MODEL THEORY OF ARITHMETIC

Lecture 14: Satisfaction classes

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> Vorliegende Arbeit ist fast gänzlich einem einzigen Problem gewidmet, nämlich dem der Definition der Wahrheit; sein Wesen besteht darin, dass man — im Hinblick auf diese oder jene Sprache — eine sachlich zutreffende und formal korrecte Definition des Terminus „wahre Aussage" zu konstruieren hat. Dieses Problem, welches zu den klassischen Fragen der Philosophie gezählt wird, verursacht bedeutende Schwierigkeiten. Obgleich nämlich die Bedeutung des Terminus „wahre Aussage" in der Umgangssprache recht klar und verständlich zu sein scheint, sind alle Versuche einer genaueren Präzisierung dieser Bedeutung bis nun erfolglos geblieben und manche Untersuchungen, in welchen dieser Terminus verwendet wurde und welche von scheinbar evidenten Prämissen ausgingen, haben oft zu Paradoxien und Antinomien gefürt [...].
>
> Alfred Tarski [11, Einleitung]

Tarski's theorem on the undefinability of truth [11] tells us that the satisfaction predicate is not definable in any model of arithmetic. It is then natural to ask whether one can strengthen a theory by adding a satisfaction predicate. The answer depends on what one means by satisfaction predicates. The aim of this lecture is to show that satisfaction defined by Tarski's inductive conditions [11] alone does not lead to any extra strength, in accordance with the deflationary theory of truth in philosophy.

Definition. All of following make sense over $\mathrm{I}\Delta_0+\exp$. Let $\mathscr{L}$ be a recursive language extending $\mathscr{L}_\mathrm{A}$ which has no new function symbol. Fix an $\mathscr{L}_\mathrm{A}$-formula $\mathrm{Term}_\mathscr{L}(t)$ that expresses 't is an $\mathscr{L}$-term', an $\mathscr{L}_\mathrm{A}$-formula $\mathrm{Fma}_\mathscr{L}(\theta)$ that expresses 'θ is an $\mathscr{L}$-formula', and an $\mathscr{L}_\mathrm{A}$-formula $\mathrm{VAsn}(\varepsilon,\theta)$ that expresses 'ε is a variable assignment for θ'. Here, a *variable assignment* for an $\mathscr{L}$-formula θ is simply a function whose domain is precisely the set of free variables in θ. If t is a term and ε is a variable assignment, then $\mathrm{eval}(t,\varepsilon)$ denotes the evaluation of t under ε. This can also be defined within $\mathscr{L}_\mathrm{A}$ with $\mathrm{I}\Delta_0+\exp$. We formulate *Tarski's (inductive) clauses for satisfaction* in $\mathscr{L}_\mathrm{A}$ as follows. The predicate $S(\theta,\varepsilon)$ is intended to mean 'θ is true under the variable assignment ε'.

(T0) $S(\theta,\varepsilon)\rightarrow\mathrm{VAsn}(\varepsilon,\theta)\wedge\mathrm{Fma}_\mathscr{L}(\theta)$.

(T1) Whenever R is a relation symbol of arity n in $\mathscr{L}$,

$$\forall t_1,t_2,\dots,t_n\left(\bigwedge_{i=1}^{n}\mathrm{Term}_\mathscr{L}(t_i)\rightarrow\left(\begin{array}{l}S(R(t_1,t_2,\dots,t_n),\varepsilon)\\ \quad\leftrightarrow\left(\begin{array}{l}\mathrm{VAsn}(\varepsilon,R(t_1,t_2,\dots,t_n))\\ \wedge R\big(\mathrm{eval}(t_1,\varepsilon),\mathrm{eval}(t_2,\varepsilon),\dots,\mathrm{eval}(t_n,\varepsilon)\big)\end{array}\right)\end{array}\right)\right).$$

(T2) $S(\neg\eta,\varepsilon)\leftrightarrow\mathrm{VAsn}(\varepsilon,\neg\eta)\wedge\neg S(\eta,\varepsilon)$.

(T3) $S(\varphi\vee\psi,\varepsilon)\leftrightarrow\mathrm{VAsn}(\varepsilon,\varphi\vee\psi)\wedge\exists\varepsilon'{\subseteq}\varepsilon\ \big(S(\varphi,\varepsilon')\vee S(\psi,\varepsilon')\big)$.

(T4) $S(\exists v\ \eta,\varepsilon)\leftrightarrow\mathrm{VAsn}(\varepsilon,\exists v\ \eta)\wedge\exists\varepsilon'{\supseteq}\varepsilon\ S(\eta,\varepsilon')$.

Tarski's clauses are weak because they are *local*, in the sense that the Tarski clause for a formula is about its finitely many direct subformulas only. This weakness will be exploited in our proofs in important ways.

Definition (Krajewski [7], Enayat–Visser [1])**.** Let $M \models \mathrm{I}\Delta_0 + \exp$ and $F \subseteq \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$, where

$$\mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}} = \{x \in M : M \models \mathrm{Fma}_{\mathscr{L}_\mathrm{A}}(x)\}.$$

Then an *F-satisfaction class* for M is some $S \subseteq F \times M$ that satisfies Tarski's clauses (T0)–(T4) restricted to $\eta, \varphi, \psi \in F$. A *full satisfaction class* for M is a $\mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$-satisfaction class.

Note that nonstandard formulas exist in nonstandard models. The class of formulas F above, however, does not need to be definable or internal in the model in any way. In view of clause (T2), every F-satisfaction class decides all formulas in F.

Example 14.1. Let $M \models \mathrm{I}\Delta_0 + \exp$. Then the $\mathscr{L}_\mathrm{A}$-formula Σ_n-Sat from Lecture 7 defines a Σ^M_n-satisfaction class for M for every $n \in \mathbb{N}$. The elementary diagram ElemDiag(M) is essentially the unique $\mathrm{Fma}^{\mathbb{N}}_{\mathscr{L}_\mathrm{A}}$-satisfaction class for M. In particular, the standard model of arithmetic $\mathbb{N}$ has a full satisfaction class.

We will see in the rest of the lecture that a countable nonstandard model of PA has a full satisfaction class if and only if it is recursively saturated. On the one hand, this tells us some nonstandard models of PA fail to have a full satisfaction class. So full satisfaction classes have nontrivial consequences on a model of arithmetic. On the other hand, it tells us every model of PA is elementarily equivalent to one that carries a full satisfaction class. So full satisfaction classes have no influence on the $\mathscr{L}_\mathrm{A}$-theory of a model of arithmetic.

14.1 Constructing full satisfaction classes

Theorem 14.2 (Kotlarski–Krajewski–Lachlan [4])**.** Every countable recursively saturated $M \models \mathrm{I}\Delta_0 + \exp$ has a full satisfaction class.

This theorem was originally proved using a version of M-logic that operates also on nonstandard formulas. In that setting, maximally consistent sets of formulas are precisely the full satisfaction classes. Here we follow an alternative approach via resplendency, which is simpler. The key auxiliary theorem is the following.

Theorem 14.3 (Enayat–Visser [1])**.** Fix $M \models \mathrm{I}\Delta_0 + \exp$. Let $F_0 \subseteq \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$ that is closed under taking direct subformulas, and S_0 be an F_0-satisfaction class for M. Then there is $K \succcurlyeq M$ which has a $\mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$-satisfaction class $S \supseteq S_0$.

This theorem says we can always find a satisfaction class deciding all the existing formulas if we move to an elementary extension. Of course, there are new formulas in the elementary extension, but then we can apply this theorem again, and again. After ω-many steps, we obtain a model with a full satisfaction class. Notice there are only finitely many Tarski clauses. So if the model we started with is resplendent, then it already has a full satisfaction class. Since all countable recursively saturated structures are resplendent by Theorem 7.6, the Kotlarski–Krajewski–Lachlan theorem follows. Notice, however, that there is an uncountable non-resplendent model of PA which admits a full satisfaction class [8, p. 296]. There are also uncountable recursively saturated models of PA that do not have full satisfaction classes [10].

It remains to establish the auxiliary theorem by Enayat and Visser. The main idea of the proof is that, since Tarski's clauses are local properties, we can deal with any finitely many of them easily; so by compactness, we can put all of them together in an elementary extension.

Proof. Let $\mathscr{L}^*_\mathrm{A} = \mathscr{L}_\mathrm{A} \cup \{U_\theta : \theta \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}\}$, where each U_θ is a new unary predicate symbol. In the elementary extension K, we will have

$$S = \{(\theta, \varepsilon) : \theta \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}} \text{ and } \varepsilon \in U_\theta\},$$

so that each U_θ is to be interpreted by the set of all satisfying assignments for θ. Therefore, we want each U_θ to satisfy the corresponding Tarski clause τ_θ, as defined below.

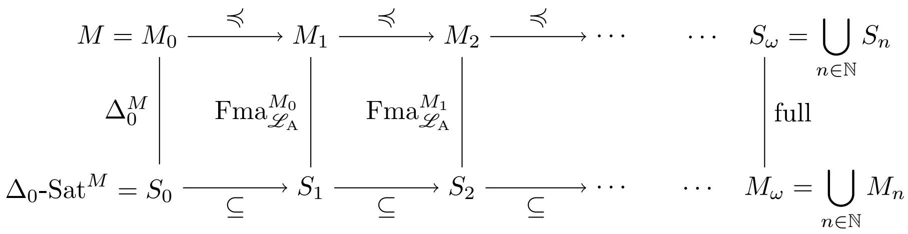

Figure 14.1: From Enayat–Visser to Kotlarski–Krajewski–Lachlan

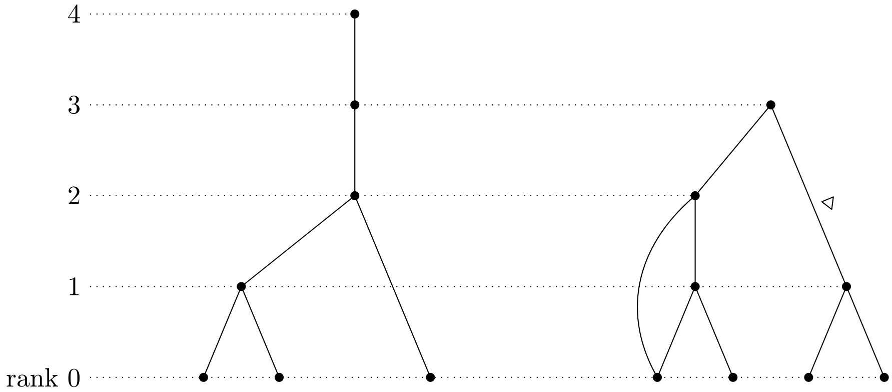

Figure 14.2: Ranks in the partially ordered sets $(\Theta, \lhd)$

(U1) If $\theta = R(t_1, t_2, \ldots, t_n)$, where R is a relation symbol of arity n in $\mathscr{L}_\mathrm{A}$ and $t_1, t_2, \ldots, t_n$ are $\mathscr{L}_\mathrm{A}^M$-terms, then τ_θ is

$$\forall\varepsilon\ \Big(U_\theta(\varepsilon) \leftrightarrow \mathrm{VAsn}(\varepsilon, \theta) \wedge R\big(\mathrm{eval}(t_1, \varepsilon), \mathrm{eval}(t_2, \varepsilon), \ldots, \mathrm{eval}(t_n, \varepsilon)\big)\Big).$$

(U2) If $\theta = \neg\eta$, where $\eta \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$, then τ_θ is

$$\forall\varepsilon\ \big(U_\theta(\varepsilon) \leftrightarrow \mathrm{VAsn}(\varepsilon, \theta) \wedge \neg U_\eta(\varepsilon)\big).$$

(U3) If $\theta = \varphi \vee \varphi$, where $\varphi, \psi \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$, then τ_θ is

$$\forall\varepsilon\ \big(U_\theta(\varepsilon) \leftrightarrow \mathrm{VAsn}(\varepsilon, \theta) \wedge \exists\varepsilon'{\subseteq}\varepsilon\ \big(U_\varphi(\varepsilon') \vee U_\psi(\varepsilon')\big)\big).$$

(U4) If $\theta = \exists v\ \eta$, where $\eta \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}$, then τ_θ is

$$\forall\varepsilon\ \big(U_\theta(\varepsilon) \leftrightarrow \mathrm{VAsn}(\varepsilon, \theta) \wedge \exists\varepsilon'{\supseteq}\varepsilon\ U_\eta(\varepsilon')\big).$$

We are done if we can show the consistency of the $\mathscr{L}_\mathrm{A}^*(M)$-theory

$$T = \mathrm{ElemDiag}(M) + \{\tau_\theta : \theta \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}}\} + \{U_\theta(\varepsilon) : (\theta, \varepsilon) \in S_0\} + \{\neg U_\theta(\varepsilon) : (\neg\theta, \varepsilon) \in S_0\}.$$

Take a finite $T_0 \subseteq T$. Let $\Theta = \{\theta \in \mathrm{Fma}^M_{\mathscr{L}_\mathrm{A}} : U_\theta \text{ appears in } T_0\}$. If $\eta, \theta \in \Theta$, then $\eta \lhd \theta$ means

$$\tau_\theta \in T_0 \text{ and } \eta \text{ is a direct subformula of } \theta.$$

The $\lhd$-related pairs are precisely those of which we will need to make Tarski's clauses true. Since Θ is finite, every $\theta \in \Theta$ has a well-defined $\lhd$-rank in $\mathbb{N}$ satisfying

$$\mathrm{rank}_\Theta(\theta) = \begin{cases} \max\{\mathrm{rank}_\Theta(\eta) + 1 : \theta \rhd \eta \in \Theta\}, & \text{if this set is nonempty;} \\ 0, & \text{otherwise.} \end{cases}$$

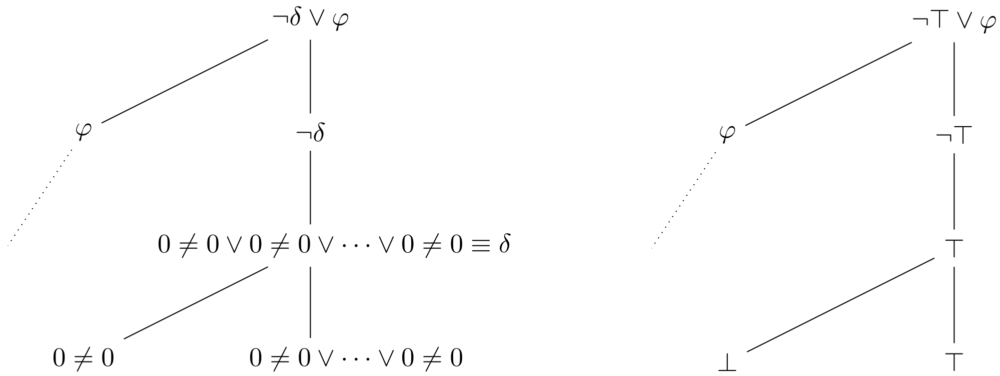

Figure 14.3: A $\triangleleft$-component in Θ containing δ, and a valid truth assignment for it

We will define $U_\theta \subseteq M$ for each $\theta \in \Theta$ such that $(M, U_\theta)_{\theta\in\Theta} \models T_0$. This will finish the proof. These U_θ's are defined by recursion along $\triangleleft$. Notice, by the shape of τ_θ, if $\varphi, \psi \triangleleft \theta \in \Theta$ and $\varphi \in \Theta$, then $\psi \in \Theta$ too. Here $\mathrm{VAsn}(M, \theta) = \{\varepsilon \in M : M \models \mathrm{VAsn}(\varepsilon, \theta)\}$.

(1) If $\mathrm{rank}_\Theta(\theta) = 0$, then

$$U_\theta = \begin{cases} \{\varepsilon \in \mathrm{VAsn}(M, \theta) : (\theta, \varepsilon) \in S_0\}, & \text{if } \theta \in F_0 \text{ or if } \theta \text{ is atomic;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

(2) If $\mathrm{rank}_\Theta(\theta) = i + 1$ and $\theta = \neg\eta$, then $U_\theta = \{\varepsilon \in \mathrm{VAsn}(M, \theta) : \varepsilon \notin U_\eta\}$.

(3) If $\mathrm{rank}_\Theta(\theta) = i + 1$ and $\theta = \varphi \vee \varphi$, then $U_\theta = \{\varepsilon \in \mathrm{VAsn}(M, \theta) : \exists\varepsilon'{\subseteq}\varepsilon\ (\varepsilon' \in U_\varphi \cup U_\psi)\}$.

(4) If $\mathrm{rank}_\Theta(\theta) = i + 1$ and $\theta = \exists v\ \eta$, then $U_\theta = \{\varepsilon \in \mathrm{VAsn}(M, \theta) : \exists\varepsilon'{\supseteq}\varepsilon\ (\varepsilon' \in U_\eta)\}$.

The expansion $(M, U_\theta)_{\theta\in\Theta}$ clearly satisfies $\mathrm{ElemDiag}(M)$. We defined the U_θ's according to Tarski's clauses; so τ_θ is satisfied for each $\theta \in \Theta$. The U_θ's with $\mathrm{rank}_\Theta(\theta) = 0$ are defined to agree with S_0, and S_0 obeys the Tarski clauses by assumption; so all our U_θ's agree with S_0. □

It may seem that formulas made true by a satisfaction class should look at least plausible. This is *entirely* false, as observed already in the Kotlarski–Krajewski–Lachlan paper [4]. Consider

$$\underbrace{0 \neq 0 \vee 0 \neq 0 \vee \cdots 0 \neq 0}_{a\text{-many disjuncts}},$$

where a is nonstandard. Call this formula δ. We adapt the proof of the Enayat–Visser theorem to make this true in a satisfaction class. Let F_0 be the set of standard $\mathscr{L}_\mathrm{A}$-formulas, and let S_0 be the standard satisfaction class for M, i.e.,

$$S_0 = \{(\theta, \varepsilon) \in F_0 \times M : \theta \text{ is true in } M \text{ under the variable assignment } \varepsilon\}.$$

Add the sentence $U_\delta(0)$ to T. Let $\theta \in \Theta$ in the $\triangleleft$-component containing δ. If θ has lower $\triangleleft$-rank than δ, then either it is $0 \neq 0$, in which case it must be false, or it is a nonstandard disjunction of $0 \neq 0$'s, in which case it can be set true. If θ has higher $\triangleleft$-rank than δ, then its truth value can be settled by treating δ as the constant $\top$. All other parts of the proof goes through as before. Therefore, every resplendent $M \models \mathrm{I}\Delta_0 + \exp$ has a full satisfaction class that makes such a δ true. If the satisfaction class satisfies some induction, then pathologies of this kind cannot occur.

Notice the Enayat–Visser theorem does *not* say that we can have $(K, S) \succcurlyeq (M, S_0)$ in general.

14.2 Using satisfaction classes

Theorem 14.4 (Lachlan [8])**.** Every nonstandard $M \models \mathrm{PA}$ that admits a full satisfaction class is recursively saturated.

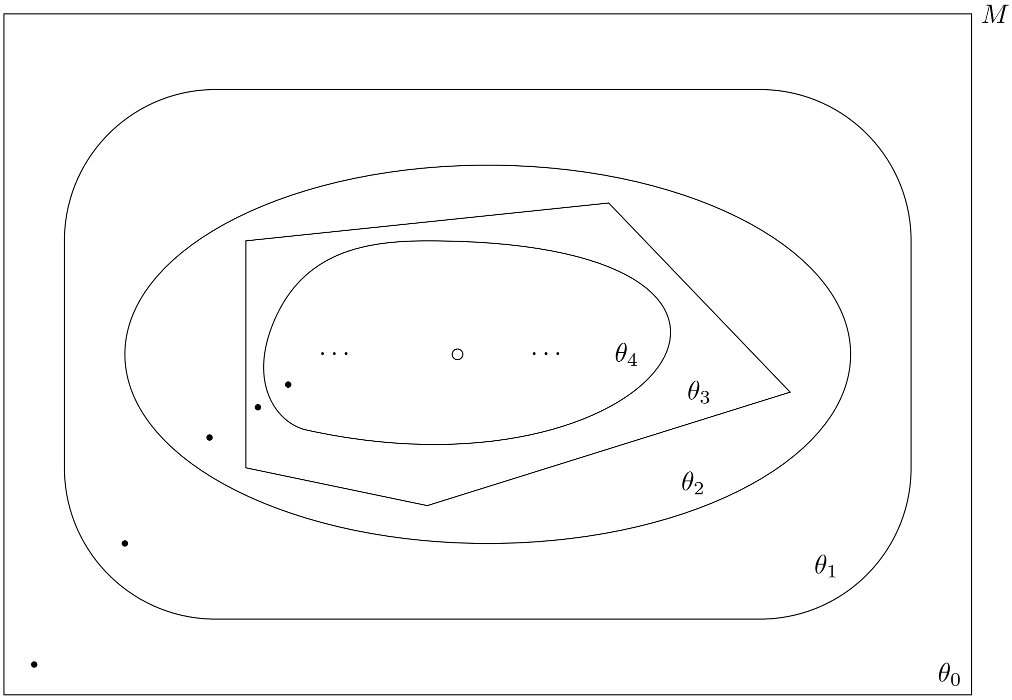

Figure 14.4: Our type $p(v) = \{\theta_i(v) : i \in \mathbb{N}\}$ over M

Notice if the satisfaction class is additionally required to satisfy induction, then the proof is only a simple overspill similar to what we did when showing Proposition 13.1(c). However, it is known [5, 7] that if a model of $\mathrm{I}\Delta_0 + \exp$ has a full satisfaction class satisfying Δ_0^0-induction, then it satisfies $\mathrm{PA} + \mathrm{Con(PA)}$. So such satisfaction classes are too strong for this lecture.

Proof. Let $p(v)$ be a recursive type over M. Without loss of generality, we can assume $p(v) = \{\theta_i(v) : i \in \mathbb{N}\}$ such that

(i) $(\theta_i(v))_{i\in\mathbb{N}}$ is recursive;

(ii) $\theta_0(v)$ is $v = v$;

(iii) $M \models \forall v\ \big(\theta_{i+1}(v) \to \theta_i(v)\big)$ for all $i \in \mathbb{N}$;

(iv) for every $i \in \mathbb{N}$, there are infinitely $v \in M$ such that $M \models \theta_i(v)$, because otherwise, we already know p is realized in M;

(v) $M \models \exists v\ \big(\theta_i(v) \wedge \neg\theta_{i+1}(v)\big)$ for all $i \in \mathbb{N}$, replacing $\theta_{i+1}(v)$ by $\theta'_{i+1}(v) = \theta_{i+1}(v) \wedge \exists v'{<}v\ \theta'_i(v')$ if necessary;

(vi) p is not realized in M.

We used our assumption that $M \models \mathrm{PA}$ when ensuring condition (v). Define $A_0 = \varnothing$ and

$$A_{i+1} = \{v \in M : M \models \theta_i(v) \wedge \neg\theta_{i+1}(v)\}$$

for each $i \in \mathbb{N}$. The A_{i+1}'s are disjoint and nonempty by (v). Thus (vi) implies

$$\{A_{i+1} : i \in \mathbb{N}\} \text{ is a partition of } M. \qquad (*)$$

Define $B_0 = \varnothing$ and for each $i \in \mathbb{N}$,

$$B_{i+1} = \begin{cases} A_1, & \text{if } B_i = \varnothing; \\ A_{k+1}, & \text{if } B_i \neq \varnothing \text{ and } k \leqslant i \text{ is least such that } A_k \cap B_i \neq \varnothing; \\ \varnothing, & \text{if none of the above happens.} \end{cases}$$

An external induction shows $B_i = A_i$ for all $i \in \mathbb{N}$. Notice, however, that these two sequences of sets are *intentionally* different. On the one hand, let $\alpha_0(v)$ be $v \neq v$ and $\alpha_{i+1}(v)$ be $\theta_i(v) \wedge \neg\theta_{i+1}(v)$ for every $i \in \mathbb{N}$, so that $\alpha_0, \alpha_1, \ldots$ define $A_0, A_1, \ldots$ respectively. On the other hand, let $\beta_0(v)$ be $v \neq v$ and $\beta_{i+1}(v)$ be

$$\begin{aligned} &\neg\exists u\ \beta_i(u) \wedge \alpha_1(v) \\ &\vee \bigvee_{k \leqslant i} \left(\begin{aligned} &\exists u\ \beta_i(u) \wedge \bigwedge_{j<k} \neg\exists u\ \big(\beta_i(u) \wedge \alpha_j(u)\big) \\ &\wedge \exists u\ \big(\beta_i(u) \wedge \alpha_k(u)\big) \wedge \alpha_{k+1}(v) \end{aligned} \right) \end{aligned}$$

for every $i \in \mathbb{N}$, so that $\beta_0, \beta_1, \ldots$ define $B_0, B_1, \ldots$ respectively. Here the big disjunctions and conjunctions are defined as follows.

- $\bigvee_{k<0} \eta_k = \bot$ and $\bigvee_{k<i+1} \eta_k = \eta_0 \vee \bigvee_{k<i} \eta_{k+1}$.
- $\bigwedge_{k<0} \eta_k = \top$ and $\bigwedge_{k<i+1} \eta_k = \eta_0 \wedge \bigwedge_{k<i} \eta_{k+1}$.

The argument below would *not* work if we set $\bigvee_{k<i+1} \eta_k = \bigvee_{k<i} \eta_k \vee \eta_i$ instead. It is because if we defined $\bigvee_{k<i+1} \eta_k$ in this way, then in case i is nonstandard, we would need to unravel the big disjunction nonstandardly many times to reach the η_k's with $k \in \mathbb{N}$. Since Tarski's clauses are local, we cannot guarantee a full satisfaction class to behave as it should in such unravellings. Notice both (α_i) and (β_i) are recursive as sequences of formulas. So they extend to sequences of length M via their definitions.

Suppose, towards a contradiction, that M has a satisfaction class S. Define

$$B_i = \big\{c \in M : S\big(\beta_i, \{\langle v, c\rangle\}\big)\big\}$$

for all $i \in M$. These agree with our previously defined B_i's because the Tarski clauses for S at finite levels are the same as the real Tarski clauses. Observe that

$$\forall i{\in}M\ \exists k{\in}\mathbb{N}\ B_{i+1} = A_{k+1}, \tag{†}$$

because

- if $B_i = \varnothing$, then $B_{i+1} = A_1$; and
- if $B_i \neq \varnothing$, then B_i meets some A_k, where $k \in \mathbb{N}$, by $(*)$, and so $B_{i+1} = A_{k+1}$ in this case.

Therefore, we can rewrite the definition of B_{i+1} as

$$B_{i+1} = \begin{cases} A_1, & \text{if } B_i = A_0; \\ A_{k+1}, & \text{if } B_i \neq A_0 \text{ and } k \leqslant i \text{ is least such that } B_i = A_k. \end{cases}$$

This implies, whenever $i \in M$ and $k \in \mathbb{N}$,

$$B_{i+1} = A_{k+1} \quad \Leftrightarrow \quad B_i = A_k. \tag{‡}$$

Take $\nu \in M \setminus \mathbb{N}$. Using (†), find $k \in \mathbb{N}$ such that

$$B_{\nu+1} = A_{k+1}$$

Then

$$\begin{array}{lcr} & B_\nu = A_k & \text{by (‡)} \\ \therefore & B_{\nu-1} = A_{k-1} & \text{by (‡)} \\ & \vdots & \\ \therefore & B_{\nu-k} = A_1 & \text{by (‡)} \\ \therefore & B_{\nu-(k+1)} = A_0 = \varnothing & \text{by (‡).} \end{array}$$

However, condition (†) also implies $B_{\nu-(k+1)} = A_{k'+1} \neq \varnothing$ for some $k' \in \mathbb{N}$. This gives the required contradiction. □

As observed by Smith [10], the θ_i's in the proof above need not be standard. However, if they are indeed all standard, then a Σ^M_ν-satisfaction class for some nonstandard ν is enough to make the proof go through.

Further exercises

These exercises are about an application of satisfaction classes to general model theory.

Definition. A structure M for a recursive language $\mathscr{L}$ is said to be *chronically resplendent* if

> whenever φ is a formula in a recursive language $\mathscr{L}^* \supseteq \mathscr{L}$ and $\bar{c} \in M$ such that $\varphi(\bar{c})$ is consistent with $\mathrm{ElemDiag}(M)$, there is a resplendent expansion of M satisfying $\varphi(\bar{c})$.

It is clear that chronically resplendent structures are resplendent. So, the following is a strengthening of Theorem 7.6.

Theorem 14.5 (Schlipf [9, p. 183])**.** Let M be a countable recursively saturated structure in a recursive language $\mathscr{L}$. Then M is chronically resplendent.

We will use a slight generalization of Theorem 14.4 to prove this theorem.

(a) Go through the proof of Theorem 14.4 and convince yourself that Theorem 14.4 remains true when M is replaced by an expansion M' of M in a recursive language $\mathscr{L}'$, provided M' satisfies full induction in the language $\mathscr{L}'$.

Proof of Theorem 14.5. Let φ be a formula in a recursive language $\mathscr{L}^* \supseteq \mathscr{L}$, and $\bar{c} \in M$ such that $\mathrm{ElemDiag}(M) + \varphi(\bar{c})$ is consistent. Without loss of generality, suppose $\mathscr{L}^* \cap \mathscr{L}_{\mathrm{A}} = \varnothing$. Define

$$\mathscr{L}' = \mathscr{L}^* \cup \mathscr{L}_{\mathrm{A}} \cup \{d\} \quad \text{and} \quad \mathscr{L}'' = \mathscr{L}' \cup \{S\},$$

where d is a new constant symbol and S is a new binary predicate symbol. Consider the set Ψ of $\mathscr{L}''(M)$-formulas consisting of the following:

- the axioms of PA^-;
- the induction axioms for all $\mathscr{L}''$-formulas, i.e.,

  $$\forall \bar{z}\ \big(\theta(0, \bar{z}) \land \forall x\ \big(\theta(x, \bar{z}) \to \theta(x+1, \bar{z})\big) \to \forall x\ \theta(x, \bar{z})\big)$$

  for all formulas $\theta(x, \bar{z})$ in the language $\mathscr{L}''$;
- $d > 0 + \underbrace{1 + 1 + \cdots + 1}_{n\text{-many 1's}}$ for all $n \in \mathbb{N}$;
- S satisfies the $\mathscr{L}'$-version of Tarski's clauses (T0)–(T4); and
- $\varphi(\bar{c})$.

(b) By considering a bijection $\mathbb{N} \to M$, apply resplendency from Theorem 7.6 to show that every finite subset of $\mathrm{ElemDiag}(M) + \Psi$ is consistent.

(c) Conclude, using Theorem 7.6 and part (a), that M has a resplendent expansion satisfying $\varphi(\bar{c})$. □

Theorem 14.5 remains true if we can replace $\varphi(\bar{c})$ by a recursive set of formulas, as in Theorem 7.6. It is unknown whether every resplendent structure is chronically resplendent [9, p. 192].

Further reading

Kotlarski's survey [6] and Engström's thesis [2] both contain plenty of information about satisfaction classes. For some nice applications, see Kossak [3].

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