MODEL THEORY OF ARITHMETIC

Lecture 11: Minimal types

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> Uniform extension operators are used to construct models with "nice" structural properties. Thus, one has a very simple proof of the Ehrenfeucht-Mostowski theorem concerning the existence of models with many automorphisms [...]. This theorem states that for every theory having infinite models and every ordered set there is a model of the theory whose set of elements contains the given set, so that every automorphism of the ordered set can be extended to an automorphism of the model. The original proof uses Ramsey's theorem. Here we give a very simple construction of the required model, using iterations of uniform extension operators. Once the existence of the operator is established the building of the model is fully constructive and one sees clearly what the automorphisms are. The existence of the uniform extension operator [...] boils down to the existence of an ultrafilter in the algebra of all subsets of the natural numbers.
>
> Haim Gaifman [2, page 123]

In this lecture, we look at types that give rise to minimal elementary end extensions, and show that they do much more than what they are named for.

11.1 Minimal extensions

The aim of this section is to show that every $M \models \mathrm{PA}$ has a minimal elementary end extension. In view of Proposition 10.4, if $p(v)$ is an unbounded strongly definable complete M-type, then $M(p)$ is already 'rather minimal' over M, in the sense that any elementary cut of $M(p)$ above M must be either $M(p)$ or M. This is because there can be no elementary cut strictly inside a gap, cf. the bottom of page 66. Nevertheless, conceivably one can thin out the maximum gap of such $M(p)$ without completely removing it, resulting a proper elementary substructure of $M(p)$ that is strictly bigger than M. To avoid this, we construct p more carefully so that it is realized at most once in any gap. In this case, if $d \in M(p)$ realizing p and $c \in \mathrm{gap}(d)$, then d would be unique to c, being the minimum element above c realizing p, or the maximum element below c realizing p. A compactness argument, as we will see below, then turns this uniqueness into first-order definability. The extra condition on p needed for this argument is called *rarity.*

Definition (Kossak–Kotlarski–Schmerl [4])**.** Let $M \models \mathrm{PA}$. A complete M-type $p(v)$ is *rare* if $c \notin \mathrm{gap}(d)$ for all distinct $c, d \in K \succcurlyeq M$ realizing p.

This definition can be formulated more syntactically.

Lemma 11.1. Let $M \models \mathrm{PA}$. Then a complete M-type is rare if and only if for every Skolem function t, there is $\theta(v) \in p(v)$ such that

$$M \models \forall u, v\ \big(\theta(u) \wedge \theta(v) \wedge u < v \to t(u) < v\big).$$

In this case, we may think of $\theta(v)$ as forcing elements realizing p to be t-far apart.

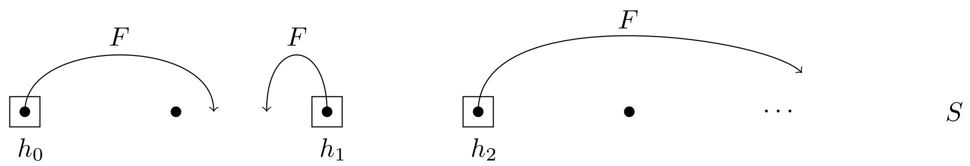

Figure 11.1: Proving FAR

Proof. One direction is straightforward. For the other, consider the theory

$$r(c,d) = p(c) + p(d) + \{c < d \wedge d < t(c)\},$$

where c, d are new constant symbols. $\square$

It follows that extensions of rare types are rare, because the formulas that guarantee rarity given by Lemma 11.1 are preserved in all elementary extensions.

Recall the close relationship between strongly definable types and COH from the last two lectures. One can devise a combinatorial principle that is in a similar relationship with rare types.

Lemma 11.2. $\mathrm{RCA}_0 \vdash \mathrm{FAR}$, where FAR says

> for every unbounded S and every total function F, there exists $H \subseteq_{\mathrm{cf}} S$ such that if $u, v \in H$ with $u < v$, then $F(u) < v$.

Proof. The set H consists of elements of the sequence (h_i) defined by

$$h_0 = \min S,$$
$$h_{i+1} = \min\{v \in S : v > h_i \text{ and } v > F(h_i)\}$$

for all i. This set is in the second-order part by Δ^0_1-CA. It is unbounded by $\mathrm{I}\Sigma^0_1$. $\square$

Remark 11.3. We see from the proof above that if S and F are both definable without parameters, then one can produce the definition of H uniformly from those of S and F.

Interleaving the use of FAR with that of COH gives us unbounded strongly definable types which are, in addition, rare.

Theorem 11.4 (Gaifman [3])**.** Every $M \models \mathrm{PA}$ admits an unbounded strongly definable rare complete M-type.

Proof. We dovetail an additional argument with the proof of Theorem 9.10. Recall the construction there involves a descending sequence

$$M = S_0 \supseteq_{\mathrm{cf}} S_1 \supseteq_{\mathrm{cf}} S_2 \supseteq_{\mathrm{cf}} \cdots$$

in $\mathrm{Def}(M)$ that are defined respectively by $\theta_0(v), \theta_1(v), \theta_2(v), \ldots \in \mathscr{L}_{\mathrm{A}}(M)$. The type $p(v)$ we want is the deductive closure of

$$p_0(v) = \mathrm{ElemDiag}(M) + \{\theta_i(v) : i \in \mathbb{N}\} + \{v > a : a \in M\}.$$

Suppose we are given an unbounded $S_i \in \mathrm{Def}(M)$. Thinning S_i out as in the proof of Theorem 9.10 if necessary, we may assume without loss that

$$M \models \forall z\ \Big(\forall^\infty v\ \big(\theta_i(v) \to \varphi_i(v,z)\big) \vee \forall^\infty v\ \big(\theta_i(v) \to \neg\varphi_i(v,z)\big)\Big),$$

where φ_i comes from some fixed enumeration of $\mathscr{L}_{\mathrm{A}}$-formulas. This ensures strong definability.

To guarantee rarity, choose $S_{i+1} \subseteq_{\mathrm{cf}} S_i$ in $\mathrm{Def}(M)$ using FAR such that

$$\text{if } u, v \in S_{i+1} \text{ with } u < v, \text{ then } F_i(u) < v,$$

where $F_i = \{\langle u, t_i(u)\rangle : u \in M\}$, and t_i comes from some fixed enumeration of Skolem functions. Then the type $p(v)$ we are constructing will be rare by Lemma 11.1 because

$$M \models \forall u, v\ \big(\theta_{i+1}(u) \wedge \theta_{i+1}(v) \wedge u < v \to t_i(u) < v\big). \quad \square$$

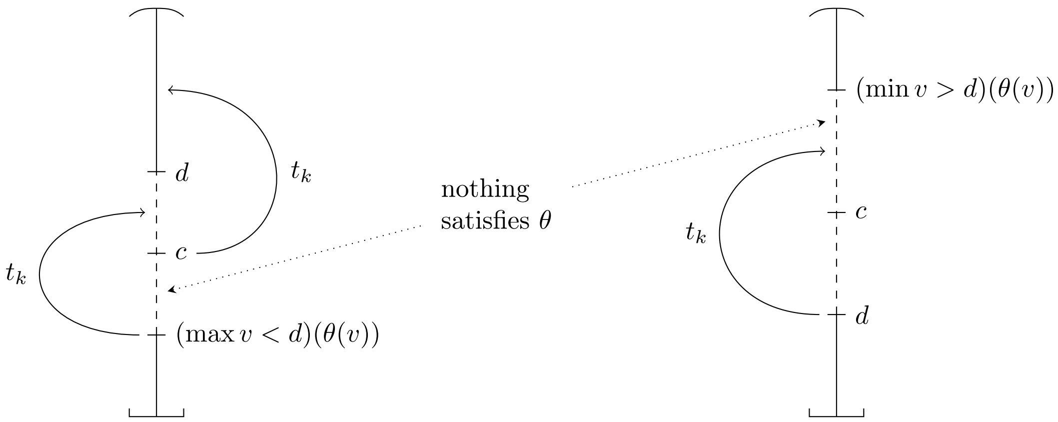

Figure 11.2: Any $c \in \mathrm{gap}(d)$ defines d when d realizes a rare type

The following notation is very useful in the study of models of PA. It describes precisely how a subset of a model *generate* an elementary substructure.

Definition. If $A \subseteq M \models \mathrm{PA}$, then the *Skolem closure of A in M*, denoted $\mathrm{cl}_M(A)$, is the closure of A under all Skolem functions in M.

Let $A \subseteq M \models \mathrm{PA}$. Then

$$\mathrm{cl}_M(A) = \{t(\bar{a}) : \bar{a} \in A \text{ and } t \text{ is a Skolem function}\},$$

because the Skolem functions are closed under composition. An application of the Tarski–Vaught Test shows $\mathrm{cl}_M(A)$ is the smallest elementary substructure of M including A. Model-theoretically, the Skolem closure of A is both the definable closure and the algebraic closure of A.

It is also convenient to have a fixed recursive sequence of Skolem functions $(t_k(v))_{k\in\mathbb{N}}$ such that

- $\mathrm{PA} \vdash \forall v\ \big(v < t_k(v) < t_k(v+1) \wedge t_k(v) < t_{k+1}(v)\big)$ for all $k \in \mathbb{N}$; and
- every Skolem function $t(v)$ is PA-*provably majorized* by some $t_k(v)$, i.e.,

$$\mathrm{PA} \vdash \forall v\ \big(t(v) \leqslant t_k(v)\big).$$

Given any recursive enumeration $(s_k(v))_{k\in\mathbb{N}}$ of Skolem functions, we can obtain such t_k's by setting

$$\begin{aligned} t_0(v) &= v+1, \\ t_{k+1}(v) &= \max\{s_k(v), t_k(v), t_{k+1}(u) : u < v\} + 1 \end{aligned}$$

for all $k \in \mathbb{N}$. These properties are analogous to those of the Grzegorczyk hierarchy, which we met in Lecture 5, when primitive recursive functions take the place of Skolem functions.

We are ready to verify the claim that if an element of a model of PA realizes a rare type, then it is first-order definable from everything in its gap.

Lemma 11.5. Let $M \models \mathrm{PA}$ and $p(v)$ be a rare complete M-type. If $d \in K \succcurlyeq M$ realizing p, and $c \in \mathrm{gap}(d)$, then $d \in \mathrm{cl}_K(M \cup \{c\})$.

Proof. There are naturally two cases.

Case 1. Suppose $c \leqslant d$. Using Lemma 10.3, find $k \in \mathbb{N}$ such that $t_k(c) \geqslant d$. Apply rarity to obtain $\theta(v) \in p(v)$ satisfying

$$M \models \forall u, v\ \big(\theta(u) \wedge \theta(v) \wedge u < v \to t_k(u) < v\big). \qquad (*)$$

Then $d = (\min v \geqslant c)(\theta(v))$ by the monotonicity of t_k.

Case 2. Suppose $d < c$. Using Lemma 10.3, find $k \in \mathbb{N}$ such that $t_k(d) \geqslant c$. Apply rarity to find $\theta(v) \in p(v)$ satisfying $(*)$. Then $d = (\max v < c)(\theta(v))$ by the monotonicity of t_k. $\square$

We now have all the ingredients for a minimal elementary end extension.

Definition. We say $K \succcurlyeq M$ of $\mathscr{L}_{\mathrm{A}}$-structures is *minimal* if there is no K_0 such that $M \precneqq K_0 \precneqq K$.

Theorem 11.6 (Gaifman [3])**.** Let $M \models \mathrm{PA}$ and $p(v)$ be an unbounded strongly definable rare complete M-type. Then $M(p)$ is a minimal elementary end extension of M.

Proof. Suppose $M(p) = M(p/d)$, and K is such that $M \preccurlyeq K \preccurlyeq M(p)$. If $K \cap \mathrm{gap}(d) = \varnothing$, then $K = M$ by Proposition 10.4. If $K \cap \mathrm{gap}(d) \neq \varnothing$, then $K = M(p)$ by Lemma 11.5. $\square$

As a consequence, every model of PA has a minimal end extension. The types involved are named accordingly.

Definition. Let $M \models \mathrm{PA}$. Then a complete M-type is *minimal* if it is unbounded, strongly definable, and rare.

11.2 Iterated extensions

The aim of this section is to investigate to what extent we can iterate our end extension constructions. Let us start with $M_0 \models \mathrm{PA}$ and an unbounded strongly definable complete M_0-type $p(v)$. We define M_α for every ordinal α by transfinite recursion. At a successor step $\alpha + 1$ when given M_α, extend $p(v)$ to an unbounded strongly definable complete M_α-type $p_\alpha(v)$, and set $M_{\alpha+1} = M_\alpha(p_\alpha)$. At a limit step λ, define $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$. A transfinite induction involving Proposition 10.4 then shows that M_α consists α-many gaps on top of M.

As in set theory, an iterated construction can be executed in one go. To do this, we exploit the *indiscernibility* of minimal types. Roughly speaking, a type $p(v)$ is indiscernible if and only if all pairs of tuples of the same length in which every element realizes p satisfy the same formulas.

Definition. A type $p(v)$ over a linearly ordered structure M is *indiscernible* if

> whenever $c_0, c_1, \ldots, c_n, d_0, d_1, \ldots, d_n \in K \succcurlyeq M$ realizing p with $c_0 < c_1 < \cdots < c_n$ and $d_0 < d_1 < \cdots < d_n$, we have $\mathrm{tp}_K(c_0, c_1, \ldots, c_n) = \mathrm{tp}_K(d_0, d_1, \ldots, d_n)$.

Proposition 11.7. Let $M \models \mathrm{PA}$. Then all minimal complete M-types $p(v)$ are indiscernible.

Proof. We show this by induction on n, following the proof of Lemma 3.1.19 in the Kossak–Schmerl book [5]. The case $n = 0$ holds because p is complete.

Let $c_0, c_1, \ldots, c_{n+1}, d_0, d_1, \ldots, d_{n+1} \in K \succcurlyeq M$ realizing p with $c_0 < c_1 < \cdots < c_{n+1}$ and $d_0 < d_1 < \cdots < d_{n+1}$. For the sake of induction, suppose $\mathrm{tp}_K(c_0, c_1, \ldots, c_n) = \mathrm{tp}_K(d_0, d_1, \ldots, d_n)$. Take $\varphi(v_0, v_1, \ldots, v_{n+1}, w) \in \mathscr{L}_{\mathrm{A}}$ and $a \in M$. We will show that

$$K \models \varphi(c_0, c_1, \ldots, c_{n+1}, a) \leftrightarrow \varphi(d_0, d_1, \ldots, d_{n+1}, a).$$

In what follows, the abbreviations $\bar{v}, \bar{c}, \bar{d}$ always refer to tuples whose indices range from 0 up to n. First, use strong definability to find $\theta(v, b) \in p(v)$ such that

$$M \models \forall \bar{v} \left(\forall^\infty v_{n+1} \left(\theta(v_{n+1}, b) \to \varphi(\bar{v}, v_{n+1}, a)\right) \vee \forall^\infty v_{n+1} \left(\theta(v_{n+1}, b) \to \neg\varphi(\bar{v}, v_{n+1}, a)\right)\right),$$

where $\theta(v, z) \in \mathscr{L}_{\mathrm{A}}$ and $b \in M$. Then set

$$t(\bar{v}, w, z) = (\min u) \begin{pmatrix} \forall v_{n+1} \geqslant u \left(\theta(v_{n+1}, z) \to \varphi(\bar{v}, v_{n+1}, w)\right) \\ \vee \forall v_{n+1} \geqslant u \left(\theta(v_{n+1}, z) \to \neg\varphi(\bar{v}, v_{n+1}, w)\right) \end{pmatrix}$$

and $t'(\bar{v}) = \max\{t(\bar{v}, w, z) : w, z \leqslant v_0\}$, so that $t'(\bar{v}) \geqslant t(\bar{v}, w, z)$ whenever $w, z \leqslant v_0$. Notice the unboundedness of p implies $a, b \in M < c_0, d_0$. So if $\bar{v}$ is $\bar{c}$ or $\bar{d}$, then

$$\begin{aligned} K \models\ & \forall v_{n+1} \geqslant t'(\bar{v}) \left(\theta(v_{n+1}, b) \to \varphi(\bar{v}, v_{n+1}, a)\right) \\ & \vee \forall v_{n+1} \geqslant t'(\bar{v}) \left(\theta(v_{n+1}, b) \to \neg\varphi(\bar{v}, v_{n+1}, a)\right). \end{aligned} \tag{†}$$

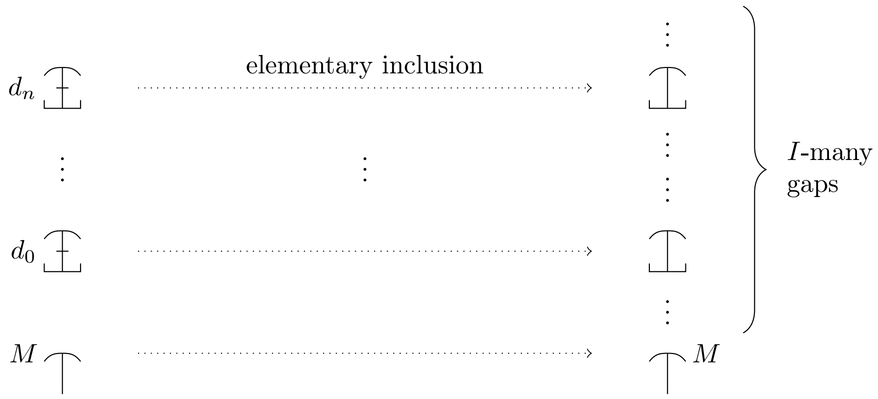

Figure 11.3: Iterating minimal end extensions along a linear order $(I, <)$

Notice also that rarity implies $t'(\bar{c}) \leqslant c_{n+1}$ and $t'(\bar{d}) \leqslant d_{n+1}$. Therefore, if $K \models \varphi(\bar{c}, c_{n+1}, a)$, then

$$\begin{array}{lll} & K \models \forall v_{n+1} \geqslant t'(\bar{c})\ \big(\theta(v_{n+1}, b) \to \varphi(\bar{c}, v_{n+1}, a)\big) & \text{by } (\dagger)\text{, since } c_{n+1} \geqslant t'(\bar{c}), \\ \therefore & K \models \forall v_{n+1} \geqslant t'(\bar{d})\ \big(\theta(v_{n+1}, b) \to \varphi(\bar{d}, v_{n+1}, a)\big) & \text{since } \mathrm{tp}_K(\bar{c}) = \mathrm{tp}_K(\bar{d}), \\ \therefore & K \models \varphi(\bar{d}, d_{n+1}, a) & \text{since } d_{n+1} \geqslant t'(\bar{d}). \end{array}$$

Similarly $K \models \varphi(\bar{d}, d_{n+1}, a)$ implies $K \models \varphi(\bar{c}, c_{n+1}, a)$, as claimed. □

Remark 11.8. Indiscernible types are usually constructed using Ramsey's Theorem. As we saw in (the Further comments of) Lecture 9, Ramsey's Theorem has the same strength as ACA_0. In our construction of indiscernible types, apparently we only used COH and FAR, which are both much weaker than ACA_0. This is actually no improvement to the usual argument because ACA was used directly a couple of times in our proofs.

Similar to rarity, indiscernibility also admits a more syntactical form.

Lemma 11.9. Let $\mathscr{L}$ be a language containing a binary relation symbol $<$, and M be an $\mathscr{L}$-structure linearly ordered by $<$. Then a type $p(v)$ over M is indiscernible if and only if for every $\varphi(v_0, v_1, \ldots, v_n) \in \mathscr{L}(M)$, there is $\theta(v) \in p(v)$ such that

$$M \models \forall \bar{v}\ \Big(\bigwedge_{i \leqslant n} \theta(v_i) \wedge \bigwedge_{i<n} v_i < v_{i+1} \to \varphi(\bar{v})\Big) \vee \forall \bar{v}\ \Big(\bigwedge_{i \leqslant n} \theta(v_i) \wedge \bigwedge_{i<n} v_i < v_{i+1} \to \neg\varphi(\bar{v})\Big).$$

Proof. One direction is straightforward. For the other direction, consider

$$r(\bar{c}, \bar{d}) = \bigcup_{i \leqslant n} p(c_i) \cup \bigcup_{i \leqslant n} p(d_i) \cup \{c_i < c_{i+1} \wedge d_i < d_{i+1} : i < n\} \cup \{\varphi(\bar{c}) \wedge \neg\varphi(\bar{d})\},$$

where $\bar{c}, \bar{d}$ are new constant symbols. □

With minimal types, not only can we iterate our constructions along ordinals, but also along arbitrary linear orders.

Theorem 11.10 (Gaifman [3])**.** Fix $M \models \mathrm{PA}$ and a minimal complete M-type $p(v)$. Let $(I, <)$ be a linear order. Then there is $K \succcurlyeq M$ with $D = \{d_i : i \in I\} \subseteq K$ such that

(1) $K = \mathrm{cl}_K(M \cup D)$;

(2) $d_i < d_j$ if and only if $i < j$ for all $i, j \in I$;

(3) $K = M \cup \bigcup_{i \in I} \mathrm{gap}(d_i)$; and

(4) $\{v \in K : K \models p(v)\} = D$.

Proof. Consider the theory

$$T = \mathrm{ElemDiag}(M) \cup \bigcup_{i\in I} p(d_i) \cup \{d_i < d_j : i, j \in I \text{ with } i < j\},$$

where the d_i's are new constant symbols. We know T is complete by Proposition 11.7 and Lemma 11.9. Let K be the $\mathscr{L}_\mathrm{A}$-reduct of the prime of model of T as given by Proposition 9.3. Then $K \succcurlyeq M$ and (1), (2) hold by construction. By unboundedness and rarity, we know (3) $\Rightarrow$ (4). It remains to prove (3).

Pick $c \in K$. We show $c \in M \cup \bigcup_{i\in I} \mathrm{gap}(d_i)$. Find $\eta \in \mathscr{L}_\mathrm{A}(M)$ and $d_0, d_1, \ldots, d_n \in D$ such that $c = (\min x)(\eta(x, \bar{d}))$. Consider $K_0 = \mathrm{cl}_K(M \cup \{d_i : i \leqslant n\})$. By rarity, we know $\mathrm{gap}_{K_0}(d_0) < \mathrm{gap}_{K_0}(d_1) < \cdots < \mathrm{gap}_{K_0}(d_n)$. Note d_i realizes an unbounded strongly definable type over $\mathrm{cl}_{K_0}(M \cup \{d_j : j < i\})$ for each $i \leqslant n$, because strong definability is preserved in extensions. Thus $K_0 = M \cup \bigcup_{i\leqslant n} \mathrm{gap}_{K_0}(d_i)$ by $n+1$ applications of Proposition 10.4. Observe that $c \in K_0$. So either $c \in M$ or $c \in \mathrm{gap}_{K_0}(d_i)$ for some $i \leqslant n$. This transfers to K by Lemma 10.3. □

Notice our use of indiscernibility in the proof above is not necessary: we could have worked with any completion of T even if the theory T itself is not complete.

Further exercises

The Splitting Theorem from Lecture 6 implies that every minimal extension is either an end extension or a cofinal extension. We have already seen minimal end extensions. So let us see some minimal cofinal extensions.

Theorem 11.11 (Blass [1])**.** Every countable nonstandard $M \models \mathrm{PA}$ has a minimal elementary cofinal extension.

Proof. Let $M \models \mathrm{PA}$ and $\mathrm{Cod}^*(M) = \{S \in \mathrm{Cod}(M) : S \text{ is infinite}\}$. As in the other proofs, we will employ some combinatorial lemmas.

(a) Let $n \in \mathbb{N}$. Show, using $\mathrm{I}\Sigma_n + \exp$ in M, that for every $\varphi(v) \in \Sigma_n(M)$,

$$(M, \mathrm{Cod}^*(M)) \models \forall S\ \exists H{\subseteq}S\ \big(\forall v{\in}H\ \varphi(v) \vee \forall v{\in}H\ \neg\varphi(v)\big).$$

(b) Show, using $\mathrm{I}\Delta_0 + \exp$ in M, that

$$(M, \mathrm{Cod}^*(M)) \models \forall S\ \forall \text{ function } F \text{ with domain } S\ \ \exists H{\subseteq}S$$
$$(F \text{ is either constant or injective on } H).$$

The construction goes by finding $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ in $\mathrm{Cod}^*(M)$. These are respectively coded by $s_0, s_1, s_2, \ldots \in M$. The type $p(v)$ will be the deductive closure of

$$p_0(v) = \mathrm{ElemDiag}(M) + \{\theta_i(v) : i \in \mathbb{N}\},$$

where $\theta_i(v)$ is the $\Delta_0(M)$-formula $v \in \mathrm{Ack}(s_i)$. Notice $\mathrm{Cod}^*(M) \neq \varnothing$ since M is nonstandard. Pick any $S_0 \in \mathrm{Cod}^*(M)$.

(c) Explain how one can make $p(v)$ a complete M-type using (a).

Consider $M(p) = M(p/d)$. Pick $c \in M(p)$. Let $\eta(x, v) \in \mathscr{L}_\mathrm{A}(M)$ such that $c = (\min x)(\eta(x, d))$. As in the proof of Proposition 10.4, we may assume $M \models \forall v\ \exists! x\ \eta(x, v)$ without loss of generality.

(d) Recall from Lemma 2.6 that every element of $\mathrm{Cod}^*(M)$ has a maximum. Suppose $\eta \in \Sigma_n(M)$, where $n \in \mathbb{N}$. Use $\mathrm{B}\Sigma_n$ in M to find $b \in M$ such that $p(v) \vdash \exists x{<}b\ \eta(x, v)$.

(e) Conclude that $c < b$ for some $b \in M$.

This shows every element of $M(p)$ is bounded above by some element of M. In other words, $M(p) \supseteq_\mathrm{cf} M$. It remains to prove the minimality of $M(p)$ over M.

(f) Show how to construct $p(v)$ using (b) so that for every S_i and for every coded function $F\colon S_i \to M$, there exists $S_j \subseteq S_i$ on which F is either constant or injective.

Take any $i \in \mathbb{N}$. Then

$$F = \{\langle v, x\rangle : M \models v \in \mathrm{Ack}(s_i) \land \eta(x, v)\} \in \mathrm{Cod}^*(M)$$

by Theorem 2.8, because $M \models \mathrm{B}\Sigma_n + \exp$. Apply (f) to find S_j on which F is either constant or injective. It suffices to make $c \in M$ or $d \in \mathrm{cl}_{M(p)}(M \cup \{c\})$.

(g) Suppose F is constant on S_j. Show that $c \in M$.

(h) Suppose F is injective on S_j. Show that d is the unique v such that $M(p) \models \eta(c, v)$. □

Further comments

Rarity and minimality

Many results presented in this lecture are, in a sense, optimal. For instance, Lemma 11.5, Theorem 11.6, and Proposition 11.7 all admit converses.

Proposition 11.12. Let $M \models \mathrm{PA}$ and $p(v)$ be a complete M-type.

(a) If for every $d \in K \succcurlyeq M$ realizing p and every $c \in \mathrm{gap}(d)$, we have $d \in \mathrm{cl}_K(M \cup \{c\})$, then $p(v)$ is rare.

(b) Suppose $p(v)$ is unbounded. If for every $K \succcurlyeq M$ and every unbounded complete K-type $q(v) \supseteq p(v)$, the extension $K(q) \succcurlyeq K$ is minimal, then $p(v)$ is strongly definable and rare.

(c) If $p(v)$ is unbounded and indiscernible, then it is minimal.

Proof. (a) This is part of Theorem 3.1.16 in Kossak–Schmerl [5].

(b) For rarity, see Corollary 3.1.17 in Kossak–Schmerl [5]. For strong definability, combine Lemmas 3.2.4–3.2.6 there.

(c) The rarity and strong definability parts are respectively Lemma 3.1.18 and Lemma 3.1.13 in Kossak–Schmerl [5]. The latter depends on Theorem 3.1.9 in the Kossak–Schmerl book [5], which originally came from Theorem 4.5 in Kossak–Kotlarski–Schmerl [4]. □

Moreover, strong definability and rarity are independent of each other. Unbounded rare types that fail to be (strongly) definable can be constructed using Lemma 2.1 in Schmerl [6]; see also Corollary 3.2.12 and the theorem preceding it in the Kossak–Schmerl book [5]. The existence of unbounded strongly definable non-rare types follows from Theorem 5.2 or Theorem 5.14 in the Gaifman paper [3].

Substructure lattices and automorphism groups

Theorem 11.10 tells us a lot about the variety of models of PA. Let us mention two applications here. The first one is about the collection of all elementary substructures of a model of PA. Such a collection forms a lattice under inclusion because of our remarks about $\mathrm{cl}_M(A)$ on page 73.

Corollary 11.13 (Gaifman [3])**.** Let I be a set and T be a complete extension of PA. Then there is $K \models \mathrm{PA}$ of which the lattice of elementary substructures is isomorphic to the Boolean algebra $\mathcal{P}(I)$ of subsets of I.

Proof sketch. Take any linear order $<$ on I. Let M be the prime model of T as given by Proposition 9.3. Apply Theorem 11.10 to M and $(I, <)$ to obtain K. Then every $K_0 \preccurlyeq K$ uniquely determines $J = \{i \in I : d_i \in K_0\} \subseteq I$, because $K = \mathrm{cl}_K(M \cup D) = \mathrm{cl}_K(D)$. Conversely, every $J \subseteq I$ uniquely determines $K_0 = \mathrm{cl}_K(\{d_i : i \in J\}) \preccurlyeq K$. □

Our second application is about automorphism groups of models of PA.

Corollary 11.14 (Gaifman [3])**.** For every complete $T \supseteq \mathrm{PA}$ and every linear order $(I, <)$, there exists $K \models T$ such that $\mathrm{Aut}(K) \cong \mathrm{Aut}(I, <)$.

Proof sketch. Take the prime model $M \models T$ as given by Proposition 9.3. Apply Theorem 11.10 to this M and $(I, <)$ to obtain $K \succcurlyeq M$. Then every $g \in \mathrm{Aut}(M)$ naturally induces an automorphism of $(D, <) \cong (I, <)$. Conversely, every $f \in \mathrm{Aut}(I, <)$ induces an automorphism of M, because $K = \mathrm{cl}_K(M \cup D) = \mathrm{cl}_K(D)$. $\square$

In particular, for every infinite cardinal κ, one can find a model of PA of size κ that has no non-trivial automorphism, essentially because $\mathrm{Aut}(\kappa, <)$ is trivial. Improving on Corollary 11.14, Schmerl [6] showed that every subgroup of the automorphism group of a linearly ordered set realizes as the automorphism group of some model of PA. This, in a sense, says the variety of automorphism groups of models of PA is as rich as possible.

Minimal cofinal extensions

Recall from Remark 9.2 that it is easy to build elementary cofinal extensions of models of PA. However, problems about cofinal extensions are not always easier. For example, although we know every model of PA has a minimal end extension, we do not know whether the same is true for cofinal extensions in general [5, Chapter 12, Question 2]. Notice our argument in the Further exercises breaks down when the model involved is uncountable.

Question 11.15. Does every model of PA have a minimal cofinal extension?

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