MODEL THEORY OF ARITHMETIC

Lecture 8: Back-and-forth

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> Harvey [Friedman] was on the Flip Wilson show. It must have been in 1971 [...] since [...] Joram Hirschfeld was just finishing his thesis then. He heard Harvey talk about embedding models of PA as initial segments and that gave him an idea that ended up in his thesis.
>
> James Schmerl, as quoted by Ali Enayat [1]

8.1 The standard system

The set of complete types realized is an important invariant for a structure. Recall a *type* over a structure M for a language $\mathscr{L}$ is a set of $\mathscr{L}(M)$-formulas $p(\bar{v})$ with finitely many free variables that is consistent with ElemDiag(M). Equivalently, a set $p(\bar{v})$ of $\mathscr{L}(M)$-formulas with finitely many free variables is a *type* over M if and only if it is *finitely satisfied* in M, i.e., we have $M \models \exists \bar{v} \bigwedge\!\!\!\bigwedge p_0(\bar{v})$ for all finite $p_0(\bar{v}) \subseteq p(\bar{v})$. Quite often, the $p(\bar{v})$'s are closed under finite conjunction. In these cases, to verify the finite satisfiability of $p(\bar{v})$ in M, it suffices to show $M \models \exists \bar{v}\ \theta(\bar{v})$ for every $\theta \in p$.

Definition. Let M be a structure in a language $\mathscr{L}$.

- Let $p(\bar{v})$ be a type over M. Sometimes, we write $p(\bar{v}) = p(\bar{v}/\bar{c})$ to indicate the finitely many parameters $\bar{c} \in M$ that appear in $p(\bar{v})$. In this case, let

$$p(\bar{v}/\bar{z}) = \{\theta(\bar{v}, \bar{z}) : \theta(\bar{v}, \bar{c}) \in p(\bar{v}, \bar{c})\}.$$

- A type $p(\bar{v}/\bar{c})$ is *complete* if for all $\theta(\bar{v}, \bar{z}) \in \mathscr{L}$,

$$\text{either} \quad \theta(\bar{v}, \bar{c}) \in p(\bar{v}/\bar{c}) \quad \text{or} \quad \neg\theta(\bar{v}, \bar{c}) \in p(\bar{v}/\bar{c}).$$

- If $\bar{a}, \bar{c} \in M$, then $\text{tp}(\bar{a}/\bar{c}) = \{\theta(\bar{v}, \bar{c}) \in \mathscr{L}(\bar{c}) : M \models \theta(\bar{a}, \bar{c})\}$.

Notice a type of the form $\text{tp}(\bar{a}/\bar{c})$ is always complete and is realized in the model it comes from. In a recursively saturated model of arithmetic, the set of realized complete types has a particularly nice characterization.

Definition. Let $M \models \mathrm{I}\Delta_0$. The *standard system* of M, denoted $\mathrm{SSy}(M)$, is $\mathrm{Cod}(M/\mathbb{N})$. A type $p(\bar{v})$ is *coded* in M if it involves only finitely many parameters, say $\bar{c} \in M$, and

$$\{\ulcorner\theta(\bar{v}, \bar{z})\urcorner : \theta(\bar{v}, \bar{c}) \in p(\bar{v})\} \in \mathrm{SSy}(M).$$

Proposition 8.1. A complete type $p(\bar{v}/\bar{c})$ over a recursively saturated $M \models \mathrm{I}\Delta_0$ is realized if and only if it is coded.

Proof. First, suppose $p(\bar{v}/\bar{z}) \in \mathrm{SSy}(M)$. Let $s \in M$ such that $p(\bar{v}, \bar{z}) = \mathrm{Ack}(s/\mathbb{N})$. Then

$$p'(\bar{v}) = \{\theta(\bar{v}, \bar{c}) \leftrightarrow \ulcorner\theta\urcorner \in \mathrm{Ack}(s) : \theta \in \mathscr{L}_{\mathrm{A}}\}$$

is a recursive type that is essentially the same as p. So $p(\bar{v})$ is realized in M by recursive saturation.

Conversely, suppose $p(\bar{v})$ is realized by $\bar{a} \in M$. Then since we can coded any finite set,

$$q(w) = \{\theta(\bar{a}, \bar{c}) \leftrightarrow \ulcorner\theta\urcorner \in \mathrm{Ack}(w) : \theta \in \mathscr{L}_\mathrm{A}\}$$

is a recursive type over M. So it is realized in M by recursive saturation. Any $s \in M$ realizing q codes p. □

Remark 8.2. One sees that the completeness of the type is only used in coding realized types. One also readily sees that the proposition no longer holds if this completeness requirement is dropped.

As a corollary, a countable recursively saturated model of arithmetic is completely determined by its theory and its standard system.

Theorem 8.3. The following are equivalent for countable recursively saturated $M, N \models \mathrm{I}\Delta_0$.

(a) $M \cong N$.

(b) $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $\mathrm{SSy}(M) = \mathrm{SSy}(N)$.

Proof. The implication (a) ⇒ (b) is trivial. For the converse, suppose (b) holds. We carry out a back-and-forth argument to find an isomorphism $M \to N$. By recursion, we will define $(r_m)_{m\in\mathbb{N}}$ in M and $(s_m)_{m\in\mathbb{N}}$ in N such that $f\colon r_m \mapsto s_m$ is an isomorphism $M \to N$. At each step $m \in \mathbb{N}$, we have $r_0, r_1, \ldots, r_{m-1} \in M$ and $s_0, s_1, \ldots, s_{m-1} \in N$ satisfying the inductive condition

$$\mathrm{tp}_M(\bar{r}) = \mathrm{tp}_N(\bar{s}).$$

This ensures f preserves the interpretation of all $\mathscr{L}_\mathrm{A}$-symbols. In particular, our f will be injective on its domain.

The inductive condition is satisfied initially because $\mathrm{tp}_M(\,) = \mathrm{Th}(M) = \mathrm{Th}(N) = \mathrm{tp}_N(\,)$.

Suppose $r_0, r_1, \ldots, r_{m-1} \in M$ and $s_0, s_1, \ldots, s_{m-1} \in N$ that satisfy the inductive condition.

Forth. Suppose $m = 2\ell + 1$. We force f to be total in these steps. Consider $c_\ell \in M$, which comes from a fixed enumeration $(c_\ell)_{\ell\in\mathbb{N}}$ of M. We put $c_\ell \in \mathrm{Dom}(f)$ by setting $r_m = c_\ell$. We want $s_m \in N$ such that $\mathrm{tp}_M(\bar{r}, r_m) = \mathrm{tp}_N(\bar{s}, s_m)$. Let $p(v/\bar{r}) = \mathrm{tp}_M(r_m/\bar{r})$. Then $p(v/\bar{z}) \in \mathrm{SSy}(M) = \mathrm{SSy}(N)$ by Proposition 8.1. The set $p(v/\bar{s})$ is finitely satisfied in N because if $\theta(v, \bar{s}) \in p(v, \bar{s})$, then

$$\begin{array}{lll} & \theta(v, \bar{r}) \in p(v/\bar{r}) = \mathrm{tp}_M(r_m/\bar{r}) & \\ \therefore & M \models \exists v\ \theta(v, \bar{r}) & \text{since } M \models \theta(r_m, \bar{r}), \\ \therefore & N \models \exists v\ \theta(v, \bar{s}) & \text{by the inductive condition on } \bar{r}, \bar{s}. \end{array}$$

Since $p(v/\bar{s})$ is a recursive complete type over N, we can apply Proposition 8.1 to find $s_m \in N$ realizing it.

Back. Suppose $m = 2\ell + 2$. In these steps, we make f surjective by a symmetric argument. □

Remark 8.4. The same proof shows countable recursively saturated $M \models \mathrm{I}\Delta_0$ are *homogeneous*, in the sense that if $\bar{a}, \bar{b} \in M$ of the same type, then $(M, \bar{a}) \cong (M, \bar{b})$.

Every back-and-forth proof (we will meet) consists essentially of a suitable *inductive condition* and some *back-and-forth lemmas* which tells us we can extend the partial mappings in the ways we want while preserving the inductive condition. We will not bother ourselves with other details any more in future back-and-forth proofs.

8.2 Self-embeddings

Our next application of back-and-forth arguments is an instance of a whole range of results that was historically very influential. These results say that every nonstandard model of arithmetic is isomorphic to a proper initial segment of itself. The following also provides a partial converse to Theorem 6.3 as claimed on page 43.

Theorem 8.5 (essentially Robert Solovay [6])**.** Let $n \in \mathbb{N}$. Then every countable recursively saturated $M \models \mathrm{B}\Sigma_{n+1}$ is isomorphic to a proper n-elementary cut of itself.

Proof. We follow the proof in Chapter 12 of Kaye's book [2]. First, find $d \in M$ that realizes

$$p(v) = \{\exists \bar{x}\ \theta(\bar{x}) \rightarrow \exists \bar{x}{<}v\ \theta(\bar{x}) : \theta \in \Pi_n\}.$$

(It can be verified that any element above a proper n-elementary cut must realize this type.) We carry out a back-and-forth argument so that at each step, we have $\bar{r}, \bar{s} \in M$ of the same finite length satisfying the inductive condition

$$M \models \exists \bar{x}\ \theta(\bar{x}, \bar{r}) \rightarrow \exists \bar{x}{<}d\ \theta(\bar{x}, \bar{s}) \qquad \text{for all } \theta \in \Pi_n. \tag{$*$}$$

The required embedding f maps the r's to the corresponding s's. The inductive condition implies f is n-elementary and $\mathrm{Im}(f) < d$.

The inductive condition $(*)$ is satisfied initially because d realizes p.

Suppose $\bar{r}, \bar{s} \in M$ that satisfy $(*)$.

Forth. We ensure f is total. Take any $r' \in M$. We will make $r' \in \mathrm{Dom}(f)$. It suffices to realize

$$q(v) = \{\exists \bar{x}\ \theta(\bar{x}, \bar{r}, r') \rightarrow \exists \bar{x}{<}d\ \theta(\bar{x}, \bar{s}, v) : \theta \in \Pi_n\}.$$

This q is finitely satisfied in M because if $\theta \in \Pi_n$ such that $M \models \exists \bar{x}\ \theta(\bar{x}, \bar{r}, r')$, then

$$\begin{aligned} & M \models \exists v, \bar{x}\ \theta(\bar{x}, \bar{r}, v) \\ \therefore\quad & M \models \exists v, \bar{x}{<}d\ \theta(\bar{x}, \bar{s}, v) && \text{by } (*). \end{aligned}$$

So we get what we want by recursive saturation.

Back. We ensure $\mathrm{Im}(f) \subseteq_{\mathrm{e}} M$. Take any $s' < \max\{\bar{s}\}$. We make $s' \in \mathrm{Im}(f)$ by realizing

$$q'(v) = \{\exists \bar{x}\ \theta(\bar{x}, \bar{r}, v) \rightarrow \exists \bar{x}{<}d\ \theta(\bar{x}, \bar{s}, s') : \theta \in \Pi_n\}.$$

This q' is finitely satisfied in M because if $\theta \in \Pi_n$ such that $M \models \forall \bar{x}{<}d\ \neg\theta(\bar{x}, \bar{s}, s')$, then

$$\begin{aligned} & M \models \exists v{<}\max\{\bar{s}\}\ \forall \bar{x}{<}d\ \neg\theta(\bar{x}, \bar{s}, v) \\ \therefore\quad & M \models \forall b{<}d\ \underbrace{\exists v{<}\max\{\bar{s}\}\ \forall \bar{x}{<}b\ \neg\theta(\bar{x}, \bar{s}, v)}_{\Sigma_n \text{ over } \mathrm{B}\Sigma_n} \\ \therefore\quad & M \models \forall b\ \exists v{<}\max\{\bar{r}\}\ \forall \bar{x}{<}b\ \neg\theta(\bar{x}, \bar{r}, v) && \text{by } (*), \\ \therefore\quad & M \models \exists v{<}\max\{\bar{r}\}\ \forall \bar{x}\ \neg\theta(\bar{x}, \bar{r}, v) && \text{by } \mathrm{B}\Sigma_{n+1}. \end{aligned}$$

So we are done by recursive saturation. ☐

Remark 8.6. The proof above works for all countable recursively saturated $M \models \mathrm{PA}^- + \mathrm{Coll}(\Sigma_{n+1})$.

Remark 8.7. Examining this proof, one sees that full recursive saturation is actually not necessary. The amount of saturation used can be reduced to the non-$\Delta_0(\Sigma_{n+1})$-definability of $\mathbb{N}$, which all nonstandard models of $\mathrm{I}\Sigma_{n+1}$ enjoy. It follows that every countable nonstandard model of $\mathrm{I}\Sigma_{n+1}$ is isomorphic to a proper n-elementary initial segment of itself.

As we saw, with recursive saturation, it is simply a matter of writing down a type to get the kind of elements we want.

8.3 Fixing cuts pointwise

Finally, we demonstrate how to build automorphisms using back-and-forth arguments. An *automorphism* of a structure M is, as usual, a bijection $M \to M$ that preserves the interpretations of all symbols in the language of M. The group of automorphisms of a structure M is denoted $\mathrm{Aut}(M)$.

Definition. If $M \models \mathrm{PA}^-$ and $g \in \mathrm{Aut}(M)$, then

$$\mathrm{I}_{\mathrm{fix}}(g) = \{x \in M : g(x') = x' \text{ for all } x' \leqslant x\}.$$

Let $g \in \mathrm{Aut}(M)$, where $M \models \mathrm{PA}^-$. Then $\mathrm{I}_{\mathrm{fix}}(g)$ is closed under successor, because if $g(x) = x$, then $g(x+1) = x+1$. So $\mathrm{I}_{\mathrm{fix}}(g)$ is a cut. Similarly, one can verify that $\mathrm{I}_{\mathrm{fix}}(g)$ is closed under addition and multiplication. Nevertheless, we cannot go much further than this.

Theorem 8.8 (Smoryński [7])**.** The following are equivalent for a cut I of a countable recursively saturated $M \models \mathrm{PA}$.

(a) I is exponential.

(b) $I = \mathrm{I}_{\mathrm{fix}}(g)$ for some $g \in \mathrm{Aut}(M)$.

Proof. We first prove (b) $\Rightarrow$ (a). Let $g \in \mathrm{Aut}(M)$ and $a \in \mathrm{I}_{\mathrm{fix}}(g)$. Pick any $x \leqslant 2^a$. We show $g(x) = x$. Without loss of generality, assume $x \neq 0$. From Further exercise 3(b), we see $\max \mathrm{Ack}(x) = \operatorname{len} x = \lfloor \log x \rfloor \leqslant a \in \mathrm{I}_{\mathrm{fix}}(g)$, and so $\max \mathrm{Ack}(g(x)) = g(\max \mathrm{Ack}(x)) = \max \mathrm{Ack}(x)$. Also, if $i < \operatorname{len} x$, then

$$i \in \mathrm{Ack}(x) \quad \Leftrightarrow \quad g(i) \in \mathrm{Ack}(g(x)) \quad \Leftrightarrow \quad i \in \mathrm{Ack}(g(x))$$

because $i < \operatorname{len} x \leqslant a \in \mathrm{I}_{\mathrm{fix}}(g)$. So $\mathrm{Ack}(x) = \mathrm{Ack}(g(x))$. Applying extensionality from Further exercise 3(d), we conclude $x = g(x)$.

Now, consider (a) $\Rightarrow$ (b). Suppose I is exponential. Without loss of generality, assume $I \neq M$. We carry out a back-and-forth argument so that at each step, we have $\bar{r}, \bar{s} \in M$ of the same finite length satisfying

$$\mathrm{tp}(\bar{r}, x) = \mathrm{tp}(\bar{s}, x) \tag{†}$$

for all x less than some $b \in M \setminus I$. The required automorphism g maps the r's to the corresponding s's at the end. The inductive condition ensures that g preserves the interpretation of all symbols in $\mathscr{L}_{\mathrm{A}}$, and fixes I pointwise.

Clearly, the inductive condition is initially satisfied.

During the construction, we need to make sure g is total, surjective, and moves arbitrarily small points above I. These are possible by the next two lemmas. The only point to note is that since I is exponential, if $2^{2^a} > I$, then $a > I$ too. □ modulo Lemmas 8.9 and 8.10

The first lemma ensures the function can be made total and surjective.

Lemma 8.9 (Kotlarski [5], Smoryński [7], Alena Vencovská, independently)**.** Let $\bar{r}, \bar{s}, a$ be elements of a recursively saturated $M \models \mathrm{PA}$ such that (†) holds for all $x < 2^{a^2}$, and $a > \mathbb{N}$. Then for every $r' \in M$, there exists $s' \in M$ such that

$$\mathrm{tp}(\bar{r}, r', x) = \mathrm{tp}(\bar{s}, s', x) \qquad \text{whenever } x < a.$$

Proof. We want to realize

$$p(v) = \big\{\forall x{<}a\ \big(\theta(\bar{r}, r', x) \leftrightarrow \theta(\bar{s}, v, x)\big) : \theta \in \mathscr{L}_{\mathrm{A}}\big\}.$$

Pick $\theta_0, \theta_1, \dots, \theta_{k-1} \in \mathscr{L}_{\mathrm{A}}$. We want to find v whose behaviour with respect to these formulas over $\bar{s}$ and parameters less than a is the same as that of r' over $\bar{r}$. The idea is to code such behavioural pattern of r' with a code that is small enough to be transferrable to the s's using (†). For simpler calculations, we employ a non-standard pairing function here. The details are as follows. Apply separation from Theorem 2.7 to find $c \in M$ such that

$$\mathrm{Ack}(c) = \{kx + i < ka : M \models i < k \wedge \theta_i(\bar{r}, r', x)\}.$$

Then Lemma 3.7 implies $c < 2^{1+\operatorname{len} c} \leqslant 2^{ka} < 2^{a^2}$, because $k \in \mathbb{N} < a$. Therefore, since

$$M \models \exists v\ \forall i{<}k\ \forall x{<}a\ \big(kx + i \in \mathrm{Ack}(c) \leftrightarrow \theta_i(\bar{r}, v, x)\big)$$

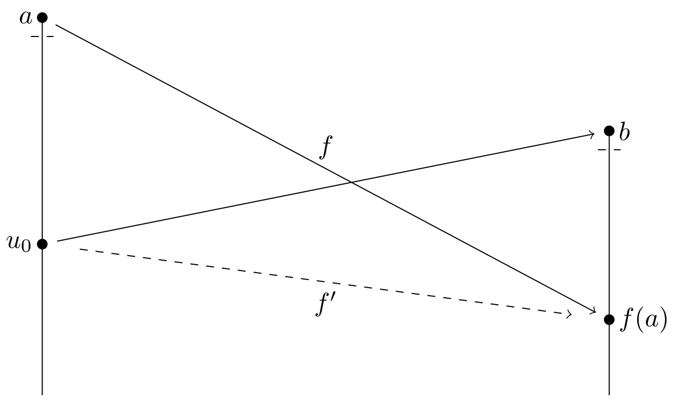

Figure 8.1: Proving the injective version of the Coded Pigeonhole Principle

as witnessed by r', we have by (†)

$$M \models \exists v\ \forall i{<}k\ \forall x{<}a\ \big(kx + i \in \operatorname{Ack}(c) \leftrightarrow \theta_i(\bar{s}, v, x)\big).$$

If $v \in M$ witnesses this, then whenever $i < k$ and $x < a$,

$$M \models \theta_i(\bar{r}, r', x) \quad \Leftrightarrow \quad kx + i \in \operatorname{Ack}(c) \quad \Leftrightarrow \quad M \models \theta_i(\bar{x}, v, x). \qquad \square$$

The second lemma ensures the function moves arbitrarily small points above the cut. We first need an alternative version of the Coded Pigeonhole Principle.

Coded Pigeonhole Principle (injective version)**.** Let $M \models \mathrm{I}\Delta_0 + \exp$. If $f\colon a \to b$ coded in M, where $a > b$, then f is not injective.

Proof. We proceed by strong induction on the code of f. The base case is true because the Pigeonhole Principle holds in $\mathbb{N}$. Now consider $f\colon a+1 \to b+1$, where $a > b$. If $\operatorname{card} f^{-1}(b) \geqslant 2$, then we are already done. If $b \notin \operatorname{Im}(f)$ or $f(a) = b$, then we can apply the induction hypothesis to $f{\restriction}a$ to get distinct $u_1, u_2 < a$ such that $f(u_1) = f(u_2)$. So suppose $f^{-1}(b) = \{u_0\}$ and $u_0 \neq a$. Consider the coded function $f'\colon a \to b$ defined by

$$f'(u) = \begin{cases} f(a), & \text{if } u = u_0; \\ f(u), & \text{otherwise.} \end{cases}$$

The induction hypothesis then gives us distinct $u_1, u_2 < a$ such that $f'(u_1) = f'(u_2)$. If $u_1 \neq u_0 \neq u_2$, then $f(u_1) = f(u_2)$. If $u_1 = u_0$, then $f(a) = f'(u_1) = f'(u_2) = f(u_2)$. Symmetrically, if $u_2 = u_0$, then $f(a) = f'(u_2) = f'(u_1) = f(u_1)$. In any case, we are done. $\square$

Notice exp is again not necessary here.

Lemma 8.10 (Smoryński [7])**.** Let $\bar{r}, \bar{s}, a$ be elements of a recursively saturated $M \models \mathrm{PA}$ such that (†) holds for all $x < 2^{a^2}$, and $a > \mathbb{N}$. Then for every $d > 2^{a^2}$, there exist distinct $r', s' \in M$ such that $r' < d$ and

$$\operatorname{tp}(\bar{r}, r', x) = \operatorname{tp}(\bar{s}, s', x) \qquad \text{whenever } x < a.$$

Proof. We want to realize

$$p(u, v) = \{u \neq v \wedge u < d\} \cup \big\{\forall x{<}a\ \big(\theta(\bar{r}, u, x) \leftrightarrow \theta(\bar{s}, v, x)\big) : \theta \in \mathscr{L}_{\mathrm{A}}\big\}.$$

Pick $\theta_0, \theta_1, \ldots, \theta_{k-1} \in \mathscr{L}_{\mathrm{A}}$. We count the behavioural patterns, in the sense described in the proof of Lemma 8.9, that are realized below d. Define $f\colon d \to 2^{ka}$ by

$$\operatorname{Ack}(f(u)) = \{kx + i < ka : M \models i < k \wedge \theta_i(\bar{r}, u, x)\}.$$

By separation, this function is coded in M. Notice $d > 2^{a^2} > 2^{ka}$ since $k \in \mathbb{N} < a$. So the injective version of the Coded Pigeonhole Principle applies. Find distinct $u_1, u_2 < d$ which satisfy $f(u_1) = f(u_2)$. As in the proof of Lemma 8.9, we get $v \in M$ such that whenever $i < k$ and $x < a$,

$$M \models \theta_i(\bar{s}, v, x) \quad \Leftrightarrow \quad kx + i \in \mathrm{Ack}(f(u_1)) \quad \Leftrightarrow \quad M \models \theta_i(\bar{r}, u_1, x) \quad \Leftrightarrow \quad M \models \theta_i(\bar{r}, u_2, x).$$

Either $u_1 \neq v$ or $u_2 \neq v$ because $u_1 \neq u_2$. So at least one of $(u_1, v), (u_2, v)$ is what we want. □

Further exercises

We investigate which automorphisms of an elementary cut can extend to the whole model. A technical condition is needed [4, Section 5].

Definition. Let $M \models \mathrm{I}\Delta_0$. A function $f : \mathbb{N} \to M$ is *coded* in M if there is $c \in M$ such that

$$\langle n, x \rangle \in \mathrm{Ack}(c) \quad \Leftrightarrow \quad f(n) = x$$

for all $n \in \mathbb{N}$ and all $x \in M$. A cut $I \subseteq_{\mathrm{e}} M$ is *ω-coded from above* if there is a coded $f : \mathbb{N} \to M$ such that

$$\inf\{f(n) : n \in \mathbb{N}\} = \{x \in M : x < f(n) \text{ for all } n \in \mathbb{N}\} = I.$$

Fix a countable recursively saturated $M \models \mathrm{PA}$, and $I \preccurlyeq_{\mathrm{e}} M$ that is not ω-coded from above.

Theorem 8.11 (Kossak–Kotlarski [3])**.** The following are equivalent for $g \in \mathrm{Aut}(I)$.

(a) Both $g(X), g^{-1}(X) \in \mathrm{Cod}(M/I)$ whenever $X \in \mathrm{Cod}(M/I)$.

(b) g extends to $\hat{g} \in \mathrm{Aut}(M)$.

Proof. (1) Show (b) ⇒ (a).

For (a) ⇒ (b), we proceed with a back-and-forth argument. At every step, we have $\bar{r}, \bar{s} \in M$ of the same finite length that satisfy the inductive condition

$$\mathrm{tp}(\bar{r}, x) = \mathrm{tp}(\bar{s}, g(x)) \qquad \text{for all } x \in I. \tag{‡}$$

This ensures the function $\hat{g}$ that maps the r's to the corresponding s's extends g, and preserves the interpretations of all symbols in $\mathscr{L}_{\mathrm{A}}$.

(2) Explain why (‡) is satisfied initially.

The back-and-forth lemma below will ensure $\hat{g}$ is total and surjective. □ modulo Lemma 8.12

Lemma 8.12 (Kossak–Kotlarski [3])**.** Let $\bar{r}, \bar{s} \in M$ that satisfy (‡), and $g \in \mathrm{Aut}(M)$ that satisfies clause (a) in Theorem 8.11. Then for every $r' \in M$, there is $s' \in M$ such that

$$\mathrm{tp}(\bar{r}, r', x) = \mathrm{tp}(\bar{s}, s', g(x)) \qquad \text{for all } x \in I.$$

Proof. Without loss, suppose $I \neq M$. Pick any $d \in M \setminus I$.

(3) Use recursive saturation to find $c \in M$ that satisfies

$$M \models \forall x{<}d\ \big(\langle \theta, x \rangle \in \mathrm{Ack}(c) \leftrightarrow \theta(\bar{r}, r', x)\big)$$

for all formulas $\theta \in \mathscr{L}_{\mathrm{A}}$.

Fix such $c \in M$. Using (a), find $c' \in M$ such that

$$g(\mathrm{Ack}(c/I)) = \mathrm{Ack}(c'/I).$$

(4) Pick $b \in I$. Let $\tilde{c}' \in M$ such that $\mathrm{Ack}(\tilde{c}') = \{\langle\theta, x\rangle \in \mathrm{Ack}(c') : \theta, x < g(b)\}$. Use (‡) to show

$$M \models \exists v\ \forall x{<}g(b)\ \big(\langle\theta, x\rangle \in \mathrm{Ack}(\tilde{c}') \leftrightarrow \theta(\bar{s}, v, x)\big)$$

for every formula $\theta \in \mathscr{L}_\mathrm{A}$.

Given a formula $\theta \in \mathscr{L}_\mathrm{A}$, define $f(\theta)$ to be the maximum $b' \leqslant d$ which makes

$$M \models \exists v\ \forall x{<}b'\ \big(\langle\theta, x\rangle \in \mathrm{Ack}(c') \leftrightarrow \theta(\bar{s}, v, x)\big).$$

(5) Show that $f(\theta) \in M \setminus I$ for all formulas $\theta \in \mathscr{L}_\mathrm{A}$.

(6) Explain why there is $b' \in M \setminus I$ that bounds $\mathrm{Im}(f)$ from below.

(7) Deduce the existence of $s' \in M$ as required by the lemma. □

Further comments

Coded types and realized types

Putting together various facts from previous lectures gives us a converse to Proposition 8.1.

Proposition 8.13. Let $M \models \mathrm{I}\Delta_0 + \exp$ in which a complete type is coded if and only if it is realized. Then M is recursively saturated.

Proof sketch. We can safely assume $M \neq \mathbb{N}$ because according to our definition, all sets in $\mathrm{SSy}(\mathbb{N})$ are finite and so cannot be a complete type. Let $p(\bar{v}/\bar{c})$ be a recursive type over M. Then it is coded in M by Proposition 7.1. The set $q(\bar{z}) = \mathrm{tp}(\bar{c})$ is also coded in M because it is realized. Apply Theorem 4.8 and Theorem 3.5 to find a complete consistent $p^*(\bar{v}/\bar{z}) \supseteq p(\bar{v}/\bar{z}) \cup q(\bar{z})$ in $\mathrm{SSy}(M)$. Then $p^*(\bar{v}/\bar{c})$ is realized in M since it is complete and coded. Any element that realizes $p^*(\bar{v}/\bar{c})$ realizes $p(\bar{v}/\bar{c})$ too. □

By previous remarks about Proposition 7.1 and Theorem 3.5, we see that actually exp is not needed here. The second half of Section 15.2 in Kaye's book [2] describes how these propositions can make sense for theories with even less coding.

References

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