MODEL THEORY OF ARITHMETIC

Lecture 5: Semiregular cuts

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> Since 1931, the year Gödel's Incompleteness Theorems were published, mathematicians have been looking for a strictly mathematical example of an incompleteness in first-order Peano arithmetic, one which is mathematically simple and interesting and does not require the numerical coding of notions from logic.
>
> Jon Barwise [8, page 1133]

We show that the totality of the Ackermann function is not provable in $\mathrm{I}\Sigma_1$.

5.1 The Grzegorczyk hierarchy

At the end of last lecture, we saw that end extensions are always Δ_0-elementary. It is thus tempting to say that all cuts are Δ_0-elementary. However, this is not strictly correct because some cuts are not even $\mathscr{L}_\mathrm{A}$-structures.

Definition. A *multiplicative cut*, or an *am-cut*, is a cut that is closed under multiplication (and hence also addition).

Proposition 4.12 implies that every end extension satisfies all Σ_1-formulas true in the ground model, and every am-cut satisfies all Π_1-formulas true in the universe it lives in.

Corollary 5.1. If $M \models \mathrm{I}\Delta_0$ and I is an am-cut of M, then $I \models \mathrm{I}\Delta_0$.

Proof. Notice $\mathrm{PA}^- \subseteq \Pi_1$. One can verify that the scheme of Δ_0-induction is equivalent over PA^- to the Π_1-sentences

$$\forall \bar{z}\ \forall b\ \big(\eta(0,\bar{z}) \wedge \forall x{<}b\ \big(\eta(x,\bar{z}) \rightarrow \eta(x+1,\bar{z})\big) \rightarrow \forall x{\leqslant}b\ \eta(x+1,\bar{z})\big),$$

where $\eta \in \Delta_0$. For instance, to prove the sentence displayed above, one applies induction to the Δ_0-formula $x < b \rightarrow \eta(x,\bar{z})$. □

Alternatively, one can show that all am-cuts of a model of $\mathrm{I}\Delta_0$ satisfy Δ_0-induction using the equivalence of $\mathrm{L}\Delta_0$ and $\mathrm{I}\Delta_0$ that we got from Theorem 2.3.

The following fast growing functions come from the *Grzegorczyk hierarchy* [4]. They can also be viewed as a version of the *Ackermann function* [1]. Notice the name 'Ackermann function' may refer to different functions for different authors. The domain of definition below is intended to be some model of arithmetic.

Definition. Set $F_0(x) = x+1$ and

$$F_{n+1}(x) = F_n^{(x+1)}(x) = \overbrace{F_n \circ F_n \circ \cdots \circ F_n}^{x+1 \text{ times}}(x)$$

Example 5.2. $F_2(2) = F_1^{(3)}(2) = F_1 \circ F_1 \circ F_1(2) = F_1 \circ F_1 \circ F_0^{(3)}(2) = F_1 \circ F_1(5)$
$= F_1 \circ F_0^{(6)}(5) = F_1(11) = F_0^{(12)}(11) = 23.$

Observation. Evaluating $F_n(x)$ is a matter of rewriting terms. So, via a suitable coding of terms, we can express $F_n(x) = y$ over $\mathrm{I}\Delta_0$ as

> there is a sequence $\langle F_n(x), \ldots, y\rangle$ of terms obeying the rewriting rules set out in the definition of the F_n's.

Such an expression has the variables n, x, y free, and is Σ_1. One can readily verify that

$$\mathrm{I}\Delta_0 \vdash \forall n, x, y, y' \ \big(F_n(x) = y \wedge F_n(x) = y' \to y = y'\big).$$

The following lemma is one of the ways to explain why it usually does not matter where the F_n's are evaluated.

Lemma 5.3. Let $n, x, y \in I \subseteq_{\mathrm{e}} M \models \mathrm{I}\Delta_0$ where I is an am-cut. For all $y' \in M$, if $I \models F_n(x) = y$ and $M \models F_n(x) = y'$, then $y = y'$.

Proof. Since $F_n(x) = y$ is Σ_1, we know $M \models F_n(x) = y$ by Proposition 4.12. Thus the observation above tells us $y = y'$. □

The F_n's have many nice monotonicity properties, only two of which are relevant to us here.

Lemma 5.4. $\mathrm{I}\Sigma_1$ proves

(a) $\forall n, x \ F_n(x) > x$; and

(b) $\forall m, n, x \ \big(m < n \to F_m(x) < F_n(x)\big)$.

Proof. First, show by Π_1-induction on k that

$$\forall n \ \big(\forall x \ F_n(x) > x \to \forall k, x \ F_n^{(k+1)}(x) > x\big).$$

Then prove (a) by induction on n leaving x universally quantified. For (b), fix m, x and show by induction on k that $F_m(x) < F_{m+k+1}(x)$. □

There is a good description of which recursive functions can be proved total in $\mathrm{I}\Sigma_1$ in terms of the Grzegorczyk hierarchy.

Proposition 5.5. $\mathrm{I}\Sigma_1 \vdash \forall x \ \exists y \ F_n(x) = y$ for every $n \in \mathbb{N}$.

Proof. Straightforward induction on n. □

Theorem 5.6. $\mathrm{I}\Sigma_1 \nvdash \forall n \ \forall x \ \exists y \ F_n(x) = y$.

The rest of the lecture is devoted to a proof of this. The first proofs of this theorem are all proof-theoretic. Here we employ a model-theoretic method that has a combinatorial flavour. In particular, we will need some Pigeonhole Principle. Recall from Example 2.5 that as in set theory, we identify a number a in a model of arithmetic with the set of its predecessors $\{0, 1, \ldots, a-1\}$.

Coded Pigeonhole Principle. Let $a, b \in M \models \mathrm{I}\Delta_0 + \exp$ where $a < b$, and let $g \in M$ code a function $a \to b$. Then $\mathrm{Im}(g) \neq b$.

Proof. Without loss, we may assume $\mathrm{Dom}(g) \neq \varnothing$. We will show by Δ_0-induction on d that

$$M \models \forall d \ \forall g{<}d \ \forall a, b{<}g \ \big(g \colon a+1 \to b+1 \wedge a < b \to \exists y{\leqslant}b \ \forall x{\leqslant}a \ (g(x) \neq y)\big).$$

Here we use abbreviations from set theory. For example, the expression $g(x) = y$ stands for $\langle x, y\rangle \in \mathrm{Ack}(g)$, which is Δ_0.

Let $g \in M$ code a function $a+1 \to b+1$, where $a < b$. Note $b > a \geqslant 0$. So $b > 0$. Take any $v < b$. If $v \notin \mathrm{Im}(g)$ or $b \notin \mathrm{Im}(g)$, then we are done. So suppose not. If $g(a) < b$, then we may as well assume $g(a) = v$. Let $U = g^{-1}(b)$ and define $g' \in M$ by

$$\mathrm{Ack}(g') = \mathrm{Ack}(g) \cup \{\langle u, v\rangle : u \in U\} \setminus \{\langle a, g(a)\rangle, \langle u, b\rangle : u \in U\}.$$

Then g' code a function $a \to b$, and $g' < g$ since $v < b$. Ignoring the trivial case when $a = 0$, we may apply the induction hypothesis to find $y < b$ that is not in $\mathrm{Im}(g')$.

To finish the proof, let us show $y \notin \mathrm{Im}(g)$. If $u \in U$, then $g(u) = b > y$. If $g(a) = b$, then $g(a) \neq y$ for the same reason. If $g(a) < b$, then $g(a) = v \neq y$ because $v \in \mathrm{Im}(g')$ but $y \notin \mathrm{Im}(g')$. If $x \leqslant a$ but $x \notin U \cup \{a\}$, then $g(x) = g'(x) \neq y$ since $y \notin \mathrm{Im}(g')$. □

By going through the proof of Theorem 2.7 carefully, one can verify that the proof above actually does not require exp.

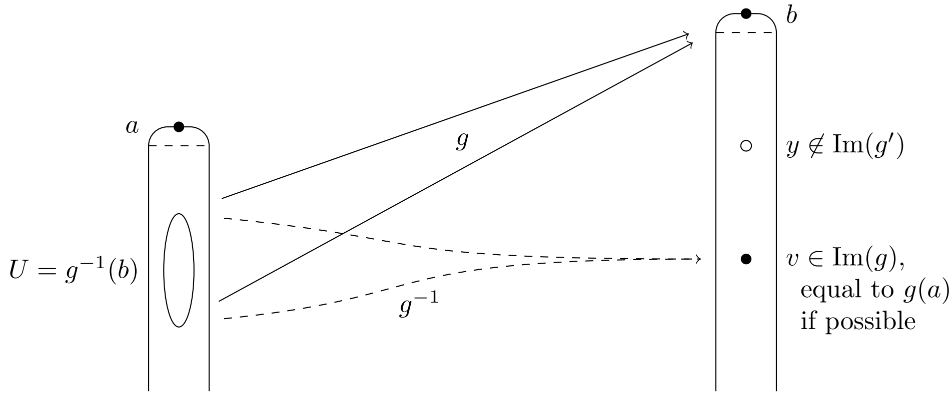

Figure 5.1: Proving the Coded Pigeonhole Principle

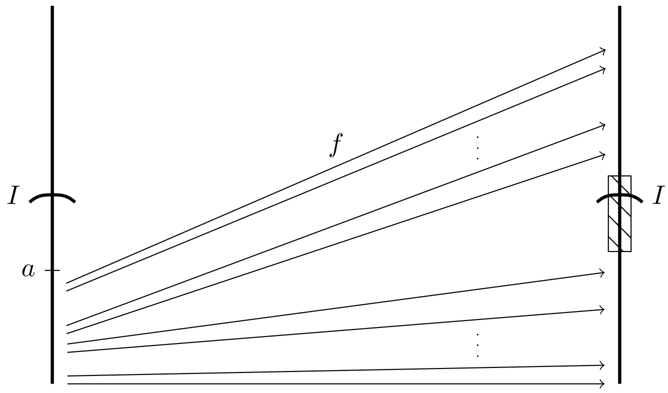

Figure 5.2: Semiregular cuts

5.2 Semiregular cuts

Since we cannot construct nonstandard models of $\mathrm{I}\Sigma_1$ from scratch, we need a way of building new models from old ones. In analogy to the argument for showing the independence of the axiom of infinity from the rest of ZFC using the hereditarily finite sets, we consider cuts that satisfy $\mathrm{I}\Sigma_1$, called *semiregular cuts*. This notion of cuts came from that of regular cardinals. Informally speaking, no coded function from a smaller initial segment can have an image unbounded in a semiregular cut.

Definition (Kirby–Paris [5]). Let $M \models \mathrm{I}\Delta_0$. A cut $I \subseteq_{\mathrm{e}} M$ is *semiregular* if

whenever $f \in M$ which codes a function with domain $a \in I$,

$$\mathrm{Im}(f) \cap I \not\subseteq_{\mathrm{cf}} I.$$

Here $X \subseteq_{\mathrm{cf}} Y$ means "X is an unbounded (or cofinal) subset of Y".

There is a notion of *regular cuts* in arithmetic that also came from the regularity of cardinals. This other notion turns out to be strictly stronger than semiregularity [5].

Example 5.7. The standard cut $\mathbb{N}$ is semiregular in all $M \models \mathrm{I}\Delta_0$, because every element of $\mathbb{N}$ has only finitely many predecessors.

Before showing that semiregular cuts satisfy $\mathrm{I}\Sigma_1$, we need to first check they are indeed $\mathscr{L}_{\mathrm{A}}$-structures.

Lemma 5.8. Semiregular cuts in models of $\mathrm{I}\Delta_0 + \exp$ are multiplicative.

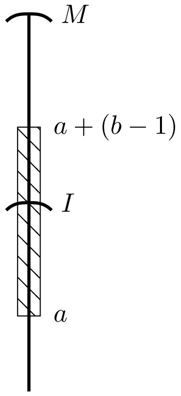

Figure 5.3: The image of $f : b \to M$ defined by $f(x) = a + x$, where $a, b \in I$ but $a + b \notin I$

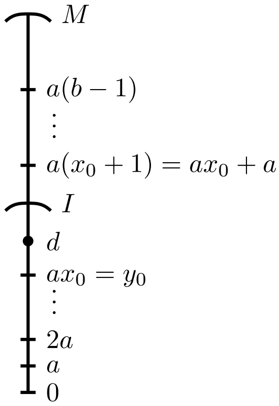

Figure 5.4: The image of $g : b \to M$ defined by $g(x) = ax$, where $a, b \in I$ but $ab \notin I$

Proof. Let $a, b \in I \subsetneq_{\mathrm{e}} M \models \mathrm{I}\Delta_0 + \exp$ where I is semiregular in M. Without loss, assume $a, b \neq 0$. Define $f : b \to M$ by $f(x) = a + x$. This is coded in M because we have Δ_0-separation from Theorem 2.7. By semiregularity,

$$\mathrm{Im}(f) \cap I = [a, a + b) \cap I \not\subseteq_{\mathrm{cf}} I,$$

so that we must have $a + b \in I$. Define $g : b \to M$ by $g(x) = ax$. This is again coded in M by Δ_0-separation. Using semiregularity, find $d \in I$ that bounds $\mathrm{Im}(g) \cap I$. Notice

$$\{y < d : M \models \exists x{<}b\ (g(x) = y)\} \in \mathrm{Cod}(M)$$

by Δ_0-separation. It contains 0 and so must be nonempty. Let y_0 be its maximum, which we know exists by Lemma 2.6(b). Find $x_0 < b$ such that $y_0 = ax_0$. We already showed I is closed under addition. So $a(x_0 + 1) = ax_0 + a \in I$. By the maximality of y_0, it must be the case that $x_0 = \max \mathrm{Dom}(g) = b - 1$. Thus $ab = a(x_0 + 1) \in I$. □

Semiregular cuts actually satisfy a bit more than just $\mathrm{I}\Sigma_1$.

Theorem 5.9 (Kirby–Paris [5])**.** The following are equivalent for $I \subsetneq_{\mathrm{e}} M \models \mathrm{I}\Delta_0 + \exp$.

(a) I is semiregular in M.

(b) I is multiplicative and $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$.

Proof. We will only show that the semiregularity of I implies $I \models \mathrm{I}\Sigma_1$, because this is the only part we will need. Suppose I is semiregular. We know from Corollary 5.1 and Lemma 5.8 that $I \models \mathrm{I}\Delta_0$. In view of Theorem 2.3, it suffices to show $I \models \mathrm{L}\Sigma_1$. Pick $\eta(x, y) \in \Delta_0(I)$ such that $I \models \exists x\ \exists y\ \eta(x, y)$. Let $a \in I \models \exists y\ \eta(a, y)$ and $c \in M \setminus I$. The plan is to use semiregularity to find a bound for the y-quantifier in $\exists y\ \eta(x, y)$ such that the Σ_1-formula becomes Δ_0 below a. Define $f : a + 1 \to M$ by

$$f(x) = \begin{cases} (\min y \leqslant c)(M \models \eta(x, y)), & \text{if it exists;} \\ c, & \text{otherwise.} \end{cases}$$

This function is coded in M by Δ_0-separation. Using semiregularity, find $d \in I$ that bounds $\mathrm{Im}(f) \cap I$. In other words, for all $x \leqslant a$,

$$f(x) \in I \quad \Leftrightarrow \quad f(x) < d. \qquad (*)$$

Now, for every $x \leqslant a$, the following implications hold.

$I \models \exists y{<}d\ \eta(x, y)$	$\Rightarrow$	$I \models \exists y\ \eta(x, y)$.	
$I \models \exists y\ \eta(x, y)$	$\Rightarrow$	$M \models \eta(x, y)$ for some $y \in I$	by Proposition 4.12.
$M \models \eta(x, y)$ for some $y \in I$	$\Rightarrow$	$M \models \eta(x, f(x))$ and $f(x) \in I$	as $f(x) = (\min x)(\eta(x, y))$.
$M \models \eta(x, f(x))$ and $f(x) \in I$	$\Rightarrow$	$M \models \exists y{<}d\ \eta(x, y)$	by $(*)$.
$M \models \exists y{<}d\ \eta(x, y)$	$\Rightarrow$	$I \models \exists y{<}d\ \eta(x, y)$	by Proposition 4.12.

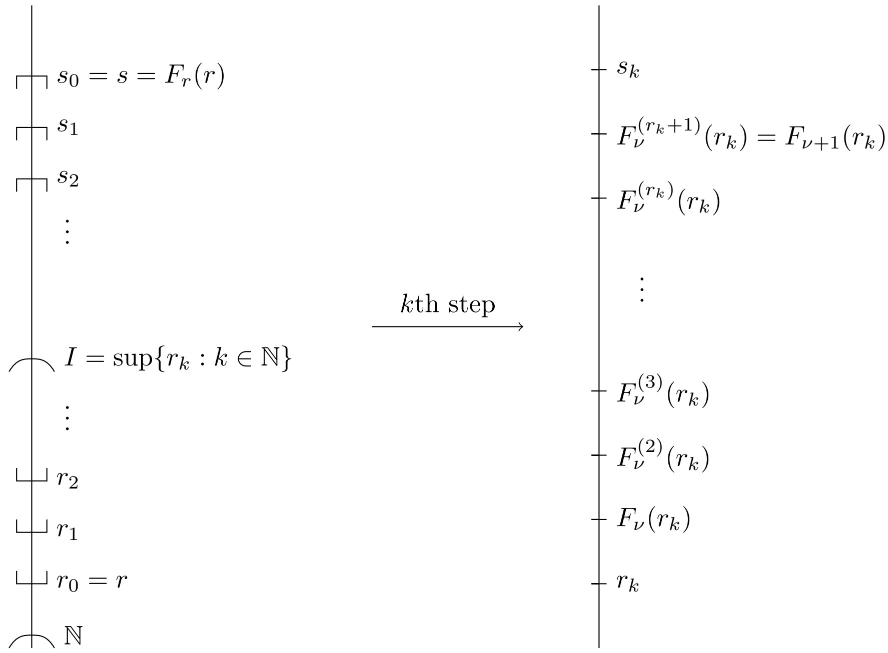

Figure 5.5: Catching a semiregular cut

It follows that $\min\{x \leqslant a : I \models \exists y\, \eta(x,y)\} = \min\{x \leqslant a : I \models \exists y{<}d\, \eta(x,y)\}$. The latter exists because of $\mathrm{L}\Delta_0$ and our choice of a. □

We are finally ready to show Theorem 5.6. The argument presented here is from Kirby–Paris [5]. The intuitive idea behind the unprovability of the totality of the Ackermann function in $\mathrm{I}\Sigma_1$ is that this function grows too fast. It grows so fast that the gap between r and $F_r(r)$ for a nonstandard r is big enough to catch a cut satisfying $\mathrm{I}\Sigma_1$. In such a cut $F_r(r)$ cannot exist.

Proof of Theorem 5.6. Work in a countable nonstandard $M \models \mathrm{I}\Sigma_1 + \forall n\, \forall x\, \exists y\, F_n(x) = y$, which we know exists by the Compactness Theorem. Unless otherwise specified, the F_n's in this proof are all evaluated in M. Pick $r \in M \setminus \mathbb{N}$ and let $s = F_r(r)$. By an external recursion, we will build

$$[r,s] = [r_0,s_0] \supseteq [r_1,s_1] \supseteq [r_2,s_2] \supseteq \cdots$$

with the induction condition that $F_n(r_k) \leqslant s_k$ for all $n,k \in \mathbb{N}$. At the end, we will make

$$I = \sup\{r_k : k \in \mathbb{N}\} = \{x \in M : x < r_k \text{ for some } k \in \mathbb{N}\}$$

semiregular, so that $I \models \mathrm{I}\Sigma_1 + \forall y\, F_r(r) \neq y$ by Theorem 5.9 and Lemma 5.3.

Notice $[r_0,s_0]$ satisfies the inductive condition by Lemma 5.4. Let $(f_k)_{k\in\mathbb{N}}$ be an enumeration of all functions $M \to M$ in $\mathrm{Cod}(M)$ in which every such function appears infinitely often. This exists because M is countable. Suppose $[r_k,s_k]$ is found. Consider f_k. If $\mathrm{Dom}(f_k) \neq a$ for any $a \leqslant r_k$, then we can set $[r_{k+1},s_{k+1}] = [r_k,s_k]$, because it does not look like f_k can harm the semiregularity of I at the current stage. Suppose $\mathrm{Dom}(f_k) = a_k$, where $a_k \leqslant r_k$. Use Σ_1-overspill to find $\nu \in M \setminus \mathbb{N}$ such that $s_k \geqslant F_{\nu+1}(r_k) = F_\nu^{(r_k+1)}(r_k)$. There are at most a_k points in $\mathrm{Im}(f_k)$, but there are $r_k + 1$ disjoint subintervals of $[r_k,s_k]$ of the form $[F_\nu^{(i)}(r_k), F_\nu^{(i+1)}(r_k))$, where $i \leqslant r_k$. The Coded Pigeonhole Principle applies to this situation because the function $g_k \colon a \to r_k + 1$ defined by

$$g_k(x) = \min\{i \leqslant r_k : F_\nu^{(i+1)}(r_k) \geqslant f_k(x)\}$$

is in $\mathrm{Cod}(M)$ by Σ_1-separation in Theorem 2.7. Let $[r_{k+1},s_{k+1}]$ be an interval of the form $[F_\nu^{(i)}(r_k), F_\nu^{(i+1)}(r_k))$, where $i \leqslant r_k$, which contains no point from $\mathrm{Im}(f_k)$. Such an interval satisfies the inductive condition because $\nu > \mathbb{N}$, and so we can move on to the next step of the construction.

Now, if $f \colon a \to M$ in $\mathrm{Cod}(M)$ where $a \in I$, then for some large enough $k \in \mathbb{N}$ we have $f = f_k$ and $a \leqslant r_k$, so that $r_{k+1} \in I$ that bounds $\mathrm{Im}(f) \cap I$ by construction. Therefore, the cut I is semiregular in M. □

Further exercises

The argument we saw in this lecture is an instance of a general way of obtaining the unprovability of Π_2-sentences originated from Kirby–Paris [5] called the method of *indicators*.

Definition. Let T be an $\mathscr{L}_\mathrm{A}$-theory extending $\mathrm{I}\Sigma_1$. An *indicator* for cuts satisfying T is a binary function Y that is Σ_1-definable over T without parameters and has the following properties.

(1) $\mathrm{I}\Sigma_1$ proves

  (i) $\forall x, y, y' \ \big(y \leqslant y' \to Y(x,y) \leqslant Y(x,y')\big)$; and

  (ii) $\forall n, x, y \ \big(Y(x,y) \geqslant n \to \exists y'{\leqslant}y \ Y(x,y') = n\big)$.

(2) For every $a, b \in M \models \mathrm{I}\Sigma_1$,

$$a \in I < b \text{ for some am-cut satisfying } T \quad \Leftrightarrow \quad Y(a,b) > \mathbb{N}.$$

In these exercises, we demonstrate how indicators give rise to unprovability results generally.

(a) Define $Y(x,y) = (\max n)(F_n(x) \leqslant y)$; more precisely, define $Y(x,y) = n$ to be the formula

$$F_n(x) \leqslant y \wedge \forall z{\leqslant}y \ (z \neq F_{n+1}(x)).$$

Using what we saw in this lecture, explain why Y is an indicator for cuts satisfying $\mathrm{I}\Sigma_1$.

Let T be a consistent $\mathscr{L}_\mathrm{A}$-theory extending $\mathrm{I}\Sigma_1$, and let Y be an indicator for cuts satisfying T.

(b) As we will see in Lecture 8, every consistent extension of $\mathrm{B}\Sigma_1$ has a model that is isomorphic (and hence elementarily equivalent) to a proper cut of itself. Use this fact to show that $T \vdash \forall x \ \exists y \ Y(x,y) \geqslant n$ for every $n \in \mathbb{N}$.

(c) Show $T \nvdash \forall n \ \forall x \ \exists y \ Y(x,y) \geqslant n$ by imitating the proof of Theorem 5.6.

Further comments

Provably total recursive functions

The class of recursive functions a true $\mathscr{L}_\mathrm{A}$-theory can prove total is a useful indication of how strong this theory is.

Definition. A *provably total recursive function* of an $\mathscr{L}_\mathrm{A}$-theory T is a formula $\varphi \in \Sigma_1$ such that

$$T \vdash \forall x \ \exists! y \ \varphi(x,y).$$

For convenience, instead of referring to these as formulas, we usually use functional notation.

The provably total recursive functions of $\mathrm{I}\Sigma_1$ have a particularly neat description.

Definition. A function $\mathbb{N} \to \mathbb{N}$ is *primitive recursive* if its values can be computed by a program built up from polynomial functions using only **for**-loops with polynomial bounds.

Notice this is not *the same as the functions with Δ_0-graphs*, because it may be hard to compute the values of a function even though it is easy to check that a value is correct.

Theorem 5.10 (Mints [7], Parsons [9], Takeuti [10], independently)**.** The provably total recursive functions in $\mathrm{I}\Sigma_1$ are exactly the primitive recursive functions.

So, Proposition 5.5 says F_n is primitive recursive for every $n \in \mathbb{N}$. In fact, more is true.

Theorem 5.11 (Grzegorczyk [4])**.** Every primitive recursive function $F\colon \mathbb{N} \to \mathbb{N}$ is *dominated* (or *majorized*) by some F_n where $n \in \mathbb{N}$, i.e., $F(x) \leqslant F_n(x)$ for all large enough $x \in \mathbb{N}$.

This can be proved using Further exercise (a), and it easily implies Theorem 5.6.

The Pigeonhole Principle

The Pigeonhole Principle usually does not appear in coded form as we presented it. More frequently, it appears as schemes.

Definition. Let Γ be a set of $\mathscr{L}_{\mathrm{A}}$-formulas. Then Γ-PHP is the scheme

$$\forall \bar{z}\ \forall a\ \big(\forall x{<}a\ \exists! y{\leqslant}a\ \theta(x,y,\bar{z}) \to \exists y{\leqslant}a\ \forall x{<}a\ \neg\theta(x,y,\bar{z})\big),$$

where $\theta \in \Gamma$.

As Dimitracopoulos and Paris [3] showed, there is a level-by-level correspondence between the hierarchy of Pigeonhole Principles and the induction–collection schemes. Putting together our Coded Pigeonhole Principle and Theorem 2.7, we see that $\mathrm{I}\Delta_0 + \exp \vdash \Delta_0$-PHP. It is, however, not known whether exp can be removed in this statement.

Question 5.12 (Angus Macintyre)**.** Does $\mathrm{I}\Delta_0 \vdash \Delta_0$-PHP?

Further reading

The first true 'mathematically interesting' statement that is unprovable in PA was discovered [8] using the method of indicators described in the Further exercises. For an accessible introduction to independence results, see Kirby–Paris [6] or Bovykin [2].

References

[1] Wilhelm Ackermann. Zum Hilbertschen Aufbau der reellen Zahlen. *Mathematische Annalen*, 99(1):118–133, 1928.

[2] Andrey Bovykin. Brief introduction to unprovability. In S. Barry Cooper, Herman Geuvers, Anand Pillay, and Jouko Väänänen, editors. *Logic Colloquium 2006*, volume 32 of *Lecture Notes in Logic*, pages 38–64. Association for Symbolic Logic, Cambridge University Press, Cambridge, 2009.

[3] Constantinos Dimitracopoulos and Jeff B. Paris. The pigeonhole principle and fragments of arithmetic. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 32(1–5):73–80, 1986.

[4] Andrzej Grzegorczyk. Some classes of recursive functions. *Rozprawy Matematyczne*, IV:1–46, 1953.

[5] Laurence A. S. Kirby and Jeff B. Paris. Initial segments of models of Peano's axioms. In Alistair Lachlan, Marian Srebrny, and Andrzej Zarach, editors. *Set Theory and Hierarchy Theory V*, volume 619 of *Lecture Notes in Mathematics*, pages 211–226. Springer-Verlag, Berlin, 1977.

[6] Laurence A. S. Kirby and Jeff B. Paris. Accessible independence results for Peano arithmetic. *Bulletin of the London Mathematical Society*, 14(4):285–293, July 1982.

[7] Grigori Mints. Quantifier-free and one-quantifier systems. *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova Akademii Nauk SSSR*, 20:115–133, 1971. Translated to English in *Journal of Soviet Mathematics*, 1(1):71–84, January–February 1973.

[8] Jeff B. Paris and Leo Harrington. A mathematical incompleteness in Peano arithmetic. In Jon Barwise, H. Jerome Keisler, Kenneth Kunen, Yiannis N. Moschovakis, and Anne S. Troelstra, editors. *Handbook of Mathematical Logic*, volume 90 of *Studies in Logic and the Foundations of Mathematics*, chapter D.8, pages 1133–1142. North-Holland Publishing Company, Amsterdam, 1977.

[9] Charles Parsons. On a number theoretic choice schema and its relation to induction. In Akiko Kino, John Myhill, and Richard E. Vesley, editors. *Intuitionism and Proof Theory*, volume 60 of *Studies in Logic and the Foundations of Mathematics*, pages 459–473. North-Holland Publishing Company, Amsterdam, 1970.

[10] Gaisi Takeuti. *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Company, Amsterdam, second edition, 1987.