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An Abstract Logical Approach to Characterizing Strong Equivalence in Logic-Based Knowledge Representation Formalisms

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Abstract

We consider knowledge representation (KR) formalisms as collections of finite knowledge bases with a model-theoretic semantics. In this setting, we show that for every KR formalism there is a formalism that characterizes strong equivalence in the original formalism, that is unique up to isomorphism and that has a model theory similar to classical logic.

Introduction

Two knowledge bases T_1 and T_2 are (ordinarily) equivalent if and only if they have the same models. Two knowledge bases T_1 and T_2 are *strongly* equivalent if and only if for any arbitrary third knowledge base U, the expansion of T_1 with U is ordinarily equivalent to the expansion of T_2 with U. In classical propositional logic, the two notions coincide, but there are many useful knowledge representation languages where they are different. Studying strong equivalence of concrete formalisms is important to gain an insight into the underlying structure and semantics of the formalism. One main aim of studying strong equivalence in a concrete formalism is to find a so-called *characterizing* formalism, that is, another language whose ordinary equivalence coincides with strong equivalence in the characterized formalism. For example, strong equivalence of normal logic programs under the stable model semantics can be characterized by the logic of here-and-there (Lifschitz, Pearce, and Valverde 2001).

However, such results about the existence of characterizing formalisms also raise a fundamental question: *Does every formalism have one?* In this paper, we answer this question with a qualified "yes". More precisely, while not every formalism has one, we show that the important case of considering only *finite* knowledge bases (but still possibly infinite languages) guarantees the existence of a characterizing formalism, and that in a very general setting. Existing results on characterizing formalisms make use of specifics of each formalism (Lifschitz, Pearce, and Valverde 2001; Turner 2001). In this paper, we completely abstract away from formalism specifics and address the core of the problem, the nature of strong equivalence itself. In fact, we will not only show the existence of just any characterizing formalism, but of characterizing formalisms whose model theory is *uniquely determined* (up to isomorphism), and structurally resembles that of classical logics. At this point, we appeal to the reader's intuition on what makes logics classical; we will later define what we mean by "classical logic" in a precise mathematical way. Still, we consider this main result of our paper a surprising and important insight, as it tells us that for the overwhelming majority of knowledge representation formalisms, strong equivalence can be approached using established techniques from classical logic.

While our work is in its essence derived from first principles, building mostly upon classical logic and lattice theory, there have been important inspirations. Foremost, Truszczyński (2006) presented a general, algebraic account of strong equivalence within approximation fixpoint theory. His setting is indeed quite general, but most of this generality derives from algebraic commonalities in the semantics of logic programs and default logic. It is not immediately clear, for example, if and how it captures Dung's abstract argumentation frameworks (AFs, 1995), another important AI formalism whose strong equivalence has been studied in the recent past (Oikarinen and Woltran 2011; Baumann 2016). More precisely, while AFs with all their semantics can be captured by approximation fixpoint theory (Strass 2013), Truszczyński's notion of expanding an operator does not coincide with the corresponding notion of expanding AFs and his results are not directly applicable.

The paper proceeds as follows. In the next section, we introduce the general setting in which we derive our results and present our conception of the term "classical logic". Afterwards, in the main part of the paper, we define characterization logics and show two classes of formalisms that always possess them.

An Abstract View on Model Theory

What is a classical logic?

We will spend this section introducing an abstract notion of logics with model-theoretic semantics and explaining when we call some of them classical. Formally, we consider logical languages $\mathcal{L}$, that is, non-empty sets of language elements. We make no assumption on the internal structure of pieces of knowledge $F \in \mathcal{L}$. These pieces of knowledge could be formulas of classical propositional logic, normal logic program rules, or attacks between arguments. A model-theoretic semantics for a language $\mathcal{L}$ uses a set $\mathcal{I}$ of interpretations and a *model* function $\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{I}}$ with the intuition that σ assigns each language subset $T \subseteq \mathcal{L}$, a *the-*

ory, the set $\sigma(T)$ of its models. We make no assumptions on the internal structure of interpretations – there need not be an underlying vocabulary of atoms or the like (although in the concrete cases we consider there often will be) that are the same among syntax and semantics. This is the main abstraction in our setting. It goes beyond what is known from classical logic in that meaning is not assigned to language elements (formulas), but only to *theories*, that is, sets of language elements. This is a necessary requirement for being able to model a number of established knowledge representation formalisms: for example, in normal logic programs, meaning is not assigned to single rules, but only to sets thereof. We illustrate our definitions so far by showing more precisely how existing formalisms can be embedded into our setting.

Example 1. Consider a set $\mathcal{A}$ of propositional atoms.

Classical propositional logic: The underlying language $\mathcal{L}_{PL}$ is the set of all classical propositional formulas over $\mathcal{A}$ and can be defined as usual by induction. The set of interpretations is then given by the set $\mathcal{I}_{PL} = \{v : \mathcal{A} \to \{\mathbf{t}, \mathbf{f}\}\}$ of all two-valued interpretations of $\mathcal{A}$. Lastly, $\sigma_{mod}(T)$ is the set of all models of the theory $T \subseteq \mathcal{L}_{PL}$, that is, the set of all interpretations satisfying all formulas in T.

Normal logic programs: The underlying language $\mathcal{L}_{LP}$ is the set of all normal logic program rules $a_0 \leftarrow a_1, \dots, a_m, \sim a_{m+1}, \dots, \sim a_n$ with $0 \le m \le n$ and $a_0, a_1, \dots, a_n \in \mathcal{A}$. The set $\mathcal{I}_{LP}$ of interpretations is then the set $\mathcal{I}_{LP} = 2^{\mathcal{A}}$ of all possible stable model candidates. Accordingly, $\sigma_{stb}(T)$ returns the set of stable models of the theory (normal logic program) $T \subseteq \mathcal{L}_{LP}$ (Gelfond and Lifschitz 1988).

Abstract argumentation frameworks (Dung 1995): The underlying language $\mathcal{L}_{AF}$ contains the fundamental building blocks of AFs, that is, arguments and attacks: $\mathcal{L}_{AF} = \{(\{a\}, \emptyset), (\{a, b\}, \{(a, b)\}) \mid a, b \in \mathcal{A}\}$. Extension-based semantics can be incorporated by setting $\mathcal{I}_F = 2^{\mathcal{A}}$ and, depending on the argumentation semantics ρ we use, we set $\sigma_\rho(T) = \rho(F_T)$, where $F_T = (\bigcup_{(A,R)\in T} A, \bigcup_{(A,R)\in T} R)$ is the AF associated to $T \subseteq \mathcal{L}_{AF}$. ◊

A consequence function for a language $\mathcal{L}$ is a function $Cn : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ that assigns a given set T of language elements another set $Cn(T)$ of language elements. Intuitively, $Cn(T)$ is understood to be the set of logical consequences of all formulas in T. Given a language, we can define the consequence function in terms of the semantics. In words, the set of consequences of a given theory T is the union of all theories S such that any model of T is a model of S.

Definition 1. Let $\mathcal{L}$ be a language and $\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{I}}$ be a model function. Define the consequence function

$$\mathcal{C}^\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{L}}, \qquad T \mapsto \bigcup_{\substack{S \subseteq \mathcal{L},\\ \sigma(T) \subseteq \sigma(S)}} S \qquad \Diamond$$

For classical logic $\mathcal{L}_{PL}$, this definition coincides with the standard notion of logical consequence. It will be of great interest in this paper that certain algebraic properties of the semantics induce certain useful properties of the consequence relation. We now introduce the most important properties.

Definition 2. Let $\mathcal{L}$ be a language.

- A model function $\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{I}}$ is *antimonotone* iff for all $T_1, T_2 \in 2^{\mathcal{L}}$: $T_1 \subseteq T_2 \implies \sigma(T_2) \subseteq \sigma(T_1)$.
- A consequence operator $Cn : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is *monotone* iff for all $T_1, T_2 \in 2^{\mathcal{L}}$: $T_1 \subseteq T_2 \implies Cn(T_1) \subseteq Cn(T_2)$.
- A consequence operator $Cn : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is *increasing* iff for all $T \in 2^{\mathcal{L}}$, we find $T \subseteq Cn(T)$. ◊

It is easy to show that for each antimonotone semantics σ, Definition 1 induces a monotone consequence function. To save some space in what follows, we define a *logic* as a tuple $(\mathcal{L}, \mathcal{I}, \sigma)$ consisting of a language $\mathcal{L}$, an interpretation set $\mathcal{I}$, and a model function $\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{I}}$.

Standard and strong equivalence

This paper is chiefly about characterizing strong equivalence in one logic via standard equivalence in another logic. We will now formally introduce these concepts.

Definition 3. Let $(\mathcal{L}, \mathcal{I}, \sigma)$ be a logic and $T_1, T_2 \subseteq \mathcal{L}$ theories. We say that T_1 and T_2 are

- *ordinarily equivalent* iff $\sigma(T_1) = \sigma(T_2)$;
- *strongly equivalent* iff $\forall U \subseteq \mathcal{L} : \sigma(T_1 \cup U) = \sigma(T_2 \cup U)$.

Model function σ has the *replacement property* if and only if ordinary equivalence implies strong equivalence. ◊

What properties must a logic possess in order for standard and strong equivalence to coincide? Maybe it suffices that the logic has a monotone consequence function?

Example 2. Consider the language $\mathcal{L} = \{a, b\}$ with interpretation set $\mathcal{I} = \{1, 2\}$ and model function σ given by

$$\sigma(\emptyset) = \sigma(\{a\}) = \{1, 2\}\,;\ \sigma(\{b\}) = \{2\}\,;\ \sigma(\{a, b\}) = \emptyset$$

It is easy to verify that the semantics σ is antimonotone and (thus) its consequence function $\mathcal{C}^\sigma$ is monotone:

$$\mathcal{C}^\sigma(\emptyset) = \mathcal{C}^\sigma(\{a\}) = \{a\}\,;\ \mathcal{C}^\sigma(\{b\}) = \mathcal{C}^\sigma(\{a, b\}) = \{a, b\}$$

However, while $\emptyset$ and $\{a\}$ are ordinarily equivalent, they are not strongly equivalent, which can be seen by extending both with the theory $\{b\}$: $\sigma(\emptyset \cup \{b\}) = \sigma(\{b\}) = \{2\}$ and $\sigma(\{a\} \cup \{b\}) = \sigma(\{a, b\}) = \emptyset$, with $\emptyset \neq \{2\}$. ◊

So having a monotone consequence function is, by itself, insufficient to guarantee the replacement property. We can however identify a property that is strong enough to guarantee replacement on its own. We call it the intersection property, because it basically says that the semantics of a theory can be obtained by only considering the semantics of the singleton sets constituting the theory.

Definition 4. Let $(\mathcal{L}, \mathcal{I}, \sigma)$ be a logic. Its model function $\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{I}}$ has the *intersection property* iff for all $T \subseteq \mathcal{L}$:

$$\sigma(T) = \bigcap_{F \in T} \sigma(\{F\}) \qquad \Diamond$$

It follows from the definition that for any two theories $T_1, T_2 \subseteq \mathcal{L}$, we have that $\sigma(T_1 \cup T_2) = \sigma(T_1) \cap \sigma(T_2)$. The intersection property is a certain locality, independence, or compositionality criterion. In particular, the intersection property entails that $\sigma(\emptyset) = \mathcal{I}$. Towards an explanation of Example 2 we can now remark that its model function σ does not have the intersection property:

$$\sigma(\{a,b\}) = \emptyset \neq \{2\} = \{1,2\} \cap \{2\} = \sigma(\{a\}) \cap \sigma(\{b\})$$

Indeed, this is necessarily so: as we will show next (and as is easy to show), satisfying the intersection property is sufficient for satisfying the replacement property.

Proposition 1. *Let $(\mathcal{L}, \mathcal{I}, \sigma)$ be a logic. If σ satisfies the intersection property, then standard equivalence coincides with strong equivalence.*

Notably, monotonicity properties were not even needed in the above result. So why is it that all formalisms we know of that have the replacement property also happen to have monotone consequence functions? It holds because σ having the intersection property implies that σ is antimonotone (and this in turn implies that $\mathcal{C}^\sigma$ is monotone).

Proposition 2. *Let $(\mathcal{L}, \mathcal{I}, \sigma)$ be a logic where σ has the intersection property. Then σ is antimonotone.*

It is easy to see that classical propositional logic $\mathcal{L}_{PL}$ has the intersection property simply *by definition*: the standard model semantics is typically firstly defined for single formulas $\varphi \in \mathcal{L}_{PL}$ and then generalized to theories T by *setting* $\sigma_{mod}(T) = \bigcap_{\varphi \in T} \sigma_{mod}(\{\varphi\})$.

Characterization Logics

From now on we omit $\mathcal{I}$ from the presentation of logics and thus write $(\mathcal{L}, \sigma)$, since concrete interpretations are immaterial for strong equivalence. Furthermore, we consider subsets of $2^{\mathcal{L}}$ as domain of σ, namely the cases $dom(\sigma) = 2^{\mathcal{L}}$ (called *full* logics) and $dom(\sigma) = \left(2^{\mathcal{L}}\right)_{\text{fin}} = \{T \in 2^{\mathcal{L}} \mid T \text{ is finite}\}$ (*finite-theory* logics), the restriction of $\mathcal{L}$ to finite knowledge bases.

Definition 5. Let $(\mathcal{L}, \sigma)$ be a logic. Define the binary relation *strong equivalence* $\equiv_s^\sigma \subseteq dom(\sigma) \times dom(\sigma)$ by $T_1 \equiv_s^\sigma T_2 \iff \forall U \in dom(\sigma) : \sigma(S \cup U) = \sigma(T \cup U)$. ◊

It is straightforward to show that $\equiv_s^\sigma$ is an equivalence relation; we denote the equivalence class of a theory $T \in dom(\sigma) \subseteq 2^{\mathcal{L}}$ by $[T]_s^\sigma$. We recall that for all theories $T_1, T_2 \subseteq \mathcal{L}$, we have $T_1 \in [T_2]_s^\sigma$ iff $[T_1]_s^\sigma = [T_2]_s^\sigma$.

Given an arbitrary logic $(\mathcal{L}, \sigma)$, we want to find a characterizing classical logic, that is, a semantics σ' that has the intersection property and whose ordinary equivalence coincides with strong σ-equivalence. Such logics get a name.

Definition 6. Let $(\mathcal{L}, \sigma)$ be a (full) logic. The logic $(\mathcal{L}, \sigma')$ is a (full) *characterization logic for* $(\mathcal{L}, \sigma)$ if and only if:

1. $\forall T_1, T_2 \subseteq \mathcal{L} : \sigma'(T_1) = \sigma'(T_2) \iff [T_1]_s^\sigma = [T_2]_s^\sigma$;
2. $\forall \mathcal{T} \subseteq 2^{\mathcal{L}} : \sigma'(\bigcup_{T \in \mathcal{T}} T) = \bigcap_{T \in \mathcal{T}} \sigma'(T)$. ◊

Property (2) is the intersection property; we refer to (1) as the *characterization property*. We will start our analysis of characterization logics with showing that they are unique up to isomorphism. More precisely, for any model function σ, the algebras corresponding to the model theories of any two characterizing model functions σ' and σ'' are isomorphic. To do that, we first show that the model theory of any characterization logic is a complete lattice, that is, a partially ordered set where each subset of the carrier set has both a greatest lower bound (glb) and a least upper bound (lub).

Proposition 3. *Let $(\mathcal{L}, \sigma)$ be a full logic with characterization logic $(\mathcal{L}, \sigma')$. The pair $\left(\sigma'\left(2^{\mathcal{L}}\right), \subseteq\right)$ is a complete lattice where glb $\bigwedge$ and lub $\bigvee$ are given such that for all $\mathcal{K} \subseteq \sigma'\left(2^{\mathcal{L}}\right)$,*

$$\bigwedge_{K \in \mathcal{K}} K = \bigcap_{K \in \mathcal{K}} K \text{ and } \bigvee_{K \in \mathcal{K}} K = \bigwedge_{L \in \mathcal{K}^u} L$$

with $\mathcal{K}^u = \left\{L \in \sigma'\left(2^{\mathcal{L}}\right) \;\middle|\; \forall K \in \mathcal{K} : K \subseteq L\right\}$.

After these necessary preliminaries, we now present the result on uniqueness of characterization logics.

Theorem 4. *Let $(\mathcal{L}, \sigma)$ be a full logic with characterization logics $(\mathcal{L}, \sigma')$ and $(\mathcal{L}, \sigma'')$. Then the complete lattices $\left(\sigma'\left(2^{\mathcal{L}}\right), \subseteq\right)$ and $\left(\sigma''\left(2^{\mathcal{L}}\right), \subseteq\right)$ are isomorphic.*

Thus if a classical characterization logic exists, it is (up to isomorphism on its model theory) uniquely determined. However, as we show next, in some cases there simply is no characterization logic.

Example 3. Let $\mathcal{L} = \mathbb{N}$ be the natural numbers and $\mathcal{I} \neq \emptyset$ arbitrary. We define the semantics $\sigma : 2^{\mathcal{L}} \to 2^{\mathcal{I}}$ such that

$$\sigma(T) = \begin{cases} \emptyset & \text{if } T \text{ is finite,} \\ \mathcal{I} & \text{otherwise.} \end{cases}$$

There are two strong equivalence classes: $[\emptyset]_s^\sigma$, the set of all finite subsets of $\mathbb{N}$, and $[\mathbb{N}]_s^\sigma$, the set of all infinite subsets of $\mathbb{N}$. Assume that $(\mathcal{L}, \sigma')$ is a characterization logic for $(\mathcal{L}, \sigma)$. By the model intersection property, we get

$$\sigma'(\mathbb{N}) = \sigma'\left(\bigcup_{n \in \mathbb{N}} \{n\}\right) = \bigcap_{n \in \mathbb{N}} \sigma'(\{n\}) = \sigma'(\emptyset)$$

in contradiction to the characterization property. ◊

We proceed with some useful properties needed to show that there is a sub-class of full logics guaranteeing the existence of a characterization logic in contrast to the unrestricted case as shown in Example 3. Most importantly, strong equivalence classes have an expansion property: It is not completely obvious, but it follows easily from the definition of strong equivalence that two strongly equivalent theories can both be expanded (via set union) with the same theory and are again strongly equivalent; the converse holds as well. Furthermore, the union of two strongly equivalent theories is again strongly equivalent to the two theories.

Lemma 5. *Let $(\mathcal{L}, \sigma)$ be a full logic and $T, T_1, T_2 \subseteq \mathcal{L}$.*

1. *Strong equivalence is invariant to expansion:*

$$[T_1]_s^\sigma = [T_2]_s^\sigma \iff \left(\forall U \subseteq \mathcal{L} : [T_1 \cup U]_s^\sigma = [T_2 \cup U]_s^\sigma\right)$$

2. *Each strong equivalence class is a join-semilattice:*

$$T_1, T_2 \in [T]_s^\sigma \implies T_1 \cup T_2 \in [T]_s^\sigma$$

It follows in particular that in the case of logics $(\mathcal{L}, \sigma)$ with $\mathcal{L}$ finite, each strong equivalence class $[T]^\sigma_s$ has a $\subseteq$-greatest element that equals the union of all elements. However, for logics with infinite $\mathcal{L}$, this need not be the case: in the logic of Example 3, the class $[\emptyset]^\sigma_s$ has no maximal elements, in particular no greatest element; the class $[\mathbb{N}]^\sigma_s$ has no minimal elements, in particular no least element. We will see that having a $\subseteq$-greatest element in each equivalence class is sufficient for the existence of a characterization logic. We therefore decided to name this class of logics.

Definition 7. Let $(\mathcal{L}, \sigma)$ be a logic.

1. For $T \subseteq \mathcal{L}$ define $\widehat{[T]^\sigma_s} = \bigcup_{S \in [T]^\sigma_s} S$.
2. $(\mathcal{L}, \sigma)$ is *covered* if and only if $\forall T \subseteq \mathcal{L} : \widehat{[T]^\sigma_s} \in [T]^\sigma_s$. ◇

Roughly, the existence of greatest elements in equivalence classes guarantees that these classes are closed under arbitrary set union. Clearly any finite logic is covered. Furthermore, two familiar representatives of covered logics are classical logic and abstract argumentation theory. In the former case, it is clear that arbitrary unions of families of equivalent theories are again theories that are equivalent to each of its members. In the latter case it is not immediately clear but can be shown with reasonable effort. We conclude this section with its main theorem showing that any full logic being covered possesses a characterization logic.

Theorem 6. *Let $(\mathcal{L}, \sigma)$ be a logic. If $(\mathcal{L}, \sigma)$ is covered then a characterization logic for $(\mathcal{L}, \sigma)$ is given by $(\mathcal{L}, \sigma')$ with*

$$\sigma' : 2^{\mathcal{L}} \to 2^{2^{\mathcal{L}}}, \quad T \mapsto \bigcup_{\substack{S \in 2^{\mathcal{L}}, \\ T \subseteq \widehat{[S]^\sigma_s}}} [S]^\sigma_s$$

Finite-Theory Characteriziation Logics

In the field of knowledge representation it is a common assumption that knowledge bases are finite. This is indeed not overly limiting, as finite knowledge bases will be most relevant for practical purposes. The following definition translates this assumption into our setting: the *finite-theory version* of a given logic (or simply, a *finite-theory logic*) considers only the finite knowledge bases of a language.

Definition 8. Given a full logic $(\mathcal{L}, \sigma)$, the finite-theory version $(\mathcal{L}, \sigma_{\text{fin}})$ of $(\mathcal{L}, \sigma)$ is defined by the semantics

$$\sigma_{\text{fin}} : \left(2^{\mathcal{L}}\right)_{\text{fin}} \to \sigma\left(2^{\mathcal{L}}\right) \qquad \text{with} \qquad \sigma_{\text{fin}}(T) = \sigma(T)$$

where $\left(2^{\mathcal{L}}\right)_{\text{fin}} = \{T \in 2^{\mathcal{L}} \mid T \text{ is finite}\}$. ◇

For finite-theory restrictions of logics, we adequately relax our requirements on characterization logics.

Definition 9. Let $(\mathcal{L}, \sigma)$ be a full logic and $(\mathcal{L}, \sigma_{\text{fin}})$ its finite-theory version. We say that $(\mathcal{L}, \sigma'_{\text{fin}})$ is a *finite-theory characterization logic for* $(\mathcal{L}, \sigma)$ if and only if:

1. $\forall T_1, T_2 \in \left(2^{\mathcal{L}}\right)_{\text{fin}} : \sigma'_{\text{fin}}(T_1) = \sigma'_{\text{fin}}(T_2)$ iff $[T_1]^{\sigma_{\text{fin}}}_s = [T_2]^{\sigma_{\text{fin}}}_s$;
2. $\forall T_1, T_2 \in \left(2^{\mathcal{L}}\right)_{\text{fin}} : \sigma'_{\text{fin}}(T_1 \cup T_2) = \sigma'_{\text{fin}}(T_1) \cap \sigma'_{\text{fin}}(T_2)$. ◇

The second item requires binary intersection only; this is due to the fact that arbitrary unions of theories are not necessarily finite, and thus their semantics might not be well-defined.

As we did in the general case before, we first analyze the algebraic structure of the resulting model theories. We show that the model theory of any finite-theory characterization logic forms a lattice, that is, a partially ordered set where each non-empty finite subset has both a greatest lower bound and a least upper bound. (This is in contrast to *complete* lattices in the general case.) The proof is, although similar in procedure, slightly more involved than in the general case.

Proposition 7. *Let $(\mathcal{L}, \sigma_{\text{fin}})$ be a finite-theory logic with characterization logic $(\mathcal{L}, \sigma'_{\text{fin}})$. Denoting $\mathcal{K} = \left\{\sigma'_{\text{fin}}(T) \mid T \in \left(2^{\mathcal{L}}\right)_{\text{fin}}\right\}$, the pair $(\mathcal{K}, \subseteq)$ is a lattice where glb and lub are given such that for all $K_1, K_2 \in \mathcal{K}$:*

$$K_1 \wedge K_2 = K_1 \cap K_2 \text{ and } K_1 \vee K_2 = \bigwedge \{K_1, K_2\}^u$$

where $\{K_1, K_2\}^u = \{K \in \mathcal{K} \mid K_1 \subseteq K, K_2 \subseteq K\}$.

As before, we can show (with reasonable effort) that finite-theory characterization logics are unique up to isomorphism.

Theorem 8. *Let $(\mathcal{L}, \sigma)$ be a finite-theory logic having two finite-theory characterization logics $(\mathcal{L}, \sigma'_{\text{fin}})$ and $(\mathcal{L}, \sigma''_{\text{fin}})$. Denoting the sets $\mathcal{K}' = \left\{\sigma'_{\text{fin}}(T) \mid T \in \left(2^{\mathcal{L}}\right)_{\text{fin}}\right\}$ and $\mathcal{K}'' = \left\{\sigma''_{\text{fin}}(T) \mid T \in \left(2^{\mathcal{L}}\right)_{\text{fin}}\right\}$, the lattices $(\mathcal{K}', \subseteq)$ and $(\mathcal{K}'', \subseteq)$ are isomorphic.*

The following theorem shows that any logic possesses a finite-theory characterization logic. This means that the most important case for knowledge representation behaves well in the sense that characterization logics always exist.

Theorem 9. *Let $(\mathcal{L}, \sigma)$ be a full logic. Then a finite-theory characterization logic for $(\mathcal{L}, \sigma)$ is given by $(\mathcal{L}, \sigma'_{\text{fin}})$ with*

$$\sigma'_{\text{fin}} : \left(2^{\mathcal{L}}\right)_{\text{fin}} \to 2^{2^{\mathcal{L}}}, \quad T \mapsto \bigcup_{\substack{S \in (2^{\mathcal{L}})_{\text{fin}}, \\ T \subseteq S}} [S]^{\sigma_{\text{fin}}}_s$$

Intuitively, in this canonical construction of a characterization semantics σ'_{fin} (akin to Herbrand interpretations in first-order logic), the model set of a theory T is the set of all theories that are strongly equivalent to some supertheory of T.

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