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Note

EDGE CYCLE EXTENDABLE GRAPHS

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Abstract

A graph is edge cycle extendable if every cycle C that is formed from edges and one chord of a larger cycle C^+ is also formed from edges and one chord of a cycle C' of length one greater than C with $V(C') \subseteq V(C^+)$. Edge cycle extendable graphs are characterized by every block being either chordal (every nontriangular cycle has a chord) or chordless (no nontriangular cycle has a chord); equivalently, every chord of a cycle of length five or more has a noncrossing chord.

Keywords: cycle extendable graph, chordal graph, chordless graph, minimally 2-connected graph.

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1. Preliminaries

A *k-cycle* is a cycle C with length $|C| = |E(C)| = k$, and a *chord* of C is an edge whose endpoints are nonconsecutive vertices of C. For any two cycles C_1 and C_2 in a graph G such that $E(C_1) \cap E(C_2)$ forms a path π with $|E(\pi)| \geq 1$ and $V(\pi) = V(C_1) \cap V(C_2)$, define their *sum*, denoted $C_1 \oplus C_2$, to be the cycle that is formed by those edges of G that are in exactly one of C_1 and C_2; thus, $E(C_1 \oplus C_2) = E(C_1) \cup E(C_2) - E(\pi)$. For instance, if G is the *house graph* (formed by inserting one chord into a 5-cycle), then each of the three occurring cycles (a 3-cycle, a 4-cycle, and a 5-cycle) is the sum of the other two. If the path π is a chord of C, then we say that the cycle C is *split into the sum* $C_1 \oplus C_2$ by that chord.

Although the edge set of one cycle cannot be contained in the edge set of another, set theoretic terminology can be borrowed to say, as in [5], that a cycle

C is *almost contained in* a cycle C^+ (and C^+ *almost contains* C) if every edge except one of C is an edge of C^+; in other words, if $|E(C) - E(C^+)| = 1$. The unique edge in $E(C) - E(C^+)$ is a chord of C, and C is split into $C \oplus (C \oplus C^+)$ by that chord. In the house graph, for instance, both the 3-cycle and the 4-cycle are almost contained in the 5-cycle, but the 3-cycle is not almost contained in the 4-cycle.

Call a graph *edge cycle extendable*—abbreviated ECE—if every cycle C that is almost contained in another cycle C^+ is almost contained in some cycle C' that has $|C'| = |C| + 1$ with $V(C) \subset V(C') \subseteq V(C^+)$ and $E(C) \cap E(C') \subset E(C^+)$. Figure 1 illustrates this, where $\{xy\} = E(C) - E(C^+)$ and $z \in V(C^+)$ and $E(C^+)$ might contain xz or yz (or both, if $|C^+| = |C| + 1$).

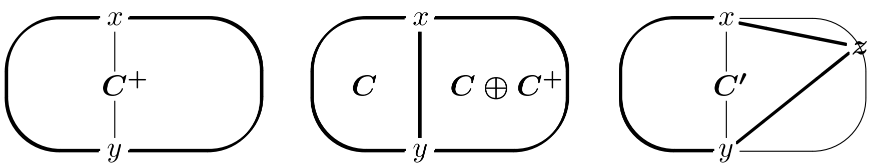

Figure 1. Illustrating the definition of edge cycle extendable graphs.

Every acyclic graph G is trivially ECE, as is $G = C_n$ for every $n \geq 3$. The house graph is not ECE, since its 3-cycle is almost contained in the 5-cycle, but not in a 4-cycle. The graph on the left in Figure 2 is not ECE (take C to be the top triangle and C^+ to be the 5-cycle that contains two edges of C and three peripheral edges), but the graph on the right is ECE.

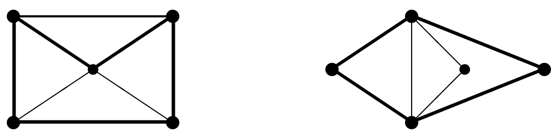

Figure 2. The graph on the left is not ECE; the graph on the right is ECE.

Lemma 1. *Every induced subgraph of an edge cycle extendable graph is edge cycle extendable.*

Proof. Suppose H is an induced subgraph of an ECE graph G and C is a cycle of H that is almost contained in a cycle C^+ of H. Say $E(C) - E(C') = \{xy\}$. Since C and C^+ are also cycles of the ECE graph G, cycle C is almost contained in a cycle C' of G with $|C'| = |C| + 1$ and $V(C') \subseteq V(C^+)$, which makes C' also a cycle of H. Therefore, H is ECE. ■

A graph is a *chordal graph* if every cycle C with $|C| \geq 4$ has a chord—in other words, every cycle long enough to have a chord does have a chord. Along with the many significant and useful characterizations of chordal graphs in [1, 6], the following very simple characterizations follow recursively from the definition:

(C1) A graph is chordal if and only if, for every cycle C and $xy \in E(C)$, there exists $z \in V(C)$ such that xyz is a triangle.

(C2) A graph is chordal if and only if every cycle C with $|C| \geq 4$ almost contains another cycle C' that has $|C'| = |C| - 1$.

Characterization (C2) shows how the concepts of chordal and ECE seem to go in opposite directions—indeed, chordal graphs could be described as 'edge cycle retractable' graphs. Lemma 2 will show, however, that every chordal graph is ECE. (The cycle C_4 is ECE, but not chordal.)

Lemma 2. *Every chordal graph is edge cycle extendable.*

Proof. Suppose G is chordal and cycle C is almost contained in cycle C^+ with $E(C) - E(C^+) = \{xy\}$. By characterization (C1), there exists $z \in V(C \oplus C^+)$ such that $\Delta = xyz$ is a triangle. Cycle C is almost contained in $C' = C \oplus \Delta$ where $|C'| = |C| + 1$ and $V(C') \subseteq V(C^+)$. Therefore, G is ECE. ■

Additional motivation for ECE graphs comes from the definition in [3] of the class of 0-*chord extendable graphs*—graphs in which every nonhamiltonian cycle C is almost contained in a cycle C' with $|C'| = |C| + 1$. This class is incomparable to the class of ECE graphs: the graph on the left in Figure 2 is 0-chord extendable but not ECE, while the graph on the right is ECE but not 0-chord extendable (the peripheral 4-cycle is nonhamiltonian but is not almost contained in a 5-cycle). Also, [5] defines the class of *strongly pancyclic graphs*—graphs for which every nontriangular cycle C almost contains a cycle C' and every nonhamiltonian cycle C is almost contained in a cycle C'' (requiring in addition that $|C'| = |C| - 1$ and $|C''| = |C| + 1$ would give the same class). This class is narrower than the class of ECE graphs: the graph on the right in Figure 2 is not strongly pancyclic but is ECE; every strongly pancyclic graph is chordal, and so is ECE by Lemma 2.

A graph is 2-*connected* if every two vertices are in a common cycle (or, equivalently, if every two edges are in a common cycle). A *block* of a graph is an inclusion-maximal induced subgraph that has no cut vertex (in other words, a block is either an edge that is in no cycle or an inclusion-maximal 2-connected subgraph).

A graph is a *chordless graph* if no cycle long enough to have a chord does have a chord—every cycle is either a triangle or an induced cycle. (The classes of chordal graphs and chordless graphs are Aristotelian 'contraries' of each other.) Because of the fundamental role that chordless graphs will play in Section 2, the following characterization is worth noting, even though it will not be used in this paper. A graph is *minimally* 2-*connected* if it is 2-connected but deleting any one edge would leave a graph that is not 2-connected.

Proposition 3 [2, 7]**.** *A 2-connected graph is a chordless graph if and only if it is minimally 2-connected.*

Chordless graphs also appear in [4].

2. Characterizations

Every chordless graph is trivially ECE. Theorem 4 will show a much more intimate relationship between chordless graphs and ECE graphs (and chordal graphs). Note that K_3 is the only 2-connected graph that is both chordal and chordless.

Theorem 4. *A 2-connected graph is edge cycle extendable if and only if it is either a chordal graph or a chordless graph.*

Proof. First suppose that G is 2-connected and ECE, yet G is neither chordal nor chordless [arguing by contradiction]. Since G is not chordless, G contains a minimum-length cycle C that has a chord; say C is split into the sum $C_1 \oplus C_2$ by a chord. The minimality of C and G being ECE imply $|C| = 4$ and $|C_1| = |C_2| = 3$, and so $V(C)$ induces a 2-connected chordal subgraph of G.

Let H be an inclusion-maximal induced 2-connected chordal subgraph of G. Since G is 2-connected but not chordal, G contains a cycle C^* with no chords and $|C^*| \geq 4$ such that C^* contains an edge vw of H. Since H is 2-connected and chordal, vw is also in a cycle of H and, by characterization (C1), vw is in a triangle Δ of H. Thus $V(C^*) \cup V(\Delta)$ induces a subgraph H' of G in which Δ is almost contained in $C^+ = C^* \oplus \Delta$, yet (because C^* has no chords and $|C^*| \geq 4$) triangle Δ is not almost contained in a 4-cycle whose vertices are all in C^+. Therefore, H' is not ECE [contradicting Lemma 1].

Conversely, if G is 2-connected and either a chordal or a chordless graph, then G is ECE either by Lemma 2 or because G has no cycle that is almost contained in another cycle. ■

Corollary 5. *A graph is edge cycle extendable if and only if every block is either a chordal graph or a chordless graph.*

Proof. This follows directly from Theorem 4, since every cycle is in a unique block and K_2 blocks are chordal (as well as chordless) graphs. ■

Chords ab and cd of a cycle C are *noncrossing chords* of C if $|\{a, b, c, d\}| \geq 3$ and the vertices come in the order a, b, c, d around C. Clearly, C can only have noncrossing chords when $|C| \geq 5$.

Theorem 6. *The following are equivalent for every graph*:

(6.1) *The graph is edge cycle extendable.*

(6.2) *If a cycle C is split into the sum $C_1 \oplus C_2$ by a chord, then both $V(C_1)$ and $V(C_2)$ induce chordal subgraphs.*

(6.3) *Every chord of a cycle of length at least five has a noncrossing chord.*

Proof. **(6.1) $\Rightarrow$ (6.2)**: Suppose G is ece, cycle C of G is split into the sum $C_1 \oplus C_2$ by a chord, and H is the block of G that contains C. Since H is 2-connected but not chordless, Theorem 4 implies H is chordal. Therefore, both $V(C_1)$ and $V(C_2)$ induce chordal subgraphs of G.

(6.2) $\Rightarrow$ (6.3): Suppose G satisfies (6.2) and has a cycle C with $|C| \geq 5$ that has a chord xy. Suppose C is split into the sum $C_1 \oplus C_2$ by xy with $|C_1| \leq |C_2|$. Thus $|C_2| \geq 4$. Let H be the subgraph of G that is induced by $V(C_2)$. Since H is chordal by (6.2), C_2 has a chord zw. Therefore, xy has the noncrossing chord zw with respect to C.

(6.3) $\Rightarrow$ (6.1): Suppose G satisfies (6.3) and some cycle C is almost contained in a cycle C^+ with $E(C)-E(C^+) = \{xy\}$. Suppose C_1 is a cycle with minimum length such that $V(C_1) \subseteq V(C)$ with $xy \in E(C_1)$, and suppose C_1^+ is a cycle with minimum length such that $V(C_1^+) \subseteq V(C^+)$ and $E(C_1) - \{xy\} \subset E(C_1^+)$ and $xy \in E(C_1 \oplus C_1^+)$. Cycle C_1^+ has the chord xy and—because of the minimality of $|C_1|$ and $|C_1^+|$—chord xy has no noncrossing chord with respect to C^+. By (6.3), $|C_1^+| = 4$ and $|C_1| = 3 = |C_1 \oplus C_1^+|$. If $C' = C \oplus (C_1 \oplus C_1^+)$, then $|C'| = |C| + 1$ and $V(C') \subseteq V(C^+)$ and $E(C) \cap E(C') \subset E(C^+)$. Therefore, G is ece. ∎

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