

## TOTAL DOMINATION EDGE CRITICAL GRAPHS WITH MAXIMUM DIAMETER

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### Abstract

Denote the total domination number of a graph  $G$  by  $\gamma_t(G)$ . A graph  $G$  is said to be total domination edge critical, or simply  $\gamma_t$ -critical, if  $\gamma_t(G + e) < \gamma_t(G)$  for each edge  $e \in E(\overline{G})$ . For  $3_t$ -critical graphs  $G$ , that is,  $\gamma_t$ -critical graphs with  $\gamma_t(G) = 3$ , the diameter of  $G$  is either 2 or 3. We characterise the  $3_t$ -critical graphs  $G$  with  $\text{diam } G = 3$ .

## 1. Introduction

Let  $G = (V, E)$  be a graph with order  $|V| = n$ . The *open neighbourhood* of a vertex  $v$  is the set of vertices adjacent to  $v$ , that is,  $N(v) = \{w \mid vw \in E(G)\}$ , and the *closed neighbourhood* of  $v$  is  $N[v] = N(v) \cup \{v\}$ . For  $S \subseteq V(G)$  we define the *open* and *closed neighbourhoods*  $N(S)$  and  $N[S]$  of  $S$  by  $N(S) = \bigcup_{v \in S} N(v)$  and  $N[S] = \bigcup_{v \in S} N[v]$ , respectively. The *private neighbourhood* of  $x \in S$ ,  $S \subseteq V(G)$ , consists of all vertices in the closed neighbourhood of  $x$  but not in the closed neighbourhood of  $S - \{x\}$ , and is denoted by  $pn(x, S)$ , that is,  $pn(x, S) = N[x] - N[S - \{x\}]$ . If  $v \in pn(x, S)$ , then  $v$  is called a *private neighbour of  $x$  relative to  $S$* , or simply a *private neighbour of  $x$* , if confusion is unlikely. If  $G$  is a graph with  $\text{diam } G = k$  and  $d(u, v) = k$ , then

we say that  $u$  and  $v$  are *diametrical vertices*. A shortest  $u$ - $v$  path in  $G$  is a *diametrical path*. Two subsets  $X$  and  $Y$  of  $V$  are called *diametrical sets* if  $d(x, y) = \text{diam } G$  for each  $x \in X$  and  $y \in Y$ . If  $X$  and  $Y$  are diametrical sets, then  $(X, Y)$  is a *maximal diametrical pair* if for each  $z \in V - (X \cup Y)$ ,  $d(x, z) < \text{diam } G$  for some  $x \in X$  and  $d(y, z) < \text{diam } G$  for some  $y \in Y$ .

For sets  $S, X \subseteq V$ , if  $N[S] = X$  ( $N(S) = X$ , respectively), we say that  $S$  *dominates*  $X$ , written  $S \succ X$  ( $S$  *totally dominates*  $X$ , respectively, written  $S \succ_t X$ ). If  $S = \{s\}$  or  $X = \{x\}$ , we also write  $s \succ X$ ,  $S \succ_t x$ , etc. If  $S \succ V$  ( $S \succ_t V$ , respectively), we say that  $S$  is a *dominating set* (*total dominating set*) of  $G$ , and we also write  $S \succ G$  ( $S \succ_t G$ , respectively). The cardinality of a minimum dominating (minimum total dominating) set of  $G$  is called the *domination number* (*total domination number*) of  $G$  and is denoted by  $\gamma(G)$  ( $\gamma_t(G)$ , respectively); if  $S$  is a minimum dominating (minimum total dominating) set, we also call  $S$  a  $\gamma$ -*set* ( $\gamma_t$ -*set*) of  $G$ . We note that the parameter  $\gamma_t(G)$  is only defined for graphs  $G$  without isolated vertices. Domination-related concepts not defined here can be found in [2].

The addition of an edge to a graph can change the domination number by at most one. Sumner and Blich [5, 6] studied *domination edge critical graphs*  $G$ , that is, graphs  $G$  for which  $\gamma(G) = \gamma(G) - 1$  for each  $e \in E(\overline{G})$ . We consider the same concept for total domination. A graph  $G$  is *total domination edge critical* or just  $\gamma_t$ -*critical* if  $\gamma_t(G + e) < \gamma_t(G)$  for any edge  $e \in E(\overline{G}) \neq \emptyset$ . It is shown in [3] that the addition of an edge to a graph can change the total domination number by at most two.

**Proposition 1** [3]. *For any edge  $e \in E(\overline{G})$ ,*

$$\gamma_t(G) - 2 \leq \gamma_t(G + e) \leq \gamma_t(G).$$

Graphs  $G$  with the property  $\gamma_t(G + e) = \gamma_t(G) - 2$  for any  $e \in E(\overline{G})$  are called *supercritical* and are characterised in [4].

In this paper, we restrict our attention to  $3_t$ -critical graphs  $G$ , that is,  $\gamma_t$ -critical graphs  $G$  with  $\gamma_t(G) = 3$ . Note that since  $\gamma_t(G) \geq 2$  for any graph  $G$ , the addition of an edge to a  $3_t$ -critical graph reduces the total domination number by exactly one. Also, observe that any graph  $G$  with  $\gamma_t(G) = 3$  is connected. Sharp bounds on the diameter of a  $3_t$ -critical graph are determined in [3].

**Proposition 2** [3]. *If  $G$  is a  $3_t$ -critical graph, then*

$$2 \leq \text{diam } G \leq 3.$$

The graphs in Figures 1 and 2 illustrate sharpness of these bounds. Our goal is to investigate the  $3_t$ -critical graphs with diameter three.

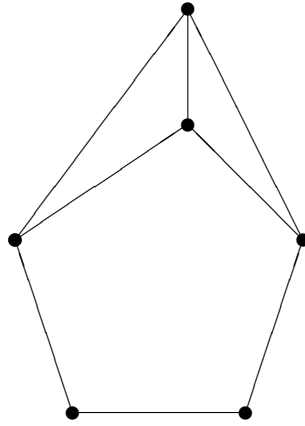


Figure 1. A  $3_t$ -critical graph  $G$  with  $\text{diam } G = 2$

## 2. $3_t$ -Critical Graphs with Diameter Three

In [3] the authors showed that any  $3_t$ -critical graph  $G$  with a cutvertex has exactly one cutvertex and it is adjacent to an endvertex. Moreover, they proved that such graphs  $G$  have  $\text{diam } G = 3$  and are the only  $3_t$ -critical graphs with an endvertex. Figure 2 illustrates a  $3_t$ -critical graph with an endvertex.

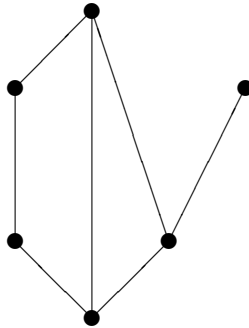


Figure 2. A  $3_t$ -critical graph with an endvertex

**Theorem 3** [3]. *A graph  $G$  with a cutvertex  $v$  is  $3_t$ -critical if and only if  $v$  is adjacent to an endvertex  $x$ , and for  $W = N(v) - \{x\}$  and  $Y = V - N[v]$ ,*

- (1)  $\langle W \rangle$  is complete and  $|W| \geq 2$ ,
- (2)  $\langle Y \rangle$  is complete and  $|Y| \geq 2$ ,

and

- (3) every vertex in  $W$  is adjacent to  $|Y| - 1$  vertices in  $Y$  and every vertex in  $Y$  is adjacent to at least one vertex in  $W$ .

We begin with a straightforward but useful observation.

**Observation 4.** *For any  $3_t$ -critical graph  $G$  and non-adjacent vertices  $u$  and  $v$ , either*

- (1)  $\{u, v\}$  dominates  $G$   
or
- (2) (without loss of generality)  $\{u, w\}$  dominates  $G - v$ , but not  $v$ , for some  $w \in N(u)$ . In this case, we write  $uw \mapsto v$ .

Next we develop some structural properties of  $3_t$ -critical graphs  $G$  with  $\text{diam } G = 3$ . Although it is possible in a  $3_t$ -critical graph  $G$  of diameter two for every pair of nonadjacent vertices to dominate  $G$  (see Figure 1, for example), we now show this is not possible if  $\text{diam } G = 3$ .

**Proposition 5.** *If  $G$  is a  $3_t$ -critical graph with  $\text{diam } G = 3$ , then  $G$  has a pair of nonadjacent vertices that does not dominate  $G$ .*

**Proof.** Let  $G$  be a  $3_t$ -critical graph with  $\text{diam } G = 3$  and suppose that every pair of nonadjacent vertices of  $G$  dominates  $G$ . Let  $x$  and  $y$  be diametrical vertices of  $G$  where  $x, a, b, y$  is a shortest  $x$ - $y$  path. Since  $\{x, b\} \succ G$ , every neighbour of  $y$  is also dominated by  $b$ . Similarly, every neighbour of  $x$  is dominated by  $a$ . Hence  $\{a, b\}$  is a total dominating set of  $G$ , contradicting the fact that  $\gamma_t(G) = 3$ . ■

Also, it is possible for a  $3_t$ -critical graph  $G$  with  $\text{diam } G = 2$  to have the property that for every pair of nonadjacent vertices  $u$  and  $v$ , there is a vertex  $x$  such that  $ux \mapsto v$ , and there is a vertex  $y$  such that  $vy \mapsto u$ . See Figure 3 for an example. We now show that a  $3_t$ -critical graph with diameter three cannot have this property.

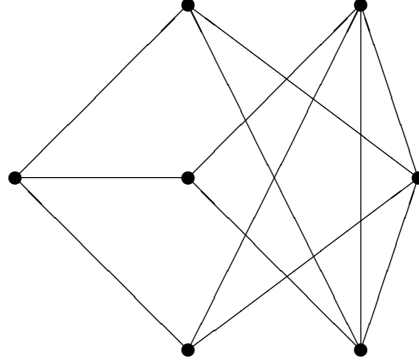


Figure 3. A  $3_t$ -critical graph with  $\text{diam } G = 2$

**Proposition 6.** *If  $G$  is a  $3_t$ -critical graph with  $\text{diam } G = 3$ , then  $G$  has a pair of nonadjacent vertices  $u$  and  $v$  such that  $ux \mapsto v$ , for some  $x \in V$ , but there is no vertex  $y$  such that  $vy \mapsto u$ .*

**Proof.** Let  $G$  be a  $3_t$ -critical graph with diameter three. Let  $x$  and  $y$  be diametrical vertices of  $G$  where  $x, a, b, y$  is a shortest  $x$ - $y$  path. By the proof of Proposition 5, at least one of  $\{x, b\}$  and  $\{a, y\}$  does not dominate  $G$ . Assume then, without loss of generality, that  $\{x, b\}$  does not dominate  $G$ . If  $xw \mapsto b$ , then  $w \in N(x)$  by Observation 4 and  $w \in N(y)$  to dominate  $y$ , thus  $d(x, y) \leq 2$ , a contradiction. Hence the only possibility is that  $bw \mapsto x$ . ■

It is useful to know more about the diametrical sets of vertices of a  $3_t$ -critical graph with diameter three.

**Theorem 7.** *If  $G$  is a  $3_t$ -critical graph with  $\text{diam } G = 3$ , then  $G$  has a unique maximal diametrical pair  $(X, Y)$ . Moreover,  $X$  (say) has cardinality one and  $\langle Y \rangle$  is complete.*

**Proof.** Let  $G$  be a  $3_t$ -critical graph with  $\text{diam } G = 3$ . The proof of the theorem is a direct consequence of the following three lemmas.

**Lemma 8.** *For any maximal diametrical pair  $(Y_1, Y_2)$  of  $G$ ,  $\langle Y_i \rangle$  is complete for each  $i$  and  $|Y_i| = 1$  for at least one  $i$ .*

**Proof.** Let  $(Y_1, Y_2)$  be a maximal diametrical pair of  $G$ . First we show that if  $|Y_i| \geq 2$ , then  $\langle Y_i \rangle$  is complete. Let  $x \in Y_1$  and  $\{y, z\} \subseteq Y_2$  and

suppose that  $yz \notin E(G)$ . Since  $\{y, z\} \not\succeq G$ , we may assume without loss of generality that  $yw \mapsto z$  for some vertex  $w$ , contradicting the fact that  $d(x, y) = 3$ . Hence  $\langle Y_2 \rangle$  is complete. A similar argument shows that  $\langle Y_1 \rangle$  is complete.

Next we show that  $|Y_i| = 1$  for at least one  $i$ . Suppose to the contrary that both  $Y_1$  and  $Y_2$  have cardinality at least two. Let  $x \in Y_1$  and  $y \in Y_2$  and consider  $\{x, y\}$ . Since  $|Y_i| \geq 2$  for  $i \in \{1, 2\}$ , there is no vertex  $w$  such that  $xw \mapsto y$  or  $yw \mapsto x$ . It follows that  $\{x, y\} \succ G$ . This is the case for every  $x \in Y_1$  and every  $y \in Y_2$ . Let  $A$  ( $B$ , respectively) be the set of vertices that are distance one from every vertex of  $Y_1$  ( $Y_2$ , respectively). If both  $\langle A \rangle$  and  $\langle B \rangle$  are complete, then  $\gamma_t(G) = 2$ , a contradiction. Thus let  $a, b \in A$  where  $ab \notin E(G)$ . Consider  $\{a, y\}$ . Since neither  $a$  nor  $y$  is adjacent to  $b$ ,  $\{a, y\} \not\succeq G$ . Hence,  $yc \mapsto a$  or  $ac \mapsto y$ . Since no vertex in  $N[y]$  dominates  $Y_1$ ,  $ac \mapsto y$ . Therefore,  $c$  dominates  $Y_2 - \{y\}$ . Furthermore, since  $\{x, y\} \succ G$ ,  $c$  is adjacent to  $x$ , implying that  $y$  is the only vertex at distance three from  $x$ , contradicting the fact that  $|Y_i| > 1$  for  $i \in \{1, 2\}$ . ■

Consider the maximal diametrical pair  $(\{x\}, Y)$  of  $G$ . Note that by Lemma 8 and the definition of maximal diametrical pair,  $Y = \{y \in V \mid d(x, y) = 3\}$ .

**Lemma 9.** *For every vertex  $u \in V - \{x\}$ ,  $d(u, y) \leq 2$  for every  $y \in Y$ .*

**Proof.** If  $|Y| = 1$ , then  $x$  is the only vertex at distance three from  $Y$ . Assume then that  $|Y| \geq 2$ . Let  $y, z \in Y$  and suppose there is a vertex  $u$  such that  $d(u, y) = 3$  and  $d(u, z) = 2$ ; note that  $u \neq x$ . Let  $uaby$  be a  $u$ - $y$  path and let  $ucz$  be a  $u$ - $z$  path ( $c$  may equal  $a$ ). Note that  $cy \notin E(G)$ . Since neither  $x$  nor  $y$  is adjacent to  $c$ ,  $xw \mapsto y$  or  $yw \mapsto x$ . If  $xw \mapsto y$ , then  $d(x, z) = 2$ , contradicting that  $z \in Y$  and that  $\{x\}$  and  $Y$  are diametrical sets. If  $yw \mapsto x$ , then  $d(u, y) = 2$ , again a contradiction. ■

**Lemma 10.**  *$(\{x\}, Y)$  is the unique maximal diametrical pair of  $G$ .*

**Proof.** Consider any maximal diametrical pair  $(\{u\}, W)$  of  $G$ . If  $u = x$ , then  $W = \{w \in V \mid d(u, w) = 3\} = \{w \in V \mid d(x, w) = 3\} = Y$  and we are done. If  $u \in Y$ , then  $d(x, u) = 3$ , i.e.,  $x \in W$  and by Lemma 9,  $d(u, z) \leq 2$  for each  $z \in V - \{x\}$ . Hence  $W = \{x\}$  and since  $(\{u\}, \{x\})$  is a maximal diametrical pair, it follows that  $Y = \{u\}$  and the result follows. Hence we may assume that  $u \notin Y \cup \{x\}$ . It follows from Lemma 9 that  $W \cap (Y \cup \{x\}) = \emptyset$ .

Consider any  $w \in W$  and suppose firstly that  $\{u, w\} \succ G$ . Note that no vertex is adjacent to  $x$  as well as to a vertex in  $Y$ . Hence either  $ux \in E(G)$  and  $wy \in E(G)$  for each  $y \in Y$ , or  $wx \in E(G)$  and  $uy \in E(G)$  for each  $y \in Y$ . Suppose the former case holds and consider an arbitrary vertex  $y \in Y$ . By Lemma 9,  $d(u, y) = 2$  and  $d(w, x) = 2$ . Let  $uay$  and  $wbx$  be a  $u$ - $y$  path and a  $w$ - $x$  path, respectively and note that  $\{ub, yb\} \cap E(G) = \emptyset$ . Thus  $\{u, y\} \not\succeq G$  and so  $uc \mapsto y$  or  $yc \mapsto u$  for some vertex  $c$ . If  $uc \mapsto y$ , then  $cw \in E(G)$  and so  $d(u, w) = 2$ , a contradiction since  $u$  and  $w$  are diametrical vertices. If  $yc \mapsto u$ , then  $d(x, y) = 2$ , also a contradiction. Similarly, the case  $wx \in E(G)$  and  $uy \in E(G)$  for each  $y \in Y$  is impossible. We conclude that  $\{u, w\} \not\succeq G$ .

Thus there is some vertex  $d$  such that  $\{u, w, d\}$  is independent. Since neither  $d$  nor  $u$  is adjacent to  $w$ ,  $uc \mapsto d$  or  $dc \mapsto u$ . If  $uc \mapsto d$ , then  $d(u, w) = 2$ , a contradiction. Thus we may assume that  $dc \mapsto u$ . Then without loss of generality,  $d \in N(Y)$  and  $c \in N(x)$ . Now we consider  $\{x, d\}$ . Since  $d$  is not adjacent to  $u$  or  $w$ , and  $x$  cannot be adjacent to both  $u$  and  $w$ ,  $xd$  is not a dominating edge for  $G + xd$ . Then  $xs \mapsto d$  or  $ds \mapsto x$ . If  $xs \mapsto d$ , then  $d(x, y) = 2$ , a contradiction. If  $ds \mapsto x$ , then  $s$  is adjacent to both  $u$  and  $w$ , contradicting the fact that  $d(u, w) = 3$ . Hence  $(\{x\}, Y)$  is the unique diametrical pair of  $G$ . ■

### 3. Characterisation

In the rest of this paper we characterise the  $3_t$ -critical graphs with diameter three. We introduce more notation to simplify the characterisation. Let  $G$  be a graph with  $\text{diam } G = 3$  and let  $(\{x\}, Y)$  be a maximal diametrical pair of  $G$ . Let  $A = N(x)$ ,  $B = \{b \mid b \notin Y \text{ and } b \succ Y\}$ , and  $C = V - (A \cup B \cup Y \cup \{x\})$ . Note that at least one of  $B$  and  $C$  is not empty. Let  $\mathcal{F}$  be the family of all graphs  $G$  with  $\text{diam } G = 3$  and the maximal diametrical pair  $(\{x\}, Y)$ . Then  $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$ , where

$$G \in \mathcal{F}_1 \text{ if } C = \emptyset \text{ and } |Y| \geq 2,$$

$$G \in \mathcal{F}_2 \text{ if } C = \emptyset \text{ and } |Y| = 1,$$

$$G \in \mathcal{F}_3 \text{ if } B = \emptyset,$$

$$G \in \mathcal{F}_4 \text{ if } B \neq \emptyset \text{ and } C \neq \emptyset.$$

To characterise the  $3_t$ -critical graphs with diameter 3, we characterise the  $3_t$ -critical graphs in each family  $\mathcal{F}_i$ ,  $1 \leq i \leq 4$ . We begin with a lemma.

**Lemma 11.** *Let  $G \in \mathcal{F}$  be  $3_t$ -critical with  $|Y| \geq 2$ . If either  $B = \emptyset$  or  $C = \emptyset$ , then  $\langle A \rangle$  is complete.*

**Proof.** Let  $G \in \mathcal{F}$  with  $|Y| \geq 2$  and suppose that  $\langle A \rangle$  is not complete. First assume that  $C = \emptyset$ . Let  $u, v \in A$  with  $uv \notin E(G)$ . Consider  $\{u, y\}$  for some vertex  $y \in Y$ . Since neither  $u$  nor  $y$  is adjacent to  $v$ ,  $uw \mapsto y$  or  $yw \mapsto u$  for some vertex  $w$ . If  $uw \mapsto y$ , then  $w \in A \cup \{x\}$  since  $w \notin N(y)$ . But then  $Y - \{y\}$  is not dominated by  $\{u, w\}$ , a contradiction. If  $yw \mapsto u$ , then  $d(x, y) \leq 2$ , again a contradiction. Next assume that  $B = \emptyset$ . Since  $\{u, v\} \not\subseteq G$ , we may assume, without loss of generality, that  $uw \mapsto v$ . But this implies that  $w \succ Y$ , contradicting the fact that  $B = \emptyset$ . ■

Lemma 11 requires that the graph  $G$  has a diametrical set  $Y$  with cardinality greater than one. (See Figure 4(b)). The graph in Figure 4(a) is an example of a graph with a diametrical set  $Y$  such that  $|Y| = 1$  and  $\langle A \rangle$  complete. However, the condition of the lemma is necessary as can be seen by the  $3_t$ -critical graph in Figure 5 that has  $|Y| = 1$  and  $\langle A \rangle$  is not complete.

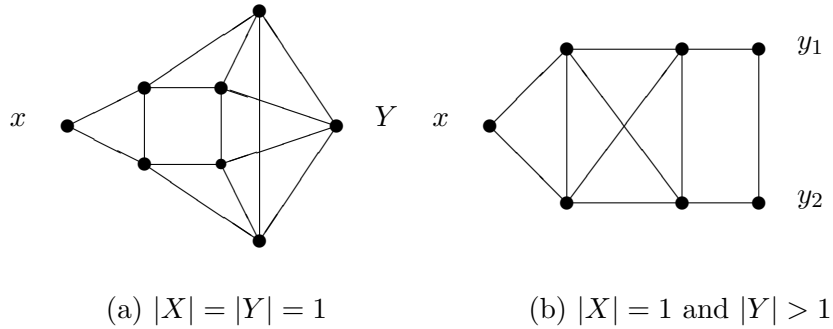


Figure 4. Two  $3_t$ -critical graphs with diameter three

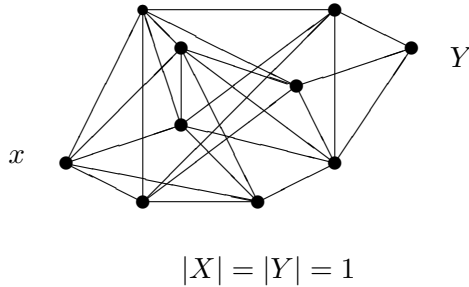


Figure 5. A  $3_t$ -critical graph with  $\langle A \rangle$  not complete



We first characterise the  $3_t$ -critical graphs  $G \in \mathcal{F}_1$ .

**Theorem 12.** *A graph  $G \in \mathcal{F}_1$  is  $3_t$ -critical if and only if the following conditions hold:*

- (1)  $(\{x\}, Y)$  is the unique maximal diametrical pair of  $G$  and  $\langle Y \rangle$  is complete.
- (2)  $\langle A \rangle$  is complete.
- (3) For every nonadjacent pair  $u, v \in B$ , there is a vertex  $a \in A$  such that  $ua \mapsto v$ . Also, no pair of adjacent vertices dominates  $G$ .
- (4) For every vertex  $b \in B$ , there is a vertex  $d \in B \cup Y$  such that  $bd \mapsto x$ .
- (5) For every pair  $\{a, b\}$  of nonadjacent vertices where  $a \in A$  and  $b \in B$ ,  $\{a, b\} \succ G$  or  $aw \mapsto b$  for some  $w \in B$ .

**Proof.** Let  $G \in \mathcal{F}_1$  be  $3_t$ -critical. By Theorem 10,  $(\{x\}, Y)$  is the unique maximal diametrical pair of  $G$  and  $\langle Y \rangle$  is complete.

Since  $C = \emptyset$ , it follows that  $\{x, y\} \succ G$  for every  $y \in Y$ . From Lemma 11 we have that  $\langle A \rangle$  is complete. Furthermore, since  $(\{x\}, Y)$  is a maximal diametrical pair, each  $b \in B$  is adjacent to at least one vertex  $a \in A$ . If there is a vertex  $b \in B$  that dominates  $B$ , then  $\{a, b\} \succ_t G$  for an  $a \in A$ , contradicting the fact that  $\gamma_t(G) = 3$ . Let  $u, v \in B$  with  $uv \notin E(G)$ . Obviously,  $\{u, v\} \not\succeq x$ , so without loss of generality, there is a vertex  $a \in A$  such that  $au \mapsto v$ . Since  $\gamma_t(G) = 3$ , no pair of adjacent vertices dominates  $G$ . To show that (4) holds, let  $b$  be any vertex in  $B$ . Since there is at least one vertex in  $B$  not adjacent to  $b$ ,  $\{x, b\} \not\succeq G$ . No vertex in  $N[x]$  dominates  $Y$ , so  $bd \mapsto x$  for some  $d \in B \cup Y$ . Condition (5) follows directly from Observation 4 and the fact that if  $bw \mapsto a$ , then  $w \in A$  to dominate  $x$ ; hence  $w \succ a$  since  $\langle A \rangle$  is complete, a contradiction.

Conversely, let  $G \in \mathcal{F}_1$  such that the stated properties hold. Since no pair of adjacent vertices dominates  $G$ ,  $\gamma_t(G) \geq 3$ . Further,  $\{a, b, y\}$  is a  $\gamma_t$ -set for every  $a \in A$ ,  $b \in B$ ,  $y \in Y$  where  $ab \in E(G)$ , implying that  $\gamma_t(G) \leq 3$ . Hence  $\gamma_t(G) = 3$ . To show that  $G$  is  $3_t$ -critical we consider first  $\{x, y\}$  for  $y \in Y$ . Since  $C = \emptyset$ ,  $\{x, y\} \succ G$ . Similarly,  $\{a, y\} \succ G$  for every  $a \in A$ . We next consider  $\{x, b\}$ . Since condition (4) holds, there is a vertex  $d \in B \cup Y$  such that  $bd \mapsto x$ . We also consider  $\{a, b\}$ ,  $a \in A$  and  $b \in B$ . Property (5) implies that either  $\{a, b\} \succ G$  or there is a vertex  $w \in B$  such that  $aw \mapsto b$ . Finally we consider  $\{b, c\}$ , where  $b, c \in B$ . Since condition (3) holds, there is a vertex  $a \in A$  such that  $ab \mapsto c$ . Thus  $G$  is  $3_t$ -critical. ■

Note that  $\{x, y\} \succ G$  for every  $y \in Y$ . We state this result as a corollary.

**Corollary 13.** *If  $G \in \mathcal{F}_1$  is  $3_t$ -critical, then  $\gamma(G) = 2$ .*

We now give a more descriptive characterisation of the  $3_t$ -critical graphs  $G \in \mathcal{F}_1$  with  $\delta(G) = 2$ . We first show that if  $\delta(G) = 2$ , then  $\deg(x) = 2$ . Recall that  $\langle A \rangle$  is complete.

**Lemma 14.** *If  $G \in \mathcal{F}_1$  and  $G$  is  $3_t$ -critical with  $\delta(G) = 2$ , then  $\deg(x) = 2$  and  $\deg(v) \geq 3$  for all  $v \in V(G) - \{x\}$ .*

**Proof.** Let  $G \in \mathcal{F}_1$  be  $3_t$ -critical. Since  $G$  has no cutvertices (Theorem 3),  $|A|, |B| \geq 2$ . Every vertex  $b \in B$  is adjacent to some vertex  $a \in A$  and to every vertex  $y \in Y$ . Thus  $\deg(b) \geq 3$  for every  $b \in B$ , since  $|Y| \geq 2$ . By Theorem 10,  $\langle Y \rangle$  is complete. Therefore  $\deg(y) \geq 3$  for each  $y \in Y$ . Finally, every vertex  $a \in A$  has at least one neighbour in  $A$ , implying that  $\deg(a) \geq 3$ . ■

We use the following notation for the characterisation. Let  $A = N(x) = \{x_1, x_2\}$  and  $B_1 = (N(x_1) \cap N(x_2)) - \{x\}$ ,  $B_2 = N(x_1) - (B_1 \cup \{x, x_2\})$ , and  $B_3 = N(x_2) - (B_1 \cup \{x, x_1\})$ . Recall that  $C = \emptyset$  and hence  $B = B_1 \cup B_2 \cup B_3$ .

We need the following lemmas for the characterisation. To simplify notation we refer to the  $3_t$ -critical graphs  $G \in \mathcal{F}_1$  with  $\delta(G) = 2$  as family  $\mathcal{G}_2$ .

**Lemma 15.** *If  $G \in \mathcal{G}_2$  and  $B_i \neq \emptyset$ , then  $\langle B_i \rangle$  is complete for  $i \in \{1, 2, 3\}$ .*

**Proof.** Let  $G \in \mathcal{G}_2$  and assume that  $B_i \neq \emptyset$ . Suppose that  $u, v \in B_i$  and  $uv \notin E(G)$ . Since neither  $u$  nor  $v$  dominates  $x$ , without loss of generality,  $uw \mapsto v$ . Then  $w \in N(u) \cap N(x)$ . But since  $u$  and  $v$  are in  $B_i$ ,  $v \in N(w)$ , contradicting that  $uw \mapsto v$ . ■

**Lemma 16.** *If  $G \in \mathcal{G}_2$  and  $B_1 \neq \emptyset$ , then each vertex in  $B_1$  dominates exactly  $|B_i| - 1$  vertices in  $B_i$  for  $i \in \{2, 3\}$ .*

**Proof.** It is easy to see that no vertex  $b \in B_1$  dominates  $B_2$  or  $B_3$ . Suppose, without loss of generality, a vertex  $b \in B_1$  is not adjacent to two vertices in  $B_2$ , say  $u$  and  $v$ , and consider  $\{b, u\}$ . Since neither  $b$  nor  $u$  dominates  $x$ ,  $\{b, u\} \not\succeq G$ . Furthermore,  $ux_1 \not\mapsto b$  since  $x_1 \in N(b)$ . Hence  $bx_2 \mapsto u$ , implying that  $v \in B_3$ , a contradiction. ■

**Lemma 17.** *If  $G \in \mathcal{G}_2$ , then  $|B_i| \geq 2$  for  $i \in \{2, 3\}$ .*

**Proof.** Let  $G \in \mathcal{G}_2$ . Since  $(\{x\}, Y)$  is a maximal diametrical pair, each  $a \in A$  is adjacent to some  $b \in B$ . Hence  $B_1 \cup B_i \neq \emptyset$  for  $i \in \{2, 3\}$ . If  $B_2 = \emptyset$  (or  $B_3 = \emptyset$ , respectively), then  $\{x_2, b_3\} \succ_t G$  for  $b_3 \in B_1 \cup B_3$  ( $\{x_1, b_2\} \succ_t G$  for  $b_2 \in B_1 \cup B_2$ , respectively). Hence neither  $B_2$  nor  $B_3$  is empty. Suppose without loss of generality that  $|B_2| = 1$ , say  $B_2 = \{b_2\}$ . By Lemma 16,  $b_2$  is not adjacent to any vertex in  $B_1$ . Also,  $b_2$  is not adjacent to any vertex in  $B_3$ , for otherwise  $\{x_2, b_3\} \succ_t G$  for some  $b_3 \in B_3 \cup N(b_2)$ . Now consider  $\{b_2, x\}$ . Since  $\{b_2, x\} \not\succeq B_3 \neq \emptyset$  and  $\{x, x_i\} \not\succeq Y$ , there exists a vertex  $w$  such that  $b_2 w \mapsto x$ . But no vertex adjacent to  $b_2$  dominates  $x_2$  as well as  $B_3$ , a contradiction. Hence  $|B_i| \geq 2$  for  $i \in \{2, 3\}$ . ■

**Lemma 18.** *If  $G \in \mathcal{G}_2$ , then  $\overline{\langle B_2 \cup B_3 \rangle}$  is the disjoint union of non-trivial stars.*

**Proof.** Note that  $\overline{\langle B_2 \cup B_3 \rangle}$  has no isolates, for if  $u \in B_2$  (say) dominates  $B_3$ , then  $\{u, x_1\} \succ_t G$ , contradicting the fact that  $\gamma_t(G) = 3$ . Assume without loss of generality that a vertex  $u \in B_2$  is not adjacent to vertices  $b_1, \dots, b_k \in B_3$ , where  $k \geq 2$  and where  $b_1$  (say) is not adjacent to  $v \in B_2$ ,  $v \neq u$ . Since  $\{u, b_1\} \not\succeq x$ , we may assume without loss of generality that  $uw \mapsto b_1$  for some vertex  $w$ . Then  $w = x_1$  to dominate  $x$ , but  $\{u, x_1\} \not\succeq b_2$ , a contradiction. The result follows since  $\langle B_i \rangle$  is complete for  $i = 2, 3$ . ■

**Theorem 19.** *A graph  $G \in \mathcal{G}_2$  if and only if the following conditions hold:*

- (1)  $(\{x\}, Y)$  is the unique maximal diametrical pair and  $\langle Y \rangle$  is complete.
- (2)  $\deg(x) = 2$  and  $\langle A \rangle$  is complete.
- (3)  $B_1 = \emptyset$  or  $\langle B_1 \rangle$  is complete.
- (4)  $|B_i| \geq 2$  and  $\langle B_i \rangle$  for  $i \in \{2, 3\}$  is complete.
- (5)  $\overline{\langle B_2 \cup B_3 \rangle}$  is the disjoint union of non-trivial stars.
- (6) If  $B_1 \neq \emptyset$ , then every vertex in  $B_1$  dominates exactly  $|B_i| - 1$  vertices in  $B_i$  for  $i \in \{2, 3\}$ . Also, if  $u \in B_2$  ( $u \in B_3$ , respectively) does not dominate  $B_1$ , then there is a vertex  $v \in B_1 \cup B_3$  ( $v \in B_1 \cup B_2$ , respectively) such that  $\{u, v\} \succ_t B$ .

**Proof.** Let  $G \in \mathcal{G}_2$ . By Theorem 12,  $(\{x\}, Y)$  is the unique maximal diametrical pair of  $G$ ,  $\langle Y \rangle$  is complete, and  $\langle A \rangle$  is complete. By Lemma 14,  $\deg(x) = 2$ . By Lemmas 15, 17, and 18, conditions (3), (4), and (5) hold. Assume without loss of generality that  $u \in B_2$  does not dominate  $B_1$ . Since

$\{x, u\} \not\succeq G$  and  $\{x, x_i\} \not\succeq Y$ , it follows that  $uv \mapsto x$  for some  $v$ . To dominate  $x_2$  but not  $x$ ,  $v \in B_1 \cup B_3$ , and clearly  $\{u, v\} \succ_t B$ . Thus by Lemma 16, condition (6) holds.

Conversely, let  $G$  be graph such that all the conditions of the theorem hold. There is no edge  $uv \in E(G)$  such that  $\{u, v\} \succ G$ . Hence  $\gamma_t(G) \geq 3$ . The path  $x_1, x_2, b_i$ , for  $b_i \in B$ , is a total dominating set. Therefore  $\gamma_t(G) = 3$ .

To show that  $G$  is  $\gamma_t$ -critical we first consider  $\{x, y\}$  for any  $y \in Y$ . Since  $C = \emptyset$ ,  $\{x, y\} \succ G$  for every  $y \in Y$ . Next consider  $\{x, b\}$  for any  $b \in B_1$ . Since  $b \succ A \cup Y$ ,  $by \mapsto x$  for any  $y \in Y$ . Now consider  $\{x, u\}$  for any  $u \in B_2$ . If  $u$  is not adjacent to any vertex in  $B_3$ , then by (5), every  $c \in B_2 - \{u\} \neq \emptyset$  is adjacent to all vertices in  $B_3$ , i.e.,  $\{x_1, c\} \succ_t G$ , a contradiction. So, if  $B_1 = \emptyset$  or  $u \succ B_1$ , let  $v \in B_3$  be adjacent to  $u$ . Clearly,  $uv \mapsto x$ . If  $u \not\succeq B_1$ , then by (6) there is a vertex  $v \in B_1 \cup B_3$  such that  $\{u, v\} \succ_t B$  and it is easy to see that  $uv \mapsto x$ . The set  $\{x, u\}$  for any  $u \in B_3$  is dealt with in exactly the same way. Further, it is easy to see that  $\{x_1, v\}$  and  $\{x_2, u\}$  dominate  $G$  for every  $v \in B_3$  and every  $u \in B_2$ . Also,  $\{x_i, y\} \succ G$  for  $i = 1, 2$  and every  $y \in Y$ . By Condition (6) a vertex  $b \in B_1$  dominates exactly  $|B_i| - 1$  vertices in  $B_i$ ,  $i = 2, 3$ . Let  $u \in B_2$  be non-adjacent to  $b \in B_1$ . Then  $bx_2 \mapsto u$ . Similarly,  $bx_1 \mapsto v$ , for  $v \in B_3$  and  $bv \notin E(G)$ . Finally, we consider  $\{u, v\}$  with  $u \in B_2$  and  $v \in B_3$ , where  $uv \notin E$ . Since  $\overline{\langle B_2 \cup B_3 \rangle}$  is the disjoint union of non-trivial stars, we may assume without loss of generality that  $u$  has degree 1 in  $\overline{\langle B_2 \cup B_3 \rangle}$ . Then  $ux_1 \mapsto v$ . It now follows that  $G \in \mathcal{G}_2$ . ■

For an example of a  $3_t$ -critical graph  $G \in \mathcal{G}_2$ , see Figure 6.

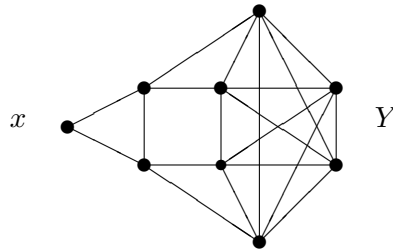


Figure 6. A  $3_t$ -critical graph  $G \in \mathcal{G}_2$

For  $3_t$ -critical graphs  $G \in \mathcal{F}_1$ , the cardinality of  $Y$  is greater than one. A necessary condition for these graphs is that  $\langle A \rangle$  is complete. However, when

the cardinality of  $Y$  is equal to one, this condition is no longer required. Figure 4(a) is an example of  $G \in \mathcal{F}_2$  and  $3_t$ -critical with  $\langle A \rangle$  complete and Figure 5 is an example of a graph  $G \in \mathcal{F}_2$  and  $3_t$ -critical with  $|Y| = 1$  and  $\langle A \rangle$  not complete.

**Theorem 20.** *A graph  $G \in \mathcal{F}_2$  is  $3_t$ -critical if and only if the following conditions hold:*

- (1)  $(\{x\}, \{y\})$  is the unique diametrical pair of  $G$ .
- (2) For each  $a \in A$  and  $b \in B$  with  $ab \in E(G)$  there exists a vertex  $w \notin N(a) \cup N(b)$ .
- (3) For each  $a, a' \in A$ , with  $aa' \notin E(G)$ , there exists  $b' \in B$  such that  $ab' \mapsto a'$ . A similar statement holds for each  $b, b' \in B$  with  $bb' \notin E(G)$ .
- (4) For every  $a \in A$ ,  $\{a, y\} \succ G$  or there exists  $a' \in A$  such that  $aa' \mapsto y$ . A similar statement holds for every  $b \in B$  and  $\{x\}$ .
- (5) For each  $a \in A$  and  $b \in B$  with  $ab \notin E(G)$ ,  $\{a, b\} \succ G$  or, without loss of generality, there exists  $b' \in B$  such that  $ab' \mapsto b$ .

**Proof.** Let  $G \in \mathcal{F}_2$  be  $3_t$ -critical. By Theorem 7  $(\{x\}, \{y\})$  is the unique diametrical pair of  $G$ . Condition (2) follows from the fact that  $\gamma_t(G) = 3$ . Since  $\langle A \rangle$  and  $\langle B \rangle$  cannot both be complete, let  $a, a' \in A$  with  $aa' \notin E(G)$ . Neither  $a$  nor  $a'$  is adjacent to  $y$ . Therefore without loss of generality there exists  $b' \in B$  such that  $ab' \mapsto a'$ . Let  $a \in A$  be an arbitrary vertex. If  $\{a, y\} \succ G$ , then Condition (4) holds. Otherwise there exists  $w$  such that  $yw \mapsto a$  or  $aw \mapsto y$ . If  $yw \mapsto a$ , then  $x \in N(w)$  implying  $d(x, y) = 2$ , a contradiction. Hence  $aw \mapsto y$  for some  $w \in A$ . A similar argument shows that for every  $b \in B$ ,  $\{b, x\} \succ G$  or there exists  $b' \in B$  such that  $bb' \mapsto x$ . Let  $a \in A$  and  $b \in B$  with  $ab \notin E(G)$ . If  $\{a, b\} \succ G$ , then Condition (5) holds. Otherwise, without loss of generality, there exists  $b' \in B$  such that  $ab' \mapsto b$ .

Conversely, let  $G$  be a graph such that the stated conditions hold. By Condition (2) there is no edge that dominates  $G$ . Thus,  $\gamma_t(G) \geq 3$ . Consider  $\{a, y\}$  for any  $a \in A$ . If  $\{a, y\} \succ G$ , then with  $b \in N(a) \cap N(y)$ ,  $\{a, b, y\}$  is a total dominating set. If  $\{a, y\} \not\succeq G$ , then by Condition (4) there exists  $a' \in A$  such that  $aa' \mapsto y$ . Again with  $b \in N(a) \cap N(y)$ ,  $\{a, a', b\}$  is a total dominating set, so  $\gamma_t(G) \leq 3$ . Hence  $\gamma_t(G) = 3$ . That  $G$  is  $\gamma_t$ -critical follows from the fact that  $\{x, y\} \succ G$  and from Conditions (2) through (5). ■

Two additional lemmas are needed for the remaining characterisations.

**Lemma 21.** *If  $G \in \mathcal{F}$  is  $3_t$ -critical, then every vertex in  $C$  is adjacent to exactly  $|Y| - 1$  vertices in  $Y$ .*

**Proof.** By definition, there is no vertex in  $C$  that dominates  $Y$ . Suppose there is a vertex  $c \in C$  that is not adjacent to at least two vertices in  $Y$ , say  $u$  and  $v$ . Clearly,  $\{c, u\} \not\sim G$ . Therefore  $cw \mapsto u$  or  $uw \mapsto c$  for some vertex  $w$ . If  $cw \mapsto u$ , then  $w \in N(x)$  and  $w \succ v$ , contradicting the fact that  $d(x, v) = 3$ . If  $uw \mapsto c$ , then  $w \succ x$ , again contradicting that  $d(x, u) = 3$ . ■

It was shown in Theorem 7 that  $\langle Y \rangle$  is complete. We now consider  $\langle C \rangle$ .

**Lemma 22.** *If  $G \in \mathcal{F}$  is  $3_t$ -critical and  $C \neq \emptyset$ , then  $\langle C \rangle$  is complete.*

**Proof.** Let  $u, v \in C$  and  $uv \notin E(G)$ . Since  $\{u, v\} \not\sim G$ , assume without loss of generality that  $uw \mapsto v$ . By definition there is a vertex  $y \in Y$  not adjacent to  $u$ . Therefore,  $w \succ y$  and  $w \succ x$ . But this contradicts the fact that  $d(x, y) = 3$ . ■

We now characterise the  $3_t$ -critical graphs in family  $\mathcal{F}_3$ .

**Theorem 23.** *A graph  $G \in \mathcal{F}_3$  is  $3_t$ -critical if and only if the following conditions hold:*

- (1)  $(\{x\}, Y)$  is the unique maximal diametrical pair of  $G$  and  $\langle Y \rangle$  is complete.
- (2)  $\langle A \cup C \rangle$  is complete.
- (3)  $|C| \geq 2$ ,  $|Y| \geq 2$  and every vertex in  $C$  is adjacent to exactly  $|Y| - 1$  vertices in  $Y$ .

**Proof.** Let  $G \in \mathcal{F}_3$  be  $3_t$ -critical. From Theorem 7 we have that  $(\{x\}, Y)$  is the unique maximal diametrical pair and  $\langle Y \rangle$  is complete.

By Lemmas 11 and 22,  $\langle A \rangle$  and  $\langle C \rangle$  are complete. We show that  $\langle A \cup C \rangle$  is complete. Let  $a \in A$  and  $c \in C$  with  $ac \notin E(G)$ . Since there is at least one vertex in  $Y$  not adjacent to  $c$ ,  $\{a, c\} \not\sim G$ . The only possibility is that  $aw \mapsto c$ . Thus  $w \succ Y$ , contradicting the fact that  $B = \emptyset$ .

By Lemma 21, if  $Y = \{y\}$  (say), then no vertex in  $C$  is adjacent to  $y$  and since  $B = \emptyset$ , it follows that  $y$  is isolated in  $G$ , which is impossible. Hence  $|Y| \geq 2$ . Suppose that  $|C| = 1$ . Since  $|Y| \geq 2$ , there is a vertex  $y \in Y$  that is not adjacent to a vertex of  $C$ . But then  $\text{diam}(G) > 3$ , a contradiction. Hence  $|C| \geq 2$ .

For the necessity, let  $G \in \mathcal{F}_3$  and assume that the conditions of the theorem hold. It is easy to see that there is no edge  $ac \in E(G)$  such that  $\{a, c\}$  dominates  $G$ . Thus  $\gamma_t(G) \geq 3$ . On the other hand, every shortest  $y$ - $a$  path,  $y \in Y$  and  $a \in A$ , is a total dominating set of cardinality three, implying that  $\gamma_t(G) = 3$ . We now show that  $G$  is  $3_t$ -critical. First consider  $\{x, c\}$ , for any  $c \in C$ . Since  $c \succ A \cup C$ ,  $cy \mapsto x$  for any  $y \in Y$  adjacent to  $c$ . Next, consider  $\{x, y\}$ , for any  $y \in Y$ . Here it is also easy to see that  $yc \mapsto x$  for any  $c \in N(y) \cap C$ . For any  $a \in A$  and  $y \in Y$ ,  $\{a, y\} \succ G$ . Finally we consider  $\{c, y\}$  with  $cy \notin E$ . Since  $y$  is the only vertex in  $Y$  not adjacent to  $c$ ,  $ca \mapsto y$  for any  $a \in A$ . ■

**Corollary 24.** *If  $G \in \mathcal{F}_3$  is  $3_t$ -critical, then  $\gamma(G) = 2$ .*

See Figures 2 and 4(b) for examples of  $3_t$ -critical graphs in  $\mathcal{F}_3$ . Note that this family of  $3_t$ -critical graphs includes those graphs with minimum degree one characterised in Theorem 3 where  $x$  is the endvertex of  $G$ .

Next we consider the family  $\mathcal{F}_4$ . See Figure 7 for an example.

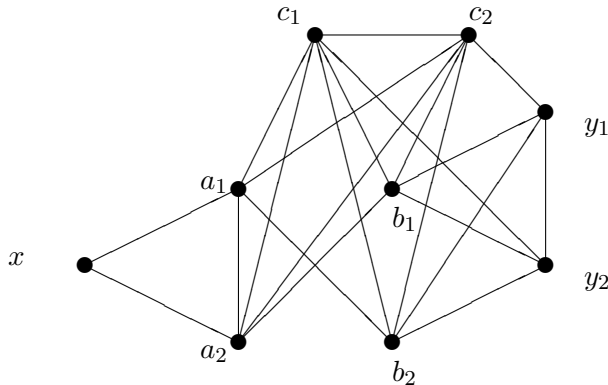


Figure 7. A  $3_t$ -critical graph  $G \in \mathcal{F}_4$

We now characterise the  $3_t$ -critical graphs  $G \in \mathcal{F}_4$  using the same notation as before.

**Theorem 25.** *A graph  $G \in \mathcal{F}_4$  is  $3_t$ -critical if and only if the following conditions hold:*

- (1)  $(x, Y)$  is the unique maximal diametrical pair of  $G$  and  $\langle Y \rangle$  is complete.
- (2)  $\langle C \rangle$  is complete and each  $c \in C$  dominates exactly  $|Y| - 1$  vertices in  $Y$ .

- (3) If  $|Y| \geq 2$ , then for every  $y \in Y$ ,  $\{x, y\} \succ G$  or there exists  $w \in B \cup C$  such that  $yw \mapsto x$ . If  $|Y| = 1$  (say  $Y = \{y\}$ ), then  $\{x, y\} \not\succeq G$  and there exists  $y' \in B$  such that  $y' \succ A \cup C$  or  $x' \in A$  such that  $x' \succ B \cup C$ .
- (4) For every  $c \in C$ , there exists  $w \in B \cup C \cup Y$  such that  $cw \mapsto x$ .
- (5) For every  $b \in B$ ,  $\{x, b\} \succ G$  or there exists  $w \in B \cup C \cup Y$  such that  $bw \mapsto x$ .
- (6) For every  $a \in A$  and  $y \in Y$ ,  $\{a, y\} \succ G$  or there exists  $w \in A \cup C$  if  $Y = \{y\}$  ( $w \in C$  if  $|Y| \geq 2$ ) such that  $aw \mapsto y$ .
- (7) For each  $a \in A$  and  $c \in C$  with  $ac \notin E(G)$ , there exists  $b \in B$  such that  $ab \mapsto c$ .
- (8) For each  $a \in A$  and  $b \in B$  with  $ab \notin E(G)$ ,  $\{a, b\} \succ G$  or there exists  $a' \in A$  such that  $a'b \mapsto a$  or  $b' \in B$  such that  $ab' \mapsto b$ . For each  $ab \in E(G)$  with  $a \in A$  and  $b \in B$ , there exists  $w \in A \cup B \cup C$  such that  $w \notin (N(a) \cup N(b))$ .
- (9) For each  $b \in B$  and  $c \in C$  with  $bc \notin E(G)$ , there exists  $a \in A$  such that  $ab \mapsto c$ .
- (10) For each  $c \in C$  and  $y \in Y$  with  $cy \notin E(G)$ , there exists  $a \in A$  such that  $ac \mapsto y$ .

**Proof.** Let  $G \in \mathcal{F}_4$  be  $3_t$ -critical. Condition (1) follows directly from Theorem 7. By Lemma 22,  $\langle C \rangle$  is complete. By Lemma 21, each vertex in  $C$  is adjacent to exactly  $|Y| - 1$  vertices in  $Y$ .

Consider arbitrary  $y \in Y$ . If  $\{x, y\} \succ G$ , then  $|Y| \geq 2$  since  $C \neq \emptyset$  and  $y$  must dominate  $C$ . Hence Condition (3) holds in this case. Therefore we may assume that  $\{x, y\}$  does not dominate  $G$ . Since  $G$  is  $3_t$ -critical,  $xw \mapsto y$  or  $yw \mapsto x$ . If  $xw \mapsto y$ , then  $w \in A$  implying that  $w \succ B \cup C$  and that  $Y = \{y\}$ . Thus if  $|Y| \geq 2$ , then  $yw \mapsto x$  and we have shown that Condition (3) holds if  $|Y| \geq 2$ . Therefore we may assume that  $|Y| = 1$ . Now  $G$  has the unique maximal diametrical pair  $(\{x\}, \{y\})$  and neither  $x$  nor  $y$  dominates any vertex in  $C$ . Hence  $xx' \mapsto y$  with  $x' \in A$  or  $yy' \mapsto x$  with  $y' \in B$ , and Condition (3) follows.

Condition (4) follows from the fact that each  $c \in C$  dominates at most  $|Y| - 1$  vertices in  $Y$  and there is no  $x' \in A$  such that  $xx' \mapsto c$  for any  $c \in C$ .

Let  $b$  be an arbitrary vertex in  $B$ . If  $\{x, b\} \succ G$ , then Condition (5) holds. Otherwise  $xx' \mapsto b$  for  $x' \in A$  or  $bb' \mapsto x$  for  $b' \in B \cup C \cup Y$ . If  $xx' \mapsto b$ , then  $x' \succ Y$  implying  $d(x, y) = 2$ , a contradiction. Hence  $bb' \mapsto x$ .



If for  $a \in A$  and  $y \in Y$ ,  $\{a, y\} \succ G$ , then Condition (6) holds. Otherwise  $yy' \mapsto a$  for  $y' \in N(y)$  or  $aa' \mapsto y$  for  $a' \in A \cup C$ . If  $yy' \mapsto a$ , then  $x \in N(y')$  implying  $d(x, y) < 3$ , a contradiction. Hence  $aa' \mapsto y$ .

Consider  $\{a, c\}$  where  $a \in A$  and  $c \in C$  are not adjacent. Since neither  $a$  nor  $c$  dominates  $Y$ ,  $\{a, c\} \not\succeq G$ . Therefore,  $ca' \mapsto a$  with  $a' \in A$  (to dominate  $x$ ) or  $ab' \mapsto c$  with  $b' \in B$  (to dominate  $Y$ ). If  $ca' \mapsto a$ , then  $c \succ Y$ , contradicting that each  $c \in C$  dominates at most  $|Y| - 1$  vertices in  $Y$ . Hence  $ab' \mapsto c$  and Condition (7) holds.

Condition (8) follows directly from the definition of  $3_t$ -critical graphs. If  $b \in B$  and  $c \in C$  with  $bc \notin E(G)$ , then  $\{b, c\} \not\succeq G$  since neither  $b$  nor  $c$  is adjacent to  $x$ . Since there is no  $c' \in N(c)$  such that  $cc' \mapsto b$ ,  $ba' \mapsto c$  with  $a' \in A$ . Hence Condition (9) holds.

Finally we consider  $\{c, y\}$  with  $cy \notin E(G)$ . Again since neither  $c$  nor  $y$  is adjacent to  $x$ ,  $\{c, y\} \not\succeq G$ . Also, since  $y$  has no neighbour  $y'$  such that  $y' \succ x$ ,  $ca' \mapsto y$  with  $a' \in A$ .

Let  $G$  be a graph such that the stated properties hold. By Condition (8) there is no  $ab \in E(G)$  with  $a \in A$  and  $b \in B$  such that  $\{a, b\} \succ G$ , and since no other edge dominates  $G$ ,  $\gamma_t(G) \geq 3$ . By Condition (10), there is  $a \in A$  for every  $c \in C$  such that  $ac \mapsto y$  for some  $y \in Y$ . Further, each  $a \in A$  is adjacent to some  $b \in B$  since  $(\{x\}, Y)$  is the unique maximal diametrical pair. Therefore,  $\{a, b, c\}$  is a total dominating set of  $G$ , implying that  $\gamma_t(G) \leq 3$ . Hence  $\gamma_t(G) = 3$ . That  $G$  is  $\gamma_t$ -critical, follows from Conditions (3) through (10). ■

Finally we consider a subclass of the family  $\mathcal{F}_4$ .

**Lemma 26.** *If  $G \in \mathcal{F}_4$  is  $3_t$ -critical and  $\langle A \rangle$  is not complete, then every  $y \in Y$  dominates at most  $|C| - 1$  vertices in  $C$ .*

**Proof.** Let  $u, v \in A$  with  $uv \notin E(G)$  and suppose there is a vertex  $y \in Y$  such that  $y \succ C$ . Consider  $\{u, y\}$ . Since  $\{u, y\} \not\succeq G$  and there is no vertex  $c \in C$  such that  $uc \mapsto y$ , there must be a vertex  $w \in N(y)$  such that  $yw \mapsto u$ . But then  $d(y, x) \leq 2$ , contradicting  $\text{diam}(G) = 3$ . ■

**Lemma 27.** *If  $G \in \mathcal{F}_4$  is  $3_t$ -critical and  $\langle A \rangle$  is not complete, then  $|C| \geq |Y|$ .*

**Proof.** Let  $|C| = k$  and  $|Y| = p$ . Since every vertex in  $C$  is adjacent to exactly  $|Y| - 1$  vertices in  $Y$ , there are exactly  $k(p - 1)$  edges from  $C$  to  $Y$ . By Lemma 26, every  $y \in Y$  dominates at most  $|C| - 1$  vertices in  $C$ .

Therefore there are at most  $p(k-1) - s$  edges from  $Y$  to  $C$ ,  $s \geq 0$ . Thus

$$p(k-1) - s = k(p-1),$$

hence

$$k - s = p$$

and it follows that  $k \geq p$ . ■

Restricting our attention to the graphs  $G \in \mathcal{F}_4$  with  $\langle A \rangle$  not complete and  $|Y| = |C|$ , we are able to obtain a more concise and descriptive characterisation than the one given for the family  $\mathcal{F}_4$ .

**Theorem 28.** *Let  $G$  be a graph in  $\mathcal{F}_4$  with  $\langle A \rangle$  not complete and  $|Y| = |C|$ . Then  $G$  is  $3_t$ -critical if and only if the following conditions hold:*

- (1)  $(\{x\}, Y)$  is the unique maximal diametrical pair of  $G$  and  $\langle Y \rangle$  is complete.
- (2)  $\langle C \rangle$  is complete and  $\langle C \cup Y \rangle$  is complete minus a perfect matching between  $C$  and  $Y$ .
- (3) Every vertex  $c \in C$  dominates  $A \cup B$ .
- (4) For every  $ab \in E(\langle A \cup B \rangle)$ , there is a vertex  $a_i \in A$  or  $b_j \in B$  not adjacent to  $a$  and  $b$  and if  $a_1, a_2$  ( $b_1, b_2$ , respectively) are nonadjacent vertices in  $A$  ( $B$ , respectively), then there is a vertex  $b \in B$  ( $a \in A$ , respectively) such that  $a_1b \mapsto a_2$  ( $b_1a \mapsto b_2$ , respectively). Also for every  $a \in A$  and  $b \in B$  that are not adjacent,  $\{a, b\} \succ G$  or there is a vertex  $w$  such that  $aw \mapsto b$  or  $bw \mapsto a$ .

**Proof.** Let  $G \in \mathcal{F}_4$  with  $\langle A \rangle$  not complete and  $|Y| = |C|$  be  $3_t$ -critical. Condition (1) follows directly from Theorem 7. By Lemma 22,  $\langle C \rangle$  is complete. By Lemmas 21 and 26, we have that each vertex in  $C$  is adjacent to  $|Y| - 1$  vertices in  $Y$  and if  $\langle A \rangle$  is not complete, then each vertex in  $Y$  is adjacent to at most  $|C| - 1$  vertices in  $C$ . Thus there are  $|C|(|Y| - 1)$  edges from  $C$  to  $Y$  and at most  $|Y|(|C| - 1)$  edges from  $Y$  to  $C$ . Since  $|C| = |Y|$ , there are exactly  $|Y|(|C| - 1)$  edges from  $Y$  to  $C$  and so every vertex in  $Y$  is adjacent to exactly  $|C| - 1$  vertices in  $C$ . Therefore, all edges minus a perfect matching are present between  $C$  and  $Y$ .

To show that (3) holds, suppose that  $ac \notin E(G)$ ,  $a \in A$  and  $c \in C$ . Consider  $\{a, y\}$  where  $y \in Y$  and  $cy \notin E(G)$ . Obviously,  $\{a, y\} \not\succeq G$ . Since no vertex in  $N[y]$  dominates  $x$ , it follows that  $az \mapsto y$  and  $z \in C$ . But this contradicts condition (2). Hence  $c \succ A$  for each  $c \in C$ .

Now suppose that  $cb \notin E(G)$ ,  $c \in C$  and  $b \in B$ , and consider  $\{b, c\}$ . Since neither  $b$  nor  $c$  is adjacent to  $x$ ,  $\{c, b\} \not\prec G$ . Therefore  $cw \mapsto b$  or  $bw \mapsto c$  for  $w \in A$  (to dominate  $x$ ). But if  $cw \mapsto b$ , then  $Y$  is not dominated, a contradiction. And if  $bw \mapsto c$ , then  $w \notin N(c)$ , contradicting the fact that every vertex  $c \in C$  dominates  $A$ . Hence  $c \succ B$  for each  $c \in C$ . Thus, condition (3) holds. Condition (4) follows from the fact that every  $b \in B$  dominates  $C \cup Y$  and every  $a \in A$  dominates  $C$ .

Let  $G \in \mathcal{F}_4$  with  $\langle A \rangle$  not complete and  $|Y| = |C|$  and assume that the conditions of the theorem hold. Since no pair of adjacent vertices dominate  $G$ ,  $\gamma_t(G) \geq 3$ . Further,  $\{a, b, c\}$ , where  $a \in A$ ,  $b \in B$  and  $c \in C$ , is a total dominating set, so  $\gamma_t(G) = 3$ . To show that  $G$  is  $3_t$ -critical, we first consider  $\{x, y\}$  for  $y \in Y$ . Then  $cy \mapsto x$  where  $c \in N(y)$ . A similar argument holds for  $\{x, c\}$ . Next consider  $\{x, b\}$  for  $b \in B$ . Then  $bc \mapsto x$  for any  $c \in C$ . For  $\{a, y\}$ ,  $ac \mapsto y$  where  $c \in C - N(y)$ . It now follows from condition (4) that  $G$  is  $3_t$ -critical. ■

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