MODEL THEORY OF ARITHMETIC

Lecture 12: Preservation

Tin Lok Wong

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> One of the original applications of the Omitting Types Theorem is the ω-completeness Theorem [...].
>
> H. Jerome Keisler [5, page 117]

Recall from Lemma 4.12 that all end extensions are Δ_0-elementary. Therefore, all Π_1-formulas are preserved in initial segments. More generally, for every $n \in \mathbb{N}$, all Π_{n+1}-formulas are preserved in n-elementary initial segments. The aim of this lecture is to show that these are the only formulas preserved in such initial segments.

The usual proof of the Łoś–Tarski preservation theorem for universal formulas [6] involves constructing a suitable extension using simply the compactness theorem. More care is needed in our case because the extension constructed has to be an end extension. To achieve this, we employ the Omitting Types Theorem. It turns out that the version for complete theories is sufficient for our purposes. With this restriction, we can also make the definitions more intuitive.

Definition. Let T be a complete theory.

- A *type* over T is a set of formulas with finitely many free variables that is consistent with T.
- A formula $\eta(\bar{v})$ is said to *isolate* a type $p(\bar{v})$ over T if
  - (i) $T \vdash \exists\bar{v}\ \eta(\bar{v})$; and
  - (ii) $T \vdash \forall\bar{v}\ \big(\eta(\bar{v}) \rightarrow \theta(\bar{v})\big)$ for all $\theta(\bar{v}) \in p(\bar{v})$.

  In this case, we say $\eta(\bar{v})$ is a *support* for $p(\bar{v})$ over T.
- A type is *omitted* in a model if it is not realized.

Remark 12.1. Isolated types are sometimes called *principal* types.

It is evident that if a type $p(\bar{v})$ is isolated over a complete theory T, then it is realized in every model of T. The Omitting Types Theorem says the converse is also true, provided the language involved is countable.

Omitting Types Theorem. Let T be a complete theory in a countable language $\mathscr{L}$, and $P = \{p_i(\bar{v}) : i \in \mathbb{N}\}$ be a countable set of types over T. Then the following are equivalent.

(a) All types in P are non-isolated over T.

(b) There is a (countable) model of T that omits all the types in P.

Proof sketch. We already talked about (b) $\Rightarrow$ (a). So let us concentrate on the converse. Suppose (a) holds. Let $\mathscr{L}^* = \mathscr{L} \cup C$, where C is a countably infinite set of new constant symbols. In $\mathscr{L}^*$, we build by recursion finite extensions $T_0 \supseteq T_1 \supseteq \cdots$ of T with the aim that

(1) $T^* = \bigcup_{k\in\mathbb{N}} T_k$ is complete and consistent; and

(2) for every formula $\varphi \in \mathscr{L}^*$, there is $c \in C$ such that $T^* \vdash \exists x\ \varphi(x) \rightarrow \varphi(c)$.

Then the $\mathscr{L}$-reduct K of the term model of T^* satisfies T. (See Lecture 4 for the definition of term models.) Consistency is preserved in every step. Completeness is ensured by putting in σ or $\neg\sigma$ for every sentence $\sigma \in \mathscr{L}^*$. Condition (2) can be satisfied by putting in $\varphi(c)$ for some fresh $c \in C$ when the theory proves $\exists x\ \varphi(x)$. (The argument we have so far is the same as the usual Henkin proof of the completeness theorem.)

It remains to make all types in P omitted in K. By (2), it suffices to make sure $\bar{c}$ does not satisfy $p_i(\bar{v})$ for every $\bar{c} \in C$ and every $i \in \mathbb{N}$. Since there are only countably many such pairs $(\bar{c}, i)$, we can deal with them separately and then dovetail together the arguments to achieve what we want. Fix one of these $(\bar{c}, i)$'s. Suppose $T_k = T \cup \{\eta(\bar{c}, \bar{d})\}$, where $\eta \in \mathscr{L}$ and $\bar{d} \in C \setminus \{\bar{c}\}$. Then $T \vdash \exists\bar{v}\ \exists\bar{w}\ \eta(\bar{v}, \bar{w})$, because T is complete and T_k is consistent. By (a), we know $\exists\bar{w}\ \eta(\bar{v}, \bar{w})$ does not isolate $p_i(\bar{v})$. Find $\theta(\bar{v}) \in p_i(\bar{v})$ such that $T \vdash \exists\bar{v}\ \big(\exists\bar{w}\ \eta(\bar{v}, \bar{w}) \wedge \neg\theta(\bar{v})\big)$. Then $T_{k+1} = T_k \cup \{\neg\theta(\bar{c})\}$ is consistent, and it ensures $K \not\models p_i(\bar{c})$. □

There are a number of places in the proof above that require the countability of $\mathscr{L}$ and P. For instance, we need to enumerate the σ's, the φ's and the $(\bar{c}, i)$'s in ω-sequences, so as not to break the inductive condition that every T_k is a *finite* extension of T. This inductive condition ensures we can extract a potential support for the types.

Building end and cofinal extensions is about finding extensions that omit the appropriate types.

Example 12.2. Fix a countable $M \models \mathrm{PA}^-$ and $n \in \mathbb{N}$. Notice Π_n-Diag$(M) \vdash \Sigma_{n+1}$-Diag(M).

(1) Obtaining an n-elementary end extension of M satisfying a theory T is equivalent to finding a complete consistent theory extending $T + \Pi_n$-Diag(M) over which
$$p_a(v) = \{v < a\} \cup \{v \neq m : m \in M\}$$
is non-isolated for every $a \in M$.

(2) Obtaining an n-elementary cofinal extension of M satisfying a theory T is equivalent to finding a complete consistent theory extending $T + \Pi_n$-Diag(M) over which
$$q(v) = \{v > m : m \in M\}$$
is non-isolated.

Suppose M, K are structures and $M \subseteq K$. Let us say a formula $\varphi(\bar{z})$ is *absolute* between M and K (or *absolute* in the extension K of M, or *absolute* in the substructure M of K) if
$$M \models \varphi(\bar{c}) \quad \Leftrightarrow \quad K \models \varphi(\bar{c})$$
for all $\bar{c} \in M$. For example, if Γ is a class of formulas, then Γ-elementarity of an extension is the same as the absoluteness of all formulas in Γ. A straightforward induction shows that the set of formulas absolute in an extension is always closed under the Boolean operations. If, moreover, both M, K are linearly ordered and K is an end extension of M, then the set of formulas absolute between M and K is closed under bounded quantification too: the proof of this is similar to that of Proposition 4.12. Therefore, it is not the case that *only* the Σ_n-formulas are absolute in n-elementary end extensions.

Definition. Fix $n \in \mathbb{N}$. Let $\langle\Sigma_n\rangle_\Delta$ denote the closure of Σ_n under the Boolean operations and bounded quantification. Set
$$\Pi^*_{n+1} = \{\forall\bar{x}\ \theta(\bar{x}, \bar{z}) : \theta \in \langle\Sigma_n\rangle_\Delta\}.$$

With enough collection, all Π^*_{n+1}-formulas are Π_{n+1}.

Lemma 12.3. Let $n \in \mathbb{N}$. Every Π^*_{n+1}-formula is equivalent to a Π_{n+1}-formula over $\mathrm{Coll}(\Sigma_{n+1})$.

Proof. Every Π^*_{n+1}-formula is equivalent to one of the form
$$\underbrace{\forall\bar{x}\ \exists\bar{y}_1{<}t_1\ \forall\bar{y}_2{<}t_2\ \cdots\ \underbrace{\bigwedge_i \bigvee_j \varphi_{ij}}_{\Pi_{n+1}}}_{\Pi_{n+1}\text{ over Coll}(\Sigma_{n+1})},$$
where each $\varphi_{ij} \in \Sigma_n \cup \Pi_n$, and $t_1, t_2, \ldots$ are terms. □

The Π^*_{n+1}-formulas are precisely those preserved in n-elementary initial segments.

Definition. Let T be a theory and Γ be a class of formulas. Then $\Gamma\text{-Cn}(T) = \{\sigma \in \Gamma : T \vdash \sigma\}$.

Theorem 12.4 (essentially Paris–Kirby [8])**.** Fix $n \in \mathbb{N}$ and a countable recursively saturated $M \models \mathrm{PA}^- + \mathrm{Coll}(\Sigma_{n+1})$. The following are equivalent for a recursive $\mathscr{L}_\mathrm{A}$-theory $T \supseteq \mathrm{PA}^-$.

(a) $M \models \Pi^*_{n+1}\text{-Cn}(T)$.

(b) M has an n-elementary end extension $K \models T$.

Proof. The proof of (b) $\Rightarrow$ (a) is a straightforward induction on formulas, as discussed above. For the converse, suppose (a) holds. We define $\mathscr{L}_\mathrm{A}(M)$-sentences $\lambda_0, \lambda_1, \ldots$ by recursion such that

$$M \models \Pi^*_{n+1}\text{-Cn}\big(T + \{\lambda_i : i < k\}\big) \tag{$(*)_k$}$$

for every $k \in \mathbb{N}$. Then $T + \{\lambda_i : i \in \mathbb{N}\} + \Sigma_{n+1}\text{-Diag}(M)$ is consistent. By the Omitting Types Theorem, it suffices to make sure this theory is complete, and no $p_a(v)$ in Example 12.2(1) is isolated over it. We use the even steps to achieve the former, and the odd steps for the latter.

Suppose λ_i is found for every $i < k$, where k is even, such that $(*)_k$ is satisfied. Take $\sigma \in \mathscr{L}_\mathrm{A}(M)$. Assume $M \not\models \Pi^*_{n+1}\text{-Cn}(T + \{\lambda_i : i < k\} + \sigma)$. Pick $\xi \in \Pi^*_{n+1}$ such that

$$T + \{\lambda_i : i < k\} + \sigma \vdash \xi \quad \text{and} \quad M \models \neg\xi.$$

In this case, we can set $\lambda_k = \neg\sigma$, because if $T + \{\lambda_i : i < k\} + \neg\sigma \vdash \zeta \in \Pi^*_{n+1}$, then

$$\begin{aligned}
& T + \{\lambda_i : i < k\} \vdash \xi \vee \zeta && \text{since } T + \{\lambda_i : i < k\} \vdash (\sigma \to \xi) \wedge (\neg\sigma \to \zeta),\\
\therefore\ & M \models \xi \vee \zeta && \text{by } (*)_k, \text{ since } \xi \vee \zeta \in \Pi^*_{n+1},\\
\therefore\ & M \models \zeta && \text{since } M \models \neg\xi.
\end{aligned}$$

This ensures completeness.

For the omitting-types part, pick any $a \in M$ and $\eta(v) \in \mathscr{L}_\mathrm{A}(M)$. We prevent $\eta(v)$ from being a support for $p_a(v)$. Without loss of generality, we may suppose

$$T + \{\lambda_i : i \leqslant k\} \vdash \exists v{<}a\ \eta(v), \tag{$\dagger$}$$

because otherwise nothing needs to be done at this stage. Let $c_1, c_2, \ldots, c_\ell \in M$ be the parameters occurring in the λ_i's or in η. Write $\lambda_i = \lambda_i(\bar{c})$ and $\eta(v) = \eta(v, \bar{c})$. If we can set $\lambda_{k+1} = \eta(c_{j+1}, \bar{c})$ for some $j < \ell$, then we are done. So suppose not. Without loss, we may assume

$$\{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \bigwedge_{j<\ell} \neg\eta(c_{j+1}, \bar{c}) \tag{$\ddagger$}$$

by the previous paragraph. Consider

$$r(v) = \big\{\varphi(v, \bar{c}) \in \Pi^*_{n+1} : T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \forall v{<}a\ \big(\eta(v, \bar{c}) \to \varphi(v, \bar{c})\big)\big\} \cup \{v < a\}.$$

This set is r.e. and hence recursive by Craig's Trick, cf. Lecture 7. Let us show it is finitely satisfied in M. Take $\varphi(v, \bar{c}) \in \Pi^*_{n+1}$ such that $T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \forall v{<}a\ \big(\eta(v, \bar{c}) \to \varphi(v, \bar{c})\big)$. Then

$$T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \exists v{<}a\ \varphi(v, \bar{c})$$

by ($\dagger$). Writing $\varphi(v, \bar{c})$ as $\forall \bar{x}\ \psi(v, \bar{x}, \bar{c})$, where $\psi \in \langle\Sigma_n\rangle_\Delta$, we have

$$\begin{aligned}
& T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \exists v{<}a\ \forall \bar{x}\ \psi(v, \bar{x}, \bar{c})\\
\therefore\ & T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \underbrace{\forall b\ \underbrace{\exists v{<}a\ \forall \bar{x}{<}b\ \underbrace{\psi(v, \bar{x}, \bar{c})}_{\langle\Sigma_n\rangle_\Delta}}_{\langle\Sigma_n\rangle_\Delta}}_{\Pi^*_{n+1}} && \text{by pure logic,}\\
\therefore\ & M \models \forall b\ \exists v{<}a\ \forall \bar{x}{<}b\ \psi(v, \bar{x}, \bar{c}) && \text{by } (*)_{k+1},\\
\therefore\ & M \models \exists v{<}a\ \forall \bar{x}\ \psi(v, \bar{x}, \bar{c}) && \text{by } \mathrm{Coll}(\Sigma_{n+1}),
\end{aligned}$$

which is applicable because Lemma 12.3 implies ψ is equivalent to a Π_{n+1}-formula over M. Hence $M \models \exists v{<}a\ \varphi(v,\bar{c})$, as required.

Apply recursive saturation to find $m \in M \models r(m)$. Notice $m \notin \{\bar{c}\}$ because the formula $v \notin \{\bar{c}\}$ is in $r(v)$ by (‡). Set $\lambda_{k+1} = \eta(m,\bar{c})$. If $\varphi(v,\bar{c}) \in \Pi^*_{n+1}$ such that $T+\{\lambda_i(\bar{c}) : i \leqslant k\}+\eta(m,\bar{c}) \vdash \varphi(m,\bar{c})$, then $T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \forall v\ \big(\eta(v,\bar{c}) \to \varphi(v,\bar{c})\big)$ as $m \notin \{\bar{c}\}$, and so $\varphi(v,\bar{c}) \in r(v)$, making $M \models \varphi(m,\bar{c})$. Therefore, the inductive condition $(*)_{k+2}$ holds. Moreover,

$$T + \{\lambda_i : i \in \mathbb{N}\} \nvdash \forall v\ \big(\eta(v,\bar{c}) \to v \neq m\big),$$

so that η cannot isolate $p_a(v)$. □

Corollary 12.5. Fix $n \in \mathbb{N}$ and a recursive $T \supseteq \mathrm{PA}^- + \mathrm{Coll}(\Sigma_{n+1})$. The following are equivalent for an $\mathscr{L}_\mathrm{A}$-formula $\varphi(\bar{x})$.

(a) φ is equivalent to a Π_{n+1}-formula over T.

(b) Whenever $\bar{c} \in M \preccurlyeq_{n,\mathrm{e}} K$ with $M, K \models T$,

$$K \models \varphi(\bar{c}) \quad \Rightarrow \quad M \models \varphi(\bar{c}).$$

Proof. The implication (a) ⇒ (b) is clear. Conversely, suppose (a) fails. Consider

$$\Psi(\bar{x}) = \Pi_{n+1}\text{-}\mathrm{Cn}(T + \varphi(\bar{x})) = \big\{\psi(\bar{x}) \in \Pi_{n+1} : T \vdash \forall \bar{x}\ \big(\varphi(\bar{x}) \to \psi(\bar{x})\big)\big\}.$$

The failure of (a) implies $T + \Psi(\bar{x}) \nvdash \varphi(\bar{x})$. Take a countable recursively saturated $M \models T + \Psi(\bar{c}) + \neg\varphi(\bar{c})$. Notice $M \models \Pi^*_{n+1}\text{-}\mathrm{Cn}(T + \varphi(\bar{c}))$ by Lemma 12.3. So Theorem 12.4 implies M has an n-elementary end extension $K \models T + \varphi(\bar{c})$. This witnesses the failure of (b). □

Further exercises

We look at an analogue of Theorem 12.4 for cofinal extensions here. Recall from Lecture 10 that $\forall^\infty x\ \theta(x)$ stands for $\exists x'\ \forall x{\geqslant}x'\ \theta(x)$.

Definition. Write $\forall^\infty_{n+1} = \{\forall^\infty x_1\ \forall^\infty x_2\ \cdots\ \forall^\infty x_\ell\ \theta(\bar{x},\bar{z}) : \theta \in \Sigma_n\}$ for each $n \in \mathbb{N}$.

Theorem 12.6 (Kaye [4])**.** Fix $n \in \mathbb{N}$ and a countable recursively saturated $M \models \mathrm{PA}^-$. The following are equivalent for a recursive $\mathscr{L}_\mathrm{A}$-theory $T \supseteq \mathrm{PA}^-$.

(a) $M \models \forall^\infty_{n+1}\text{-}\mathrm{Cn}(T)$.

(b) M has an n-elementary cofinal extension $K \models T$.

Proof. (1) Prove (b) ⇒ (a).

Conversely, suppose (a) holds. We define $\mathscr{L}_\mathrm{A}(M)$-sentences $\lambda_0, \lambda_1, \ldots$ by recursion such that

$$M \models \forall^\infty_{n+1}\text{-}\mathrm{Cn}\big(T + \{\lambda_i : i < k\}\big) \tag{§$_k$}$$

for every $k \in \mathbb{N}$.

(2) Show that this ensures $T + \{\lambda_i : i \in \mathbb{N}\} + \Pi_n\text{-}\mathrm{Diag}(M)$ is consistent at the end.

By the Omitting Types Theorem, it suffices to make sure this theory is complete, and $q(v)$ in Example 12.2(2) is not isolated over it. We use the even steps to achieve the former, and the odd steps for the latter.

Suppose λ_i is found for every $i < k$, where k is even, such that $(\S)_k$ is satisfied.

(3) Show that $\forall^\infty_{n+1}$ is closed under disjunction modulo logical equivalence.

(4) Let $\sigma \in \mathscr{L}_\mathrm{A}(M)$. Show that either $\lambda_k = \sigma$ or $\lambda_k = \neg\sigma$ makes $(\S)_{k+1}$ true.

For the omitting-types part, pick any $\eta(v) \in \mathscr{L}_{\mathrm{A}}(M)$. We prevent $\eta(v)$ from being a support for $q(v)$. Without loss of generality, we may suppose

$$T + \{\lambda_i : i \leqslant k\} \vdash \exists v\ \eta(v), \tag{¶}$$

because otherwise nothing needs to be done at this stage. Let $c_1, c_2, \ldots, c_\ell \in M$ be the parameters occurring in the λ_i's or in η. Write $\lambda_i = \lambda_i(\bar{c})$ and $\eta(v) = \eta(v, \bar{c})$. If we can set $\lambda_{k+1} = \eta(c_{j+1}, \bar{c})$ for some $j < \ell$, then we are done. So suppose not. Without loss, we may assume

$$\{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \bigwedge_{j<\ell} \neg\eta(c_{j+1}, \bar{c}) \tag{$\|$}$$

by Exercise (4) above. Consider

$$s(w) = \big\{\varphi(w, \bar{c}) \in \forall^\infty_{n+1} : T + \{\lambda_i(\bar{c}) : i \leqslant k\} \vdash \forall w\ \big(\exists v{\leqslant}w\ \eta(v, \bar{c}) \to \varphi(w, \bar{c})\big)\big\}.$$

This set is r.e. and hence recursive by Craig's Trick. Let us show it is finitely satisfied in M. Take $\varphi(w, \bar{c}) \in s(w)$.

(5) Notice $\mathrm{PA}^- \vdash \forall^\infty w\ \big(\exists v\ \eta(v, \bar{c}) \to \exists v{\leqslant}w\ \eta(v, \bar{c})\big)$. Show that $M \models \exists w\ \varphi(w, \bar{c})$.

Apply recursive saturation to find $m \in M \models s(m)$. Set $\lambda_{k+1} = \exists v{\leqslant}m\ \eta(v, \bar{c})$.

(6) Show that $(\S)_{k+2}$ is satisfied.

(7) Explain why $q(v)$ is not isolated over $T + \{\lambda_i : i \in \mathbb{N}\}$. □

Corollary 12.7 (Motohashi [7])**.** Fix $n \in \mathbb{N}$ and a recursive $T \supseteq \mathrm{PA}^-$. The following are equivalent for an $\mathscr{L}_{\mathrm{A}}$-formula $\varphi(\bar{x})$.

(a) φ is equivalent to a $\forall^\infty_{n+1}$-formula over T.

(b) Whenever $\bar{c} \in M \preccurlyeq_{n,\mathrm{cf}} K$ with $M, K \models T$,

$$K \models \varphi(\bar{c}) \quad \Rightarrow \quad M \models \varphi(\bar{c}).$$ □

Further comments

Optimality of hypotheses

It can be seen from the proof of Theorem 12.4 that full recursive saturation is not necessary: short Π_{n+1}-recursive saturation is actually enough. The amount of saturation needed for Theorem 12.6 is discussed in Kaye's original paper [4]. There are versions of these theorems that do not require the theory T to be recursive; for the same proofs to go through, we need recursive-in-T saturation. In particular, Corollary 12.5 is actually true for non-recursive theories T too. If one removes the saturation condition altogether, then Theorem 12.4 becomes false; see the paper by Dimitracopoulos [1].

In general, one cannot replace $\mathrm{PA}^- + \mathrm{Coll}(\Sigma_{n+1})$ by $\mathrm{I}\Sigma_n$ in Theorem 12.4. To see this, suppose this theorem is true for $M \models \mathrm{I}\Sigma_n$. Consider $T = \mathrm{B}\Sigma_{n+1}$. Notice $\mathrm{I}\Sigma_n \vdash \Pi^*_{n+1}\text{-}\mathrm{Cn}(\mathrm{B}\Sigma_{n+1})$ by a careful analysis of our proof of Theorem 6.1. So M has an n-elementary end extension $K \models \mathrm{B}\Sigma_{n+1}$. If this extension is not proper, then trivially $M \models \mathrm{B}\Sigma_{n+1}$. If this extension is proper, then Theorem 6.3 implies $M \models \mathrm{B}\Sigma_{n+1}$ too.

One also cannot replace $\mathrm{PA}^- + \mathrm{Coll}(\Sigma_{n+1})$ by $\mathrm{I}\Sigma_n$ in Corollary 12.5 in general. To see this, consider $T = \mathrm{I}\Sigma_n + \exp + \neg\mathrm{B}\Sigma_{n+1}$, which, as Paris and Kirby [8, Proposition 7] proved, is consistent. By Theorem 6.3, no $M, K \models T$ can satisfy $M \precneqq_{n,\mathrm{e}} K$. So condition (b) is true trivially for every formula $\varphi \in \mathscr{L}_{\mathrm{A}}$. Let $\varphi = \Sigma_{n+1}\text{-Sat}$ from Lecture 7. A diagonalization argument shows φ is not equivalent to any Π_{n+1}-formula; see the Further comments in Lecture 7. So condition (a) is false for this φ.

We may as well define Π^*_{n+1} to be the closure of $\langle\Sigma_n\rangle_\Delta$ under universal quantification and bounded quantification. However, the proof of Theorem 12.4 is neater with our definition. For a similar reason, we may also define $\forall^\infty_{n+1}$ to be the closure of $\langle\Sigma_n\rangle_{\mathrm{B}}$ under conjunction, disjunction, universal quantification, and $\forall^\infty$-quantification, where $\langle\Sigma_n\rangle_{\mathrm{B}}$ denotes the closure of Σ_n under the Boolean operations. Combining Corollary 12.7 with Further exercises (a)–(d) in Lecture 6, we obtain the following.

Corollary 12.8 (Kaye [4])**.** Let $n \in \mathbb{N}$. Then every Π_{n+3}-formula is equivalent over $\mathrm{B}\Sigma_{n+1} + \exp$ to a $\forall^\infty_1$-formula. □

In both Theorem 12.4 and Theorem 12.6, the theory T can actually be in any recursive language extending $\mathscr{L}_{\mathrm{A}}$. The proofs are the same.

Generalizations

In fact, the results presented in this lecture are very general. For instance, suppose $n \in \mathbb{N}$ and $\mathscr{L}$ is a countable language containing a distinguished binary relation symbol $\sqsubseteq$. Then one can define Σ_{n+1}, Π_{n+1}, $\mathrm{Coll}(\Sigma_{n+1})$, and $\forall^\infty_{n+1}$ as in the case of arithmetic. A little more care is needed when defining end and cofinal extensions: an extension $K \supseteq M$ of $\mathscr{L}$-structures is an *end extension* if for every $k \in K$, having $k \sqsubseteq m \in M$ implies $k \in M$; the extension $K \supseteq M$ is a *cofinal extension* if for every $k \in K$, there exists $m \in M$ such that $k \sqsubseteq m$. All the results presented in this lecture, except Corollary 12.8, are true with these definitions in place.

A particularly interesting example is when $\mathscr{L}$ is some language for set theory and $\sqsubseteq\, = \,\in$. In this case, an extension $K \supseteq M$ is an end extension if and only if M is transitive in K. An extension is a cofinal extension if and only if every new set is an element of an old set. See Zarach [9] or Gitman–Hamkins–Johnstone [2], for example, for a comparison between the collection scheme and the replacement scheme in set theory.

Further reading

See Hodges's book [3] or Keisler's survey [5] for more applications of the Omitting Types Theorem.

References

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