts the fact that y_{min} is part of an equilibrium. Given that $\frac{n+1}{2}$ legislators must approve a proposal to have it be an outcome, there exists a legislator i who votes 'yes' with positive probability for both of these proposals. Let v_i denote i's expected continuation utility from the equilibrium. This means that $u_i(y_{min}) \geq v_i$ and $u_i(y_{max}) \geq v_i$. Given the single peaked preferences, this implies that $u_i(y) \geq v_i$ for any y that is approved in equilibrium, and so by the definition of v_i it follows that $u_i(y_{min}) = v_i = u_i(y_{max})$. It follows that $y_{min} < \widehat{y}_i < y_{max}$. It also follows that with probability 1, one of these two proposals is approved (i.e., no other proposals are approved since otherwise one of y_{min} and y_{max} would give i lower utility than the continuation). Now we show that one of the two proposals is $\widehat{y}_{\text{med}}$. Suppose the contrary and, without loss of generality, that $u_{\text{med}}(y_{max}) \geq u_{\text{med}}(y_{min})$. If the median proposes $\epsilon\widehat{y}_{\text{med}} + (1-\epsilon)y_{max}$

[23] These are well defined. There are at most two proposals that a player finds indifferent to the continuation, and then an interval over which they strictly prefer the proposal to the continuation, and (at most) two intervals over which they strictly prefer the continuation to the proposal. Thus, there are a finite set of intervals of proposals that will win the approval of a majority. An approved proposal must maximize the expected utility of the proposer subject to being approved and there are a finite number of solutions to this.

for small positive ϵ it will be accepted. (For the case where $y_{max} < \widehat{y}_{\text{med}}$ legislators above the median strictly prefer the new proposal to the continuation. For the case where $y_{max} > \widehat{y}_{\text{med}}$ note that legislators whose peaks are at least as large as y_{max} must have $u_j(y_{max}) > v_j$ and so for small ϵ it follows that $u_j(\epsilon\widehat{y}_{\text{med}} + (1-\epsilon)y_{max}) > v_j$. Others who were voting 'yes' for y_{max} but have lower peaks below y_{max} are made better off by the change and will still vote 'yes'.) This is a contradiction, and so the supposition was wrong. Thus, say $y_{max} = \widehat{y}_{\text{med}}$. It then follows that y_{min} should never be approved, since a majority (the median and above) prefer $y_{max} = \widehat{y}_{\text{med}}$ to $y_{min} < \widehat{y}_{\text{med}}$. Thus, there cannot be an equilibrium such that $y_{min} \neq y_{max}$.

Next, let us examine the case where $y_{min} = y_{max}$. If $y_{min} \neq \widehat{y}_{\text{med}}$, then when the median legislator is recognized the proposal $\widehat{y}_{\text{med}}$ would be approved by a majority (given the stationary continuation expectation of $y_{min} = y_{max}$ which must be worse than $\widehat{y}_{\text{med}}$ for a majority). Thus, there cannot be an equilibrium where $y_{min} = y_{max}$. We are left with the case that $y_{min} = y_{max} = \widehat{y}_{\text{med}}$. ∎

Proof: **Proof of Lemma 6:**
If $\overline{\delta} < 1$, then to verify that a set of stationary strategies is a subgame perfect equilibrium, one needs only examine single stage deviations (fixing the continuation expectation), and so the conclusion of the lemma follows from standard arguments for upper-hemi continuity of an equilibrium correspondence. (That is, given the continuity of $\overline{u}_i$ (and $\overline{v}_i$) it is clear that $\overline{\ell}$ is still an equilibrium in the first session given that agents expect $\overline{\ell}$ to be the continuation.) For the case where $\overline{\delta} = 1$ we also need to check that there is no agent who can deviate in an infinite number of sessions and be made better off. For this to be possible, it must be that there is some i who is included in the proposal of every $j \neq i$ so that i can force the game to last an infinite number of sessions by vetoing every proposal. We can follow the same steps as the proof of Proposition 4 for $\overline{d} = 1$ (see below) to show there is a positive probability that some proposal is made and approved which excludes i, and so given the stationarity of strategies, the game will end in finite time regardless of i's actions. ∎

Proof: **Proof of Proposition 4:**
We begin with the case of $\delta = 1$. Let v_j denote the expected utility of legislator j in the equilibrium. Suppose to the contrary that i is included in (almost every) proposal. Thus, $u_i(d) \geq v_i$ for (almost every) d which is made and approved in equilibrium. This implies that $u_i(d) = v_i$ for almost every d.

Let $\overline{d} = (\overline{y}, \overline{x}_1, \ldots, \overline{x}_n)$ denote the expected proposal. By concavity $u_j(\overline{d}) \geq v_j$ for all j. Suppose that $\overline{x}_j > 0$ for some $j \neq i$. Then i can propose $(\overline{y}, x_1, \ldots, x_n)$ where $x_k = \overline{x}_k + \frac{\overline{x}_j}{n_1}$ for $k \neq j$. This will be approved and makes i strictly better off. which is a contradiction. Thus $\overline{x}_j = 0$ for all $j \neq i$ and so $\overline{x}_i = X$.

Since $u_i(d) = v_i$ and $x_i = X$ for (almost every) d, it follows from single peaked preferences that either $y = \widehat{y}_i$ for every d or that there are exactly two decisions that are approved in equilibrium with corresponding y_- and y_+. Consider the second case first and label the the decisions d_- and d_+ where, without loss of generality, $y_- < \widehat{y}_i < y_+$. Suppose that $i \leq$ med (the other case is analagous). Consider C such that $j \in C$ have $u_j(d_+) \geq v_j$. For small ϵ the proposal $(y_+ - \epsilon, \overline{x}_1, \ldots, \overline{x}_n)$ would be approved by all $j \in C$ (the only j to worry about are those with $u_j(d_+) = v_j$, but this implies that also $u_j(d_-) = v_j$ and so $y_- < \widehat{y}_j < y_+$). This is a contradiction. So consider the other case where $y = \widehat{y}_i$ and $x_i = X$ for every approved proposal. Any other agent could make proposal $(\widehat{y}_i, \frac{X}{n}, \ldots, \frac{X}{n})$ which would be approved by a majority and strictly improving for that majority, which is a contradiction. Thus our original supposition was wrong.

The existence of $\overline{d} < 1$ with the claimed properties now follows from Lemma 1. ∎

Proof: **Proof of Proposition 2:**
First, consider the case where $\delta < 1$.

We provide an outline of the proof. We first represent the legislators' proposal strategies (which have finite support) by a vector ℓ listing probabilities and proposals for each legislator. Second, we define the set of proposals a legislator i might make which would be approved by a majority given that they expect ℓ to describe the continuation strategies (and they expect this to be approved). We denote this set by $A_{-i}(\ell)$ and show that it is a continuous correspondence which is nonempty and compact valued. We then define another correspondence mapping ℓ into best responses for each legislator - where each legislator chooses proposals from $A_{-i}(\ell)$ to maximize u_i. So, given ℓ we obtain a set of best proposals that i can make and have approved given the expected continuation ℓ. This correspondence is denoted $A_i^*(\ell)$ and is upper-hemicontinous and nonempty, compact and convex valued and thus has a fixed point. We then verify that this fixed point represents the proposals each legislator makes which together with proper approval strategies form the equilibrium.

As it may be that i randomizes over the C's that he or she proposes to, we keep track of the strategy of proposals that i makes by a vector $(d_{iC}, \pi_{iC})_C$, where d_{iC} is a proposal that i makes and is approved by legislators in C (and perhaps others as well) and π_{iC} is the probability that this proposal made and $\sum_C \pi_{iC} = 1$.[24] (See the proof of Proposition 3 to verify that in equilibrium with probability 1 each i will make a proposal that is approved.) Let L be the set of vectors $(\ldots, d_{iC}, \pi_{iC}, \ldots) \in (D \times [0,1])^{nM}$, such that $\sum_C \pi_{iC} = 1$ for each i. For $\alpha \in [0,1]$, $\ell = (\ldots, d_{iC}, \pi_{iC}, \ldots) \in L$ and $\widetilde{\ell} = (\ldots, \widetilde{d}_{iC}, \widetilde{\pi}_{iC}, \ldots) \in L$, let $\overline{\ell} = a\ell + (1-a)\widetilde{\ell}$ be the vector with entries $\overline{d}_{iC} = \frac{a\pi_{iC}d_{iC} + (1-a)\widetilde{\pi}_{iC}\widetilde{d}_{iC}}{a\pi_{iC} + (1-a)\widetilde{\pi}_{iC}}$ (provided $a\pi_{iC} + (1-a)\widetilde{\pi}_{iC} > 0$, and $\overline{d}_{iC} = ad_{iC} + (1-a)\widetilde{d}_{iC}$ otherwise) and $\overline{\pi}_{ic} = a\pi_{iC} + (1-a)\widetilde{\pi}_{iC}$.[25]

[24] Although in some cases i may choose to randomize over an infinite number of proposals, we show that there always exists an equilibrium where at most a finite number of proposals are approved and so this construction will suffice.

[25] This definition of a convex combination of elements in L ensures that if $\pi_{iC} = 0$ and $\widetilde{\pi}_{iC} > 0$ then the new proposal to C is still $\widetilde{d}_{iC}$, as it should be.

For any $\ell \in L$, let $E[\ell] \in D$ be defined by

$$E[\ell] = \sum_i p_i \left(\sum_C \pi_{iC} d_{iC} \right),$$

$$v_j(\ell) = \sum_i p_i \left(\sum_C \pi_{iC} u_j(d_{iC}) \right)$$

and

$$A_i(\ell) = \{d \in D \ : \ u_i(d) \geq \delta v_i(\ell)\}.$$

The set $A_i(\ell)$ is the set of acceptable decisions for i (ones that i will approve) when i expects ℓ to describe the continuation. Note that the correspondence $A_i(\ell)$ is nonempty, compact valued, and continuous in ℓ. Nonemptiness follows since by concavity $u_i(E[\ell]) \geq \delta v_i(\ell)$ and so $E[\ell] \in A_i(\ell)$. Compactness follows from the fact that $A_i(\ell)$ is closed (by the continuity of u_i) and a subset of a compact space. Continuity follows from the continuity of u_i and v_i, and the fact that u_i is locally non-satiated except at $y = \widehat{y}_i$ and $x_i = X$.[26] Let

$$A_C(\ell) = \cap_{j \in C} A_j(\ell)$$

and

$$A_{-i}(\ell) = \cup_{\{C: i \notin C, \#C \geq \frac{n-1}{2}\}} A_C(\ell).$$

Thus, $A_C(\ell)$ is the set of decisions that would be approved by a group C, and $A_{-i}(\ell)$ is the set of decisions that i can make that would be approved by at least $(n-1)/2$ other agents. Note that $A_{-i}(\ell)$ is nonempty, compact valued, and continuous in ℓ. Nonemptiness follows from the fact that $E[\ell] \in A_j(\ell)$ for each $j \neq i$. Compactness follows from the compactness of each A_j and the finite number of unions and intersections. To establish continuity, since upper hemicontinuity follows from the continuity of each A_j, we need only check lower hemicontinuity of each $A_C(\ell)$ (since lower hemicontinuity is preserved under unions, but not necessarily under intersections). Consider $\ell^n \to \ell$ and $d \in A_C(\ell)$. We show that $\exists d^n \in A_C(\ell^n)$ such that $d^n \to d$ along a subsequence, to establish the claim. Since $\delta < 1$ it follows that $u_j(E[\ell]) > \delta v_j(\ell)$, and so $u_j(E[\ell]) > \delta v_j(\ell^n)$ for large enough n. Let $d^k = a^k E[\ell] + (1 - a^k)d$, where $a^k \to 0$. It follows that for each k we can find n_k such that $u_j(d^k) > \delta v_j(\ell^n)$ for all $j \in C$ and $n \geq n_k$. Choosing successively larger n_k for each d^k provides the required subsequence.

Let $A_i^*(\ell) = \text{argmax}_{d \in A_{-i}(\ell)} u_i(d)$. By Berge's Maximum Theorem, $A_i^*(\ell)$ is nonempty, compact valued, and upper-hemicontinuous. Finally, let $H(\ell) = \{\widetilde{\ell} \in L : \widetilde{\pi}_{iC} > 0 \Rightarrow \widetilde{d}_{iC} \in A_i^*(\ell)\}$. Thus, H maps a given expectated continuation ℓ into the best response strategy profiles, where each legislator i is choosing a best proposal subject to winning approval (or randomizing over a finite number of such proposals). By the properties of each A_i^*, it follows that H is nonempty, compact valued, and upper-hemicontinuous. Also note that by the concavity of u_i, $H(\ell)$ is convex valued.[27] Thus, there exists a fixed point of H by Kakutani's fixed point theorem.

[26] This rules out 'thick' indifference sets (with nonempty interiors) and establishes lower hemicontinuity.

[27] Each $A_C(\ell)$ is convex valued and so if d and d' both maximize u_i over $A_C(\ell)$ then so does $ad+(1-a)d'$ and so by the definition of L the convexity of $H(\ell)$ follows.

Let $\ell = (\ldots, d_{iC}, \pi_{iC}, \ldots)$ be such a fixed point. Define stationary strategies as follows: When i is recognized, i makes proposal d_{iC} with probability π_{iC}. Note that for any i, each d_{iC} such that $\pi_{iC} > 0$ is such that $u_i(d_{iC}) > \delta v_i(\ell)$ since i could make propsal $E[\ell]$, and so $u_i(d_{iC}) \geq u_i(E[\ell]) > \delta v_i(\ell)$. When i is called on to vote for a proposal d, i approves d if and only if $u_i(d) \geq \delta v_i(\ell)$. By the construction of A_i^* these constitute a stationary equilibrium, as given $\delta < 1$ and the stationarity we need only to check that no agent wants to deviate in the first session.[28]

Next, consider the case where $\delta = 1$. We verify that there exists an equilibrium by verifying that a sequence of equilibria corresponding to discount factors $\delta^k \to 1$ has a convergent subsequence which is an equilibrium when $\delta = 1$. This follows from Lemma 6. ∎

Proof: **Proof of Proposition 1**
The case of $\delta < 1$ is straightforward, so consider the case of $\delta = 1$. Note that some agent can make a proposal that would be better for a majority than the status quo, so the stationary equilibrium must have some proposal approved with positive probability. If the equilibrium has only dimensions approved together, then it must end in the first session, as Proposition 4 then applies. So, suppose that there is a positive probability that a proposal is approved which involves only one dimension.

Case 1: The dimension is Y. Let the proposer be i and the proposal y. Let C be the set of legislators voting 'yes'. The continuation expectation has some distribution over x_j's and some expected $\overline{x}_j$, where $\sum_j \overline{x}_j = 1$ and $\overline{x}_j > 0$ for all j. (The game is then essentially a Baron and Ferejohn game, and so all agents must have a positive payoff in equilibrium.) Legislator i can make all agents in C better off by proposing y, x where $x_j = \overline{x}_j + \frac{\sum_{k \notin C \cup i} \overline{x}_k}{(n+1)/2}$ for $j \in C \cup i$, which would be approved, which is a contradiction.

Case 2: The dimension is X. the continuation expectation is for $\widehat{y}_{\text{med}}$ to be approved.

Thus, if a proposal of a single dimension is approved, it must be the distributive dimension and then an expectation of $\widehat{y}_{\text{med}}$ for the future. Note that in this case, there is still some agent excluded from the X split and so the approval must be in the first period (by the same reasoning as Propositions 4 and 3). ∎

Proof: **Proof of Proposition 5:**
Suppose that every proposal has the same y approved. Let $\widehat{x}_i$ be the expected private good that each legislator gets in equilibrium. By Benchmark 1 it follows that $\widehat{x}_i = \frac{X}{n}$. Suppose without loss of generality that $y \leq \widehat{y}_{\text{med}}$. Consider the case where $y < \widehat{y}_{\text{med}}$ and consider a coalition C of legislators with $\widehat{y}_i \geq \widehat{y}_{\text{med}}$, and some $j \in C$. Agent j must be making an approved proposal with y and $x_j = X - \delta \frac{(n-1)X}{2n}$ in equilibrium.[29] Instead, j

[28] The single deviation property holds according to standard arguments since $\delta < 1$.

[29] This follows from a standard argument in bargaining games: By offering y and slightly more than $\delta \frac{(n-1)X}{2n}$ to $\frac{n-1}{2}$ other legislators, the proposal would be approved and would offer j a utility higher than

can propose $\widehat{y}_{\text{med}}$ and $x_i = \delta\frac{X}{n}$ for $i \in C$, $i \neq j$, which would be approved since all $i \in C$ would be better off than in the continuation, which is a contradiction. Thus, $y = \widehat{y}_{\text{med}}$.

So, consider C from the definition of preferences that admit local compromise. Find ϵ such that $|\overline{u}_i(y) - \overline{u}_i(\widehat{y}_{\text{med}})| < \delta\frac{X}{n}$ for each $i \in C$ and y such that $|y - \widehat{y}_{\text{med}}| < \epsilon$. Pick some $j \in C$, and consider the case where j is called on to propose. Agent j must be making an approved proposal with $y = \widehat{y}_{\text{med}}$ and $x_j = X - \delta\frac{(n-1)X}{2n}$. Since preferences admit local compromise, it follows that there exists $|y - \widehat{y}_{\text{med}}| < \epsilon$, such that $\sum_{i\in C} \overline{u}_i(y) > \sum_{i\in C} \overline{u}_i(\widehat{y}_{\text{med}})$. Let $\gamma_i = \overline{u}_i(y) - \overline{u}_i(\widehat{y}_{\text{med}})$. It follows that from our choice of ϵ above, that $|\gamma_i| < \delta\frac{X}{n}$. Let j propose y and $x_i = \delta\frac{X}{n} - \gamma_i + \alpha$ for each $i \in C$, $i \neq j$. This leaves each $i \in C$, $i \neq j$ better off than the continuation for positive α. It also follows that for small enough α

$$\overline{u}_j(y) + X - \sum_{i\in C, i\neq j} x_i > \overline{u}_j(\widehat{y}_{\text{med}}) + X - \delta\frac{(n-1)X}{2n}.$$

This is a contradiction, because by deviating and proposing the suggested y and x_i's, the proposal would be approved and j would be better off. ∎

Proof: **Proof of Proposition 3:**
For the case of $\delta < 1$ this is clear since any proposer has the potential to make a proposal that makes all agents better off than the expected continuation. For the case where $\delta = 1$, this follows from Proposition 4 and the fact that any proposer thus has the potential to make a proposal that makes some exact majority (including him or herself) strictly better off than the continuation.

Let us verify that X is distributed to an exact majority in an approved decision. Suppose to the contrary. Consider an exact majority C, including the proposer, that weakly prefers the approved decision y, x to the continuation. (Such a group exists, or the sequential voting in a subgame perfect equilibrium would have led to disapproval of the proposal.) The proposer could make a proposal that makes all legislators in C strictly better off by proposing y and $\widehat{x}$ where $\widehat{x}_j = x_j + \frac{\sum_{i\notin C} x_i}{(n+1)/2}$ for $j \in C$ and $\widehat{x}_j = 0$ for $j \notin C$. This would be approved, and so the proposer's decision could not have been a best response which is a contradiction.

Let us check that the equilibrium is independent of the default. Consider any stationary equilibrium. Given the first point, every recognized proposer must be making a proposal that is approved in the first session. Change the default decision. This is still an equilibrium: given the stationary strategies the continuation expectation is the same. There are no infinite strategy deviations by any single legislator that could lead

the continuation. Thus, j can make a proposal that would be accepted and offer j a utility higher than the continuation, and so j must be making some approved proposal in equilibrium or there would be an improving deviation. The only candidate for such a proposal is to have j offering exactly $\delta\frac{(n-1)X}{2n}$ to the other agents, since relative to any proposal where j offers more, j could do better by offering slightly less and still have it approved.

to the default, since with probability 1 the game will end in finite time independent of the deviation (as the legislator is excluded from at least one approved proposal). ∎

Characterization of Equilibria for Examples 1 to 4:[30]

Case 1: $b_2 \geq 3b_1$.

In this case, legislator 1 always proposes to legislator 2, legislator 3 always proposes to legislator 1, and legislator 2 mixes over proposing to the other two.

Let y_{ij}, x_{ij} denote the decision proposed by i when proposing to j, and a_ij denote the probability that i proposes to j.

The specifics of this equilibrium are:

$y_{12} = \widehat{y}_{\text{med}} - X\frac{3(b_2)^2-3(b_1)^2}{b_2 A}$, $x_{12} = (X, 0, 0)$

$y_{21} = \widehat{y}_{\text{med}}$ $x_{21} = (X(\frac{5b_1b_2+(b_2)^2}{A}), X\frac{6b_2b_1+6(b_1)^2}{A}, 0)$ $a_{21} = \frac{3(b_1)^2-4b_1b_2+b_2^2}{b_2(5b_1+b_2)}$

$y_{23} = \widehat{y}_{\text{med}} + X\frac{b_2+5b_1}{A}$ $x_{23} = (0, X, 0)$ $a_{23} = 1 - a_{21}$

$y_{31} = \widehat{y}_{\text{med}} + X\frac{6b_2+6b_1}{A}$ $x_{31} = (X, 0, 0)$

where $A = 6(b_1)^2 + 11b_1b_2 + (b_2)^2$

Case 2: $\frac{1+\sqrt{5}}{2}b_1 \leq b_2 < 3b_1$.

Legislator 1 always proposes to legislator 2, legislator 3 mixes over proposing to the other two, and legislator 2 mixes over proposing to the other two.

The specifics of this equilibrium are:

$y_{12} = \widehat{y}_{\text{med}} - X\frac{6(b_2)^2-6b_1b_2-6(b_1)^2}{b_2 A}$, $x_{12} = (X, 0, 0)$

$y_{21} = \widehat{y}_{\text{med}}$ $x_{21} = (X(\frac{4b_1b_2+2(b_2)^2}{A}), X\frac{2b_1b_2+6(b_1)^2}{A}, 0)$ $a_{21} = \frac{3(b_1)^2+2b_1b_2-(b_2)^2}{b_2(2b_1+b_2)}$

$y_{23} = \widehat{y}_{\text{med}} + X\frac{2b_2-4b_1}{A}$ $x_{23} = (0, X, 0)$ $a_{23} = 1 - a_{21}$

$y_{31} = \widehat{y}_{\text{med}} + X\frac{8b_2+6b_1}{A}$ $x_{31} = (X, 0, 0)$

$y_{32} = \widehat{y}_{\text{med}} + X\frac{8b_2+6b_1}{A}$ $x_{32} = (0, X, 0)$

where $A = 6(b_1)^2 + 12b_1b_2 + 2(b_2)^2$

[30] For these to be equilibria it must be that X is small enough so that $0 \leq y_{ij} \leq 1$ for each ij. Corresponding bounds on X follow directly from the given expressions.

Case 3: $b_2 < \frac{1+\sqrt{5}}{2} b_1$.

Legislator 1 always proposes to legislator 2, legislator 3 mixes over proposing to the other two, and legislator 2 mixes over proposing to the other two.

The specifics of this equilibrium are:

$y_{12} = \widehat{y}_{\text{med}}$, $x_{12} = (X\frac{b_1 b_2 + 2(b_2)^2}{(b_1+b_2)^2}, X\frac{(b_1)^2 + b_1 b_2 - (b_2)^2}{(b_1+b_2)^2}, 0)$

$y_{21} = \widehat{y}_{\text{med}}$ $x_{21} = (X(\frac{(b_2)^2}{(b_1+b_2)^2}), X\frac{(b_1)^2 + 2b_1 b_2}{(b_1+b_2)^2}, 0)$ $a_{21} = \frac{b_1}{b_2}$

$y_{23} = \widehat{y}_{\text{med}} + X\frac{b_2}{(b_1+b_2)^2}$ $x_{23} = (0, X, 0)$ $a_{23} = \frac{b_2 - b_1}{b_2}$

$y_{31} = \widehat{y}_{\text{med}} + X\frac{2b_2+b_1}{(b_1+b_2)^2}$ $x_{31} = (X, 0, 0)$ $a_{31} = \frac{b_2}{b_1+b_2}$

$y_{32} = \widehat{y}_{\text{med}} + X\frac{2b_2+b_1}{(b_1+b_2)^2}$ $x_{32} = (0, X, 0)$ $a_{32} = \frac{b_1}{b_1+b_2}$

Proposals are independent of b_3. Since b_3 is highest, 3 never keeps any X, so the scale of b_3 does not change 3's relative rankings of the proposals.

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