Reasoning with Uncertain Inputs in Possibilistic Networks

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Abstract

Graphical belief models are compact and powerful tools for representing and reasoning under uncertainty. Possibilistic networks are graphical belief models based on possibility theory. In this paper, we address reasoning under uncertain inputs in both quantitative and qualitative possibilistic networks. More precisely, we first provide possibilistic counterparts of Pearl's methods of virtual evidence then compare them with the possibilistic counterparts of Jeffrey's rule of conditioning. As in the probabilistic setting, the two methods are shown to be equivalent in the quantitative setting regarding the existence and uniqueness of the solution. However in the qualitative setting, Pearl's method of virtual evidence which applies directly on graphical models disagrees with Jeffrey's rule and the virtual evidence method. The paper provides the precise situations where the methods are not equivalent. Finally, the paper addresses related issues like transformations from one method to another and commutativity.

Introduction

Belief revision and more generally belief dynamics is a fundamental task in artificial intelligence. Indeed, rational agents often need to revise their beliefs in order to take into account new information. In uncertainty frameworks, this task is often referred to as belief revision or reasoning with uncertain inputs. Belief revision has received a lot of attention in artificial intelligence especially in logic-based and some uncertainty frameworks (Benferhat et al. 2010)(Dubois and Prade 1994). In spite of the power of graphical belief models for representing and reasoning with uncertain information, belief revision and reasoning with uncertain inputs in such models is addressed only in few works mostly in the context of Bayesian networks (Chan and Darwiche 2005)(Vomlel 2004).

In this paper, we compare two methods for revising the beliefs encoded in a possibilistic framework when new and uncertain information is available. The two methods compared here are Jeffrey's rule of conditioning (Jeffrey 1965) and the virtual evidence method (Pearl 1988). They were originally proposed and studied in a probabilistic setting where they are shown to be equivalent and differ only in the way they specify the inputs (Chan and Darwiche 2005).

In the possibilistic setting, the counterparts of Jeffrey's rule are proposed in (Dubois and Prade 1997)(Dubois and Prade 1993). In (Benferhat, Tabia, and Sedki 2011), we studied the existence and the uniqueness of the solution in both the quantitative and qualitative possibilistic settings. The possibilistic counterpart of Jeffrey's rule is investigated for belief revision in possibilistic knowledge bases in (Benferhat et al. 2010) where it is claimed that this rule can successfully recover most of the belief revision kinds such as the natural belief revision, drastic belief revision, reinforcement, etc. In (Benferhat, Da Costa Pereira, and Tettamanzi 2013), a syntactic version is proposed for the possibilistic counterpart of Jeffrey's rule. In this paper, we address revising the beliefs encoded by means of possibilistic networks with uncertain inputs. More precisely, the paper provides

- Possibilistic counterparts of Pearl's method of virtual evidence and its generalization named the virtual evidence method in both the quantitative and qualitative settings. Unlike the probabilistic and quantitative possibilistic settings, the inputs for the qualitative counterparts of Pearl's methods should be possibility degrees because of the definition of the qualitative conditioning.
- An analysis of the existence and uniqueness of the solutions using the proposed possibilistic counterparts of Pearl's methods.
- Transformations from Jeffrey's rule to the virtual evidence method and vice versa and comparisons of these methods in both the quantitative and qualitative settings. As in the probabilistic setting, the two methods are shown to be equivalent in the quantitative setting regarding the existence and uniqueness of the solution. However in the qualitative setting, Pearl's method of virtual evidence is not equivalent to Jeffrey's rule since it is impossible using this method to increase the possibility degree of an event but its generalization is shown equivalent to Jeffrey's rule.

Possibility Theory and Possibilistic networks

Let us first fix the notations used in the rest of this paper. V=$\{X, Y, A_1, A_2, ..\}$ denotes a set of variables (in capital letters and indexed when necessary). D_{A_i}=$\{a_1, a_2, .., a_m\}$ denotes the domain of a variable A_i (note that D_{A_i} is assumed a finite domain). a_i denotes an instance (value) of variable A_i, namely $a_i \in D_{A_i}$. Ω=$\times_{A_i \in V} D_{A_i}$ denotes the universe of discourse (all possible states of the world). It is the cartesian product of all the variable domains involved in V. A tuple w=$(a_1, a_2, .., a_n)$ which is an instance of Ω represents a possible state of the world (also called a model or interpretation). ϕ, φ, λ denote subsets of Ω called events.

Possibility theory

Possibility theory is an alternative uncertainty theory suited for representing and reasoning with uncertain and incomplete information (Dubois and Prade 1988a; Yager 1983). The concept of possibility distribution π is an important building block of possibility theory: It is a mapping from the universe of discourse Ω to the unit scale $[0, 1]$ which can be either quantitative or qualitative (ordinal). In both these settings, a possibility degree $\pi(w_i)$ expresses to what extent it is consistent that w_i can be the actual state of the world. In particular, $\pi(w_i)$=1 means that w_i is totally plausible and $\pi(w_i)$=0 denotes an impossible event. The relation $\pi(w_i)$>$\pi(w_j)$ means that w_i is more plausible than w_j. A possibility distribution π is said to be normalized if $\max_{w_i\in\Omega}(\pi(w_i))$=1. The second important concept in possibility theory is the one of possibility measure denoted $\Pi(\phi)$ and computing the possibility degree relative to an event $\phi \subseteq \Omega$. It evaluates to what extent ϕ is consistent with the current knowledge encoded by the possibility distribution π on Ω. It is defined as follows:

$$\Pi(\phi) = \max_{w_i \in \phi}(\pi(w_i)). \tag{1}$$

The term $\Pi(\phi)$ denotes the possibility degree of having one of the events involved in ϕ as the actual state of the world.

The necessity measure is the dual of possibility measure and evaluates the certainty implied by the current knowledge of the world. Namely, $N(\phi)$=$1 - \Pi(\overline{\phi})$ where $\overline{\phi}$ denotes the complement of ϕ.

According to the interpretation underlying the possibilistic scale [0,1], there are two variants of possibility theory:

- **Qualitative (or min-based) possibility theory**: In this case, the possibility distribution is a mapping from the universe of discourse Ω to an *ordinal* scale where only the "ordering" of the values is important.
- **Quantitative (or product-based) possibility theory**: In this case, the possibilistic scale [0,1] is numerical and possibility degrees are like numeric values that can be manipulated by arithmetic operators. One of the possible interpretations of quantitative possibility distributions is viewing $\pi(w_i)$ as a degree of surprise as in Spohn's ordinal conditional functions (Spohn 1988).

The other fundamental notion in possibility theory is the one of conditioning concerned with updating the current knowledge encoded by the possibility distribution π when a completely sure event (evidence) is observed. Note that there are several definitions of the possibilistic conditioning (Hisdal 1978)(L.M. De Campos and Moral 1995)(Dubois and Prade 1988b) (Fonck 1997). In the quantitative setting, the product-based conditioning (also known as Dempster rule of conditioning (Shafer 1976)) is defined as follows:

$$\pi(w_i|_p\phi) = \begin{cases} \frac{\pi(w_i)}{\Pi(\phi)} & \text{if } w_i \in \phi; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Conditioning in the qualitative setting is defined as follows (Hisdal 1978):

$$\pi(w_i|_m\phi) = \begin{cases} 1 & \text{if } \pi(w_i)\text{=}\Pi(\phi) \text{ and } w_i \in \phi; \\ \pi(w_i) & \text{if } \pi(w_i)< \Pi(\phi) \text{ and } w_i \in \phi; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

While there are several similarities between the quantitative possibilistic and the probabilistic frameworks (conditioning is defined in the same way), the qualitative one is significantly different. Note that the two definitions of conditioning satisfy the condition: $\forall\omega\in\phi$, $\pi(\omega)$=$\pi(\omega|\phi)\otimes\Pi(\phi)$ where $\otimes$ is either the product or min-based operator.

Possibilistic networks

A possibilistic network G=$<\mathcal{G},\Theta>$ is specified by:

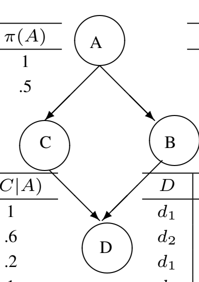
i) *A graphical component* $\mathcal{G}$ consisting in a directed acyclic graph (DAG) where vertices represent variables of interest and edges represent direct *dependence* relationships between these variables.

ii) *A quantitative component* Θ allowing to quantify the uncertainty of the relationships between domain variables using local possibility tables (CPTs). The quantitative component or G's parameters consist in a set of local possibility tables Θ_i=$\{\theta_{a_i|u_i}\}$ where $a_i \in D_i$ and u_i is an instance of U_i denoting the parent variables of A_i in $\mathcal{G}$.

Note that all the local possibility distributions Θ_i must be normalized, namely $\forall i$=$1..n$, $\forall u_i \in D_{U_i}$, $\max_{a_i\in D_i}(\theta_{a_i|u_i})$=1. The structure of G encodes a set of conditional independence relationships I=$\{I(A_i, U_i, Y)\}$ where Y is a subset of variables non descendent from A_i. For example, in the network of Figure 1, variable C is independent of B in the context of A.

Example 1. Figure 1 gives an example of a possibilistic network over four binary variables A, B, C and D.

A	$\pi(A)$
a_1	1
a_2	.5



B	A	$\pi(B\|A)$
b_1	a_1	1
b_2	a_1	1
b_1	a_2	.2
b_2	a_2	1

C	A	$\pi(C\|A)$
c_1	a_1	1
c_2	a_1	.6
c_1	a_2	.2
c_2	a_2	1

D	B	C	$\pi(D\|BC)$
d_1	b_1	c_1	0
d_2	b_1	c_1	1
d_1	b_1	c_2	1
d_2	b_1	c_2	.4
d_1	b_2	c_1	.6
d_2	b_2	c_1	1
d_1	b_2	c_2	1
d_2	b_2	c_2	.4

Figure 1: Example of a possibilistic network

In the possibilistic setting, the joint possibility distribution is factorized using the chain rule defined as follows:

$$\pi(a_1, a_2, .., a_n) = \otimes_{i=1}^{n}(\pi(a_i|u_i)), \tag{4}$$

where $\otimes$ denotes the product-based (resp. min-based) operator used in the quantitative (resp. qualitative) setting.

Example 2. In the min-based setting, the joint distribution encoded by the network of Figure 1 is derived as follows:

$$\pi(A,B,C,D) = \min(\pi(A), \pi(C|A), \pi(B|A), \pi(D|BC)).$$

Reasoning with uncertain inputs in the probabilistic setting

In the probabilistic framework, there are two main methods for revising beliefs represented using probability distributions or probabilistic models by uncertain information: Jeffrey's rule (Jeffrey 1965) and the virtual evidence methods (Pearl 1988). Let us first focus on the notions of beliefs and uncertain inputs.

Beliefs and uncertain inputs

The concept of belief used in this paper allows an agent to encode at which extent a given event is believed to be or become the actual state of the world. Generally, beliefs are specified over a universe of discourse Ω using belief measures like probability or possibility measures[1]. Then belief degrees are associated with each singleton event $\omega \in \Omega$ in the form of a belief distribution. According to the chosen setting, belief measures allow to assess the belief of any arbitrary event $\phi \subseteq \Omega$. Now given a set of initial beliefs (also called prior beliefs), an agent may have new information which can be in the form of evidence (also called hard evidence and corresponding for instance to a sure observation of the value of a variable) or in the form of uncertain or soft evidence (e.g. unreliable input) or simply new beliefs regarding some events[2]. In the uncertainty literature, belief change dealing with hard evidence is known as belief update and it is generally based on conditioning while it is known as belief revision in case of uncertain inputs.

In Jeffrey's rule and the virtual evidence methods, the uncertainty bears on an exhaustive and mutually exclusive set of events λ_1,..,λ_n (namely, $\forall \lambda_i \subseteq \Omega$ and $\forall \lambda_j \subseteq \Omega$ with $i \neq j$, we have $\lambda_i \cap \lambda_j = \emptyset$ and $\lambda_1 \cup \lambda_2 \cup .. \cup \lambda_n = \Omega$). However, the new information is expressed differently:

- In Jeffrey's rule, the new beliefs are a probability distribution over λ_1,..,λ_n and must consequently sum up to 1. The new information is expressed in the form of (λ_i, α_i) such that $P'(\lambda_i) = \alpha_i$ where p' denotes the revised probability distribution fully accepting the new beliefs.
- In Pearl's methods, the new information is expressed by specifying the amount of increase or decrease of the belief on each event λ_i moving from p to p'. This amount is called in (Darwiche 2009) the Bayes factor and corresponds to the ratio $\frac{P'(\lambda_i)}{P(\lambda_i)}$. For example, a ratio regarding an event λ_i of 2 means that the new belief regarding λ_i is twice as it was before receiving this new information.

[1]The beliefs of an agent can be encoded using other formalisms like belief bases (e.g. probabilistic or possibilistic knowledge bases), graphical belief models, etc.

[2]On the different meanings of hard, soft and uncertain evidence, see (Ma and Liu 2011)(Pan, Peng, and Ding 2006)(Bilmes 2004).

Jeffrey's rule of conditioning

Jeffrey's rule (Jeffrey 1965) is an extension of the probabilistic conditioning to the case where the evidence is uncertain. This method involves a way for:

1. **Specifying the uncertain evidence:** The uncertainty is of the form (λ_i, α_i) with $\alpha_i = P'(\lambda_i)$ meaning that after the revision operation, the posterior probability of each event λ_i must be equal to α_i (namely, $P'(\lambda_i) = \alpha_i$). The uncertain inputs are seen as a constraint or an effect once the new information is fully accepted.
2. **Computing the revised probability distributions:** Jeffrey's method assumes that although there is a disagreement about the events λ_i in the old distribution p and the new one p', the conditional probability of any event $\phi \subseteq \Omega$ given any uncertain event λ_i remains the same in the original and the revised distributions. Namely,

$$\forall \lambda_i \in \Omega, \forall \phi \subseteq \Omega, P(\phi|\lambda_i) = P'(\phi|\lambda_i). \quad (5)$$

   The underlying interpretation of the revision implied by constraint of Equation 5 is that the revised probability distribution p' must not change the conditional probability degrees of any event ϕ given the uncertain events λ_i. To revise the probability degree of any event $\phi \subseteq \Omega$, the following formula is used:

$$P'(\phi) = \sum_{\lambda_i} \alpha_i * \frac{P(\phi, \lambda_i)}{P(\lambda_i)}. \quad (6)$$

The revised distribution p' obtained using Jeffrey's rule always exists and it is unique (Chan and Darwiche 2005). In the following, we first present Pearl's method of virtual evidence applying directly on Bayesian networks then its generalization named virtual evidence method applying directly on probability distributions as in Jeffrey's rule.

Pearl's method of virtual evidence

This method is proposed in (Pearl 1988) in the framework of Bayesian networks. The main idea of this method is to cast the uncertainty relative to the uncertain evidence E on some $virtual$ sure event η: the uncertainty regarding E is specified as the likelihood of η in the context of E. In Pearl's method of virtual evidence the beliefs are encoded with a Bayesian network over a set of variables $\{A_1, .., A_n\}$. Assume that the observation regarding a variable A_i is uncertain (for instance, because of a sensor unreliability). Pearl's virtual evidence method deals with this issue by adding for each uncertain observation variable A_i a variable Z_i with an arc from A_i to Z_i. The uncertainty relative to A_i is then cast as the likelihoods of $Z_i = z_i$ in the context of A_i. Then the uncertain inputs are taken into account by observing the sure evidence $Z_i = z_i$. Doing this way, it is clear that the conditional probability of any event ϕ given A_i is the same in the old and revised distribution, namely $\forall \phi \subseteq \Omega$, $p(\phi|A_i) = p'(\phi|A_i)$. It is the d-separation[3] criterion that ensures this property. In this method, the uncertainty bears on a set of exhaustive and mutually exclusive events a_1,..,a_n (forming the domain of variable A_i). Let γ_1:..:γ_n denote the

[3]The d-separation property states that two disjoint variable sub-sets X and Y are d-separated if there exists a third variable sub-set Z such that X and Y are independent given Z.

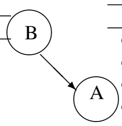
likelihood ratios encoding the new inputs. Such ratios should satisfy the following condition:

$$\gamma_1 : .. : \gamma_n = \frac{P'(a_1)}{P(a_1)} : .. : \frac{P'(a_n)}{P(a_n)} \quad (7)$$

Note that there are many solutions for the values of $\gamma_1, .. , \gamma_n$ satisfying the condition of Equation 7 (one possible solution for encoding the inputs within the network is to set $p(z|a_i)$ to $\gamma_i = \frac{p'(a_i)}{p(a_i)}$). It is worth to mention that contrary to Jeffrey's rule where the inputs α_1,..,α_n are the revised belief degrees once the revision performed, in Pearl's methods, the inputs are likelihood ratios γ_1,..,γ_n satisfying Equation 7 and they don't form a probability distribution.

Example 3. *Assume that the current beliefs about a given problem are encoded by the Bayesian network G of Figure 2 over two binary variables A and B. The joint probability distribution encoded by this network is given by the joint probability distribution $p(AB)$ of Figure 2.*

B	$p(B)$
b_1	0.75
b_2	0.25



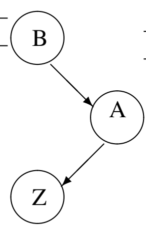
A	B	$p(A\|B)$
a_1	b_1	0.8
a_2	b_1	0.2
a_1	b_2	0.4
a_2	b_2	0.6

A	B	$p(AB)$
a_1	b_1	0.6
a_2	b_1	0.15
a_1	b_2	0.1
a_2	b_2	0.15

Figure 2: Example of an initial Bayesian network G and the joint distribution $p(AB)$ encoded by G.

Assume now that we have new inputs γ_{a_1}=.57 and γ_{a_2}= 2. Following Pearl's method of virtual evidence, this is handled by adding a variable Z as a child of A as in Figure 3.

B	$p(B)$
b_1	0.75
b_2	0.25



A	B	$p(A\|B)$
a_1	b_1	0.8
a_2	b_1	0.2
a_1	b_2	0.4
a_2	b_2	0.6

Z	A	$p(Z\|A)$
z	a_1	0.57
z	a_2	2

Figure 3: Bayesian network G': Bayesian network G of Figure 2 augmented with node Z to encode the new inputs.

Let us mention that the conditional probability table of node Z in order to encode the new inputs don't need to be normalized (it can easily be normalized but this is not needed to revise the old beliefs encoded by the initial network). Another solution satisfying Equation 7 is γ_{a_1}:γ_{a_2} = .2:.7 (since $\frac{.57}{2} = \frac{.2}{.7}$). The revised beliefs are given in Table 1.

A	B	$p(AB\|z)$
a_1	b_1	0.34
a_2	b_1	0.3
a_1	b_2	0.06
a_2	b_2	0.3

Table 1: The conditional distribution $p_{G'}(.|z)$ representing the revised distribution encoded by the network of Figure 2.

It is easy to check that the revised distribution p'=$p_{G'}(.|z)$ fully integrates the inputs.

Virtual evidence method

The virtual evidence method generalizes Pearl's method of virtual evidence and applies directly on joint probability distributions as in Jeffrey's rule.

1. **Specifying the uncertain inputs:** The new information is in the form of a set of likelihood ratios γ_1,..,γ_n such that γ_i=$P(\eta|\lambda_i)$ and
$$\gamma_1 : .. : \gamma_n = \frac{P'(\lambda_1)}{P(\lambda_1)} : .. : \frac{P'(\lambda_n)}{P(\lambda_n)},$$
where λ_1,..,λ_n denote the exhaustive and mutually exclusive set of events on which bears the uncertainty. Moreover, as a consequence of the d-separation criterion in Bayesian networks, we have the following property:
$$\forall \phi \subseteq \Omega, \forall i = 1..n, P'(\eta|\lambda_i, \phi) = P'(\eta|\lambda_i),$$
where η denotes the virtual event.
2. **Computing the revised beliefs:** The revised probability distribution p' is simply equivalent to $p(.|\eta)$ and it is computed as follows (Chan and Darwiche 2005):
$$\forall \phi \subseteq \Omega, P'(\phi) = P(\phi|\eta) = \frac{\sum_{i=1}^{n}(\gamma_i * P(\lambda_i, \phi))}{\sum_{j=1}^{n}(\gamma_j * P(\lambda_j))}. \quad (8)$$

Example 4. *Let us reuse the joint probability distribution of the example of Figure 2. Let also the likelihood ratios be $\gamma_1 = \frac{P'(a_1)}{P(a_1)}$=.57 and $\gamma_2 = \frac{P'(a_2)}{P(a_2)}$=2. The revised distribution p' is computed using Equation 8.*

A	B	$p(AB)$
a_1	b_1	0.6
a_2	b_1	0.15
a_1	b_2	0.1
a_2	b_2	0.15

A	B	$p(AB\|\eta)$
a_1	b_1	0.34
a_2	b_1	0.3
a_1	b_2	0.06
a_2	b_2	0.3

Table 2: Example of initial probability distribution p and the revised distribution $p(.|\eta)$.

From the results of Table 1 and Table 2, it is clear that the revised distributions are equivalent.

Jeffrey's rule and Pearl's methods differ only in the way they specify the inputs and the way the revised beliefs are computed (Chan and Darwiche 2005). In Jeffrey's rule, the inputs are seen as the result or the effect of the revision operation while in the virtual evidence method, the inputs only denote the relative difference between the old beliefs and the revised ones specified in terms of likelihood ratios.
In the following, we compare the two methods presented in this section in a possibilistic framework.

Reasoning with uncertain inputs in the quantitative possibilistic setting

Jeffrey's rule of conditioning in the quantitative possibilistic setting

In the possibilistic setting, given the initial beliefs encoded by a possibility distribution π and a set of inputs in the form of (α_i, λ_i) such that $\Pi'(\lambda_i)$=α_i and α_i∈[0, 1] meaning that after revising π, the new possibility degree of λ_i is α_i. The revised possibility distribution π' according to Jeffrey's rule must satisfy the following conditions:

C1: $\forall \lambda_i$, $\Pi'(\lambda_i)$=α_i.

C2: $\forall \lambda_i \subset \Omega$, $\forall \phi \subseteq \Omega$, $\Pi'(\phi|\lambda_i)$=$\Pi(\phi|\lambda_i)$.

As in the probabilistic setting, revising a possibility distribution π into π' according to the possibilistic counterpart of Jeffrey's rule must fully accept the inputs (condition C1) and preserve the fact that the uncertainty about the events λ_i must not alter the conditional possibility degree of any event $\phi \subseteq \Omega$ given any uncertain event λ_i (condition C2). The revision based on the possibilistic counterpart of Jeffrey's rule in the product-based possibilistic setting is performed as follows (Dubois and Prade 1997):

Definition 1. *Let π be a possibility distribution and (λ_1, α_1),..,(λ_n, α_n) be a set of exhaustive and mutually exclusive events where the uncertainty is of the form $\Pi'(\lambda_i)$=α_i for i=1..n. The revised possibility degree of any arbitrary event $\phi \subseteq \Omega$ is computed as follows (we assume that $\Pi(\phi)>0$):*

$$\forall \phi \subseteq \Omega, \Pi'(\phi) = \max_{\lambda_i}(\alpha_i * \frac{\Pi(\phi, \lambda_i)}{\Pi(\lambda_i)}). \tag{9}$$

It follows from Equation 9 that the revised possibility degree of any interpretation $\omega_j \in \Omega$ is computed as follows:

$$\forall \omega_j \in \lambda_i, \pi'(w_j) = \alpha_i * \frac{\pi(w_j)}{\Pi(\lambda_i)}.$$

It is shown in (Benferhat, Tabia, and Sedki 2011) that the revised possibility distribution π' computed according to Definition 1 always exists and it is unique.

Example 5. *In this example, we assume that we have beliefs over two binary variables A and B. The possibility distribution $\pi(AB)$ encodes the current beliefs. Table 3 gives the distribution π, the marginal distribution of A (namely, $\pi(A)$), the one of B (namely, $\pi(B)$) and the conditional distribution of B given A (namely, $\pi(B|A)$).*

A	B	$\pi(AB)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	0.4

A	$\pi(A)$
a_1	1
a_2	0.4

B	$\pi(B)$
b_1	1
b_2	0.4

A	B	$\pi(B|A)$
a_1	b_1	1
a_2	b_1	1
a_1	b_2	0.1
a_2	b_2	1

Table 3: Example of initial possibility distribution π and the underlying marginal and conditional distributions.

Now assume that we have new beliefs in the form (a_1, .4) and (a_2, 1). The revised distribution using Jeffrey's rule of Equation 9 is given by π' of Table 4.

According to Tables 3 and 4, it is clear that the input beliefs are fully accepted (see the marginal distribution $\pi'(A)$) and that $\forall a_i \in D_A$, $\forall b_j \in D_B$, $\Pi(b_j|a_i)$=$\Pi'(b_j|a_i)$.

Pearl's method of virtual evidence in the quantitative possibilistic setting

In Pearl's virtual evidence method, the new information is a set of likelihood ratios γ_1,..,γ_n and satisfies the following condition:

A	B	$\pi'(AB)$
a_1	b_1	0.4
a_2	b_1	1
a_1	b_2	0.04
a_2	b_2	1

A	$\pi'(A)$
a_1	**0.4**
a_2	**1**

B	$\pi'(B)$
b_1	1
b_2	1

A	B	$\pi'(B|A)$
a_1	b_1	1
a_2	b_1	1
a_1	b_2	0.1
a_2	b_2	1

Table 4: Revised beliefs of the initial distribution given in Table 3 using Jeffrey's rule of Equation 9.

C3: γ_1:..:γ_n= $\Pi(\eta|\lambda_1)$:..:$\Pi(\eta|\lambda_n)$=$\frac{\Pi(\lambda_1|\eta)}{\Pi(\lambda_1)}$:..:$\frac{\Pi(\lambda_n|\eta)}{\Pi(\lambda_n)}$.

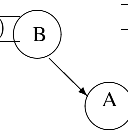
Pearl's virtual evidence method guarantees that the uncertainty bears only on the events λ_1,..,λ_n and does not concern the other events. Formally,

C4: $\forall \phi \subseteq \Omega$, $\Pi(\eta|\lambda_i, \phi)$= $\Pi(\eta|\lambda_i)$.

Pearl's method of virtual evidence applies in a quite straightforward way for quantitative possibilistic networks. Indeed, once the new inputs specified, they are integrated into the network G encoding the current beliefs in the form of a new node Z with a conditional possibility table designed in such a way that conditioning on the node Z, the conditional distribution $\pi_G(.|z)$ provides the revised joint distribution.

Example 6. *Let G be a possibilistic network over two binary variables A and B. The network G encodes the same possibility distribution as the distribution π of Table 3.*

B	$\pi(B)$
b_1	1
b_2	0.4



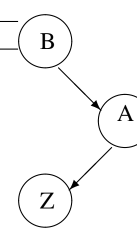
A	B	$\pi(A|B)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.25
a_2	b_2	1

A	B	$\pi(AB)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	0.4

Figure 4: Example of a possibilistic network G and the joint distribution $\pi(AB)$ encoded by G.

Let us assume now that new information says that γ_{a_1}:γ_{a_2}=.4:2.5. One solution satisfying this ratio is γ_{a_1}=.04 and γ_{a_2}=.25. Let us then add a new node Z to integrate γ_{a_1} and γ_{a_2}.

B	$\pi(B)$
b_1	1
b_2	0.4



A	B	$\pi(A|B)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.25
a_2	b_2	1

Z	A	$p(Z|A)$
z	a_1	0.04
z	a_2	0.25

Figure 5: G': The possibilistic network G of Figure 4 augmented with the node Z.

The revised beliefs are given in Table 5.

One can easily check that the revised distribution of Table 5 using Pearl's method of virtual evidence is exactly the same as the distribution π' obtained using Jeffrey's rule given in Table 4. It is also easy to check that conditions C3 and C4 are satisfied.

A	B	$\pi(AB\|z)$
a_1	b_1	0.4
a_2	b_1	1
a_1	b_2	0.04
a_2	b_2	1

Table 5: The conditional possibility distribution $\pi_{G'}(.|z)$ representing the revised distribution of the initial beliefs encoded by the network of Figure 4.

After addressing Pearl's method of virtual evidence in the quantitative possibilistic setting, let us see its generalization.

Virtual evidence method in the quantitative possibilistic setting

Here, the virtual evidence method applies on any possibility distribution exactly as Jeffrey's rule. The revised beliefs are computed according to the following definition.

Definition 2. *Let the initial beliefs be encoded by π and the new inputs be γ_1,..,γ_n. The revised possibility degree $\Pi'(\phi)$ of any event $\phi\subseteq\Omega$ is computed as follows:*

$$\forall\phi \subseteq \Omega, \Pi'(\phi) = \Pi(\phi|\eta) = \frac{\max_{i=1}^{n} \gamma_i * \Pi(\phi, \lambda_i)}{\max_{j=1}^{n} \gamma_j * \Pi(\lambda_j)}. \quad (10)$$

It is straightforward that revising the possibility degree of individual events $\omega_k\in\Omega$ is done as follows:

$$\forall\omega_k \in \lambda_i, \pi'(\omega_k) = \pi(\omega_k|\eta) = \frac{\gamma_i * \pi(\omega_k)}{\max_{j=1}^{n} \gamma_j * \Pi(\lambda_j)}. \quad (11)$$

Example 7. *Let the initial beliefs be encoded by the possibility distribution π of Table 6. Let also the likelihood ratios be γ_1=$\frac{\Pi'(a_1)}{\Pi(a_1)}$=.4 and γ_2=$\frac{\Pi'(a_2)}{\Pi(a_2)}$=2.5 as in the example of Table 5. The revised distribution π' is computed using Equation 10.*

A	B	$\pi(AB)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	0.4

A	B	$\pi(AB\|\eta)$
a_1	b_1	0.4
a_2	b_1	1
a_1	b_2	0.04
a_2	b_2	1

Table 6: Example of initial possibility distribution π and the revised distribution $\pi(.|\eta)$.

The distribution π' computed using Equation 11 always exists and it is unique according the following proposition.

Proposition 1. *Let π be the possibility distribution encoding the initial beliefs. Let also γ_1,..,γ_n be the likelihood ratios corresponding to the new inputs regarding the exhaustive and mutually exclusive set of events λ_1,..,λ_n. Then the revised possibility distribution π' computed using the formula of Equation 10 always exists and it is unique.*

Proof. Let π be the possibility distribution encoding the initial beliefs and let γ_1,..,γ_n be the likelihood ratios regarding the events λ_1,..,λ_n.

1. Let us first show that the revised possibility distribution π' computed using the formula of Equation 10 satisfies the conditions **(C3)** and **(C4)**. Let us start proving that condition **(C3)** is satisfied.
$\Pi(\eta|\lambda_1)$:..: $\Pi(\eta|\lambda_n)$ = $\frac{\Pi(\eta,\lambda_1)}{\Pi(\lambda_1)}$:..: $\frac{\Pi(\eta,\lambda_n)}{\Pi(\lambda_n)}$
$$= \frac{\Pi(\lambda_1|\eta)*\Pi(\eta)}{\Pi(\lambda_n)} :..: \frac{\Pi(\lambda_n|\eta)*\Pi(\eta)}{\Pi(\lambda_n)}$$
$$= \frac{\frac{\gamma_1*\Pi(\lambda_1)}{\max_j(\gamma_j*\Pi(\lambda_j))}*\Pi(\eta)}{\Pi(\lambda_n)} :..: \frac{\frac{\gamma_n*\Pi(\lambda_n)}{\max_j(\gamma_j*\Pi(\lambda_j))}*\Pi(\eta)}{\Pi(\lambda_n)}$$
$$= \gamma_1 * \frac{\Pi(\eta)}{\max_j(\gamma_j*\Pi(\lambda_j))} :..: \gamma_n * \frac{\Pi(\eta)}{\max_j(\gamma_j*\Pi(\lambda_j))}$$
$$= \gamma_1 :..: \gamma_n \ \square$$
Let us now prove that condition **(C4)** is satisfied.
$\forall\phi\subseteq\Omega$, $\Pi(\eta|\lambda_i,\phi)$= $\frac{\Pi(\eta,\lambda_i,\phi)}{\Pi(\lambda_i,\phi)}$
$$=\frac{\Pi(\lambda_i,\phi|\eta)*\Pi(\eta)}{\Pi(\lambda_i,\phi)} = \frac{\frac{\gamma_i*\Pi(\lambda_i,\phi)}{\max_j(\gamma_j*\Pi(\lambda_j))}*\Pi(\eta)}{\Pi(\lambda_i,\phi)}$$
$$=\frac{\gamma_i*\Pi(\eta)}{\max_j(\gamma_j*\Pi(\lambda_j))} = \frac{\gamma_i*\frac{\Pi(\lambda_i,\eta)}{\Pi(\lambda_i|\eta)}}{\max_j(\gamma_j*\Pi(\lambda_j))}$$
$$=\frac{\gamma_i*\frac{\Pi(\lambda_i,\eta)}{\frac{\gamma_i*\Pi(\lambda_i)}{\max_j(\gamma_j*\Pi(\lambda_j))}}}{\max_j(\gamma_j*\Pi(\lambda_j))}$$
$$=\frac{\Pi(\lambda_i,\eta)}{\Pi(\lambda_i)} = \Pi(\eta|\lambda_i) \quad \square$$

2. Now let us provide the proof that if a distribution π' satisfies the conditions **(C3)** and **(C4)** then π' is computed using Equation 10.
$\forall\phi\subseteq\Omega$, $\Pi'(\phi)$= $\Pi(\phi|\eta)$=$\frac{\Pi(\phi,\eta)}{\Pi(\eta)}$
$$=\frac{\max_{i=1}^{n}(\Pi(\phi,\lambda_i,\eta))}{\max_{j=1}^{n}(\Pi(\lambda_j,\eta))}$$
$$=\frac{\max_{i=1}^{n}(\Pi(\eta|\phi,\lambda_i)*\Pi(\phi,\lambda_i))}{\max_{j=1}^{n}(\Pi(\eta|\lambda_j)*\Pi(\lambda_j))}$$
$$=\frac{\max_{i=1}^{n}(\Pi(\eta|\lambda_i)*\Pi(\phi,\lambda_i))}{\max_{j=1}^{n}(\Pi(\eta|\lambda_j)*\Pi(\lambda_j))}$$
$$=\frac{\max_{i=1}^{n}(\gamma_i*\Pi(\phi,\lambda_i))}{\max_{j=1}^{n}(\gamma_j*\Pi(\lambda_j))} \ \square$$

In the following, we provide the transformations from Jeffrey's rule to the virtual evidence method and vice versa.

From Jeffrey's rule to the virtual evidence method in a quantitative possibilistic setting

The following transformations are the possibilistic counterparts of the corresponding ones proposed in the probabilistic framework in (Chan and Darwiche 2005):

Proposition 2. *Let π be a possibility distribution encoding the initial beliefs and let also λ_1,..,λ_n be an exhaustive and mutually exclusive set of events and new information in the form of (α_i, λ_i) such that for i=1..n, $\Pi'(\lambda_i)$=α_i. Let γ_1,..,γ_n be likelihood ratios such that*

$$\gamma_1 : .. : \gamma_n = \frac{\alpha_1}{\Pi(\lambda_1)} : .. : \frac{\alpha_n}{\Pi(\lambda_n)},$$

then the revised possibility distribution π'_J computed using Jeffrey's rule of Equation 9 and the revised possibility distribution π'_P computed using the virtual evidence method of Equation 10 are equivalent. Namely, $\forall\omega\in\Omega$, $\pi'_J(\omega)=\pi'_P(\omega)$.

Proof sketch. The proof is direct. Just set in the virtual evidence method of Equation 10 $\gamma_i=\frac{\alpha_i}{\Pi(\lambda_i)}$ for i=1..n, and the obtained distribution π'_P satisfies conditions C1 and C2 of Jeffrey's rule and since the revised distribution with Jeffrey's rule is unique then π'_P equals π'_J. □

From the virtual evidence method to Jeffrey's rule in a quantitative possibilistic setting

We show now how to obtain the new beliefs α_1,..,α_n needed in Jeffrey's rule from the available set of likelihood ratios:

Proposition 3. *Let π be a possibility distribution encoding the initial beliefs. Let also λ_1,..,λ_n be an exhaustive and mutually exclusive set of events and new information in the form of likelihood ratios γ_1,..,γ_n. For i=1..n, let $\alpha_i=\gamma_i*\Pi(\lambda_i)$. Then the revised possibility distribution π'_J computed using Jeffrey's rule of Equation 9 and the revised possibility distribution π'_P computed using the virtual evidence method of Equation 10 are equivalent. Namely, $\forall\omega\in\Omega$, $\pi'_J(\omega)=\pi'_P(\omega)$.*

Proof sketch. The proof is similar to the proof of Proposition 2. Using Jeffrey's rule of Equation 9 with the inputs (λ_i, α_i) for i=1..n such that $\alpha_i=\gamma_i*\Pi(\lambda_i)$ and the obtained distribution π'_J satisfies conditions C3 and C4 of the virtual evidence method and since the revised distribution is also unique then π'_J equals π'_P. □

Reasoning with uncertain inputs in the qualitative possibilistic setting

Jeffrey's rule in the qualitative possibilistic setting

In the qualitative setting, the revision according to Jeffrey's rule is performed as follows (Dubois and Prade 1997):

Definition 3. *Let π be a possibility distribution and λ_1,..,λ_n be a set of exhaustive and mutually exclusive events. The revised possibility degree of any arbitrary event $\phi\subseteq\Omega$ is computed using the following formula:*

$$\forall\phi \subseteq \Omega, \Pi'(\phi) = \max_{\lambda_i}(\min(\Pi(\phi|\lambda_i), \alpha_i)). \tag{12}$$

It is straightforward that for elementary events ω_j, the revised beliefs are computed according the following formula:

$$\forall w_j \in \lambda_i, \pi'(w_j) = \begin{cases} \alpha_i & \text{if } \pi(w_j)\geq\alpha_i \text{ or } \pi(w_j)=\Pi(\lambda_i); \\ \pi(w_j) & \text{otherwise.} \end{cases}$$

Contrary to the probabilistic and quantitative possibilistic settings, there exist situations where the revision according to Equation 12 does not guarantee the existence of a solution satisfying conditions C1 and C2 (Benferhat, Tabia, and Sedki 2011).

Example 8. *Assume that we have beliefs in the form of a possibility distribution $\pi(AB)$ over two binary variables A and B (we have the same beliefs as in Table 3). In Table 7, we have the joint distribution $\pi(AB)$, the marginal distributions $\pi(A)$ and $\pi(B)$) and the conditional one $\pi(B|A)$.*

A	B	$\pi(AB)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	0.4

A	$\pi(A)$
a_1	1
a_2	0.4

B	$\pi(B)$
b_1	1
b_2	0.4

A	B	$\pi(B|A)$
a_1	b_1	1
a_2	b_1	1
a_1	b_2	0.1
a_2	b_2	1

Table 7: Example of initial possibility distribution π and the underlying marginal and conditional distributions.

Assume now that we want to revise π of Table 7 into π' such $\pi'(a_1)$=.4 and $\pi'(a_2)$=1. The revised distribution using the qualitative counterpart of Jeffrey's rule of Equation 12 is given by π' Table 8.

A	B	$\pi'(AB)$
a_1	b_1	0.4
a_2	b_1	1
a_1	b_2	0.1
a_2	b_2	1

A	$\pi'(A)$
a_1	0.4
a_2	1

B	$\pi'(B)$
b_1	1
b_2	1

A	B	$\pi'(B|A)$
a_1	b_1	1
a_2	b_1	1
a_1	b_2	0.1
a_2	b_2	1

Table 8: Revised beliefs of the initial distribution given in Table 7 using Jeffrey's rule of Equation 12.

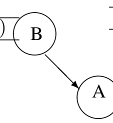
According to the results of Table 7 and 8, it is clear that in this example conditions C1 and C2 are fulfilled.

Pearl's method of virtual evidence in the qualitative possibilistic setting

As in the quantitative setting, the inputs are specified in the same way. Namely, the uncertainty bears on an exhaustive and mutually exclusive set of events λ_1,..,λ_n and the new information is specified as likelihood ratios γ_1:..:γ_n according to condition C3. As shown in the following example, unlike the probabilistic and quantitative possibilistic settings, it is not enough for the parameters γ_1,..,γ_n to satisfy condition C3 to be directly integrated into the conditional possibility table of the new node Z.

Example 9. *Let G be the possibilistic network of Figure 6 and the corresponding joint distribution π_G.*

B	$\pi(B)$
b_1	1
b_2	0.4



A	B	$\pi(A|B)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	1

A	B	$\pi_G(AB)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	0.4

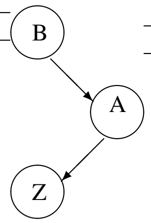
Figure 6: Example of a possibilistic network G and the joint distribution $\pi_G(AB)$ encoded by G in the qualitative setting.

Let us assume now that we want to revise the distribution π_G encoded by the network G of Figure 6 into a new distribution π'_G such that γ_{a_1}=1 and γ_{a_2}=2 meaning that the initial belief degree of a_1 is not changed while the degree of a_2 is to be doubled. The augmented network G' encoding the new inputs is shown in Figure 7.

The revised beliefs are given in Table 9.

B	$\pi(B)$
b_1	1
b_2	0.4

A	B	$\pi(A\|B)$
a_1	b_1	1
a_2	b_1	0.4
a_1	b_2	0.1
a_2	b_2	1



Z	A	$p(Z\|A)$
z	a_1	1
z	a_2	2

Figure 7: The possibilistic network G' obtained by augmenting G of Figure 6 with the node Z.

A	B	$\pi_{G'}(AB\|z)$
a_1	b_1	1
a_2	b_1	.4
a_1	b_2	.1
a_2	b_2	.4

Table 9: The conditional distribution $\pi_{G'}(.|z)$ representing the revised distribution of the initial beliefs of Figure 6.

One can notice from the results of Table 9 that condition C3 is not satisfied since $\frac{\Pi_{G'}(a_2|z)}{\Pi_G(a_2)}=\frac{.4}{.4}\neq\gamma_{a_2}=2$.

Hence, instead of using only the inputs γ_1,..,γ_n, the conditional possibility table of the new node Z must be set for each uncertain event λ_i directly to $\Pi'(\lambda_i)$=γ_i*$\Pi(\lambda_i)$ as in Jeffrey's rule. This is imposed by the min-based operator and the definition of conditioning in the qualitative possibilistic setting (see Equations 3 and 4). Clearly, if a parameter $\pi_{G'}(z|a_i)>1$ then it is not taken into account (note that in the probabilistic and quantitative possibilistic settings, the values of γ_i are not necessarily in the interval [0, 1] and they are always taken into account thanks to the definition of conditioning in these settings). Now, even when replacing γ_i by γ_i*$\Pi(\lambda_i)$ for i=1..n, it is impossible with Pearl's method of virtual evidence to increase the plausibility of an event as stated in the following proposition.

Proposition 4. *Let G be a* min*-based network and let* π_G *be the possibility distribution encoded by G. Let also* γ_1,..,γ_n *be the likelihood ratios corresponding to the new inputs regarding the exhaustive and mutually exclusive set of events* a_1,.., a_n. *Let* G' *be the augmented possibilistic network with the virtual node* Z *to encode* γ_1,.., γ_n *such that for* i=1..n, $\pi_{G'}(Z$=$z|a_i)$=$\Pi_G(a_i)$*γ_i. *Then we have two cases:*

- *If* $\forall i$=1..n, $\gamma_i\leq$*1, then conditions C3 and C4 are satisfied.*
- *Otherwise if* $\exists\gamma_i>$*1 then the revised possibility degree* $\Pi_{G'}(a_i|z)$=$\Pi_G(a_i)$ *implying that C3 is not satisfied while C4 is always satisfied.*

Proposition 4 states that associating with an uncertain event a_i a possibility degree of $\Pi_G(a_i)$*γ_i in the augmented network G', the posterior possibility degree $\Pi_{G'}(a_i|z)$ equals $\Pi_G(a_i)$ (unless $\Pi_{G'}(a_i|z)$ is the greatest one in the context of z in which case $\Pi(a_i|z)$=1 because of normalization). As a consequence of Proposition 4, it is impossible to augment the possibility degree of an event a_i unless $\Pi(a_i)\geq\Pi(a_j)$ for any $j\neq i$ meaning that condition C3 is not satisfied. Indeed, because of the idempotency of the min-based operator used in the min-based chain rule of Equation 4 and the definition of the min-based conditioning of Equation 3, applying directly Pearl's method of virtual evidence does not guarantee that condition C3 will be satisfied. However, condition C4 is always satisfied as it is implied by a graphical property.

Proof sketch. The proof follows from the min-based chain rule and the augmented network G'. Indeed,
$\Pi'_G(a_i|z)$=$\Pi(a_i,z)$=$\max_{A_1..a_i..A_n}(\Pi(A_1..a_i..A_n,z))$
= $\max_{A_1..a_i..A_n}(\min(\pi(A_1|U_1),..,\pi(a_i|U_i),..,\pi(A_n|U_n),\pi(z|a_i))$
$\leq\pi(z|a_i)$=γ_i*$\Pi_G(a_i)$ □
However, one can show that due to the encoding of the inputs by means of augmenting the network, for every event ϕ, $\forall a_i\in D_i$, $\Pi(z|a_i,\phi)$=$\Pi(z|a_i)$ (due to d-separation) meaning that condition C4 is always satisfied since it is a graphical property of the augmented network G'. □

Virtual evidence method in the qualitative possibilistic setting

The min-based counterpart of the quantitative possibilistic virtual evidence method of Definition 2 is defined as follows:

Definition 4. *Let the initial beliefs be encoded by* π *and the new inputs be* γ_1,..,γ_n *specified as likelihood ratios* γ_1:..:γ_n *such that* γ_i=$\Pi(\eta|\lambda_i)$=$\frac{\Pi(\lambda_i|\eta)}{\Pi(\lambda_i)}$. *The revised possibility degree* $\Pi'(\phi)$ *of any event* $\phi\subseteq\omega$ *is computed as follows:*

$$\forall\phi\subseteq\Omega,\Pi(\phi|\eta)=\max_{i=1}^{n}(\min(\Pi(\phi|\lambda_i),\gamma_i*\Pi(\lambda_i)) \qquad (13)$$

For single interpretations $\omega_k\in\Omega$, the revised degrees are computed as follows:

$$\forall\omega_k\in\lambda_i,\pi'(\omega_k)=\min(\Pi(\omega_k|\lambda_i),\gamma_i*\Pi(\lambda_i)) \qquad (14)$$

Example 10. *Let us reuse the beliefs given Table 7 as initial beliefs. Assume now that we want to revise* π *of Table 7 into* π' *such* γ_{a_1}=$\frac{\pi'(a_1)}{\pi(a_1)}$=*.75 and* γ_{a_2}=$\frac{\pi'(a_2)}{\pi(a_2)}$=*2.5. The revised distribution using the qualitative counterpart of the virtual evidence method of Equation 13 is given by* π' *Table 10.*

A	B	$\pi'(AB)$
a_1	b_1	0.75
a_2	b_1	1
a_1	b_2	0.1
a_2	b_2	1

A	$\pi'(A)$
a_1	0.75
a_2	1

γ_{a_1}	.75
γ_{a_2}	2.5

Table 10: Revised beliefs of the initial distribution given in Table 7 using the virtual evidence method of Equation 13.

From the results of Table 7 and 10, one can easily check that the conditions C3 and C4 are satisfied.

The distribution π' computed using Equation 13 always satisfy conditions C3 and C4 as stated in Proposition 5.

Proposition 5. *Let* π *be the possibility distribution encoding the initial beliefs. Let also* γ_1,..,γ_n *be the likelihood ratios corresponding to the new inputs regarding the exhaustive and mutually exclusive set of events* λ_1,..,λ_n. *Then the revised possibility distribution* π' *computed using the formula of Equation 13 always satisfy conditions C3 and C4.*

Proof sketch. Let π be the possibility distribution encoding the initial beliefs and let γ_1,..,γ_n be the likelihood ratios corresponding to the new inputs regarding the set of exhaustive and mutually exclusive set of events λ_1,..,λ_n such that for i=1..n, $\gamma_i=\frac{\Pi'(\lambda_i)}{\Pi(\lambda_i)}$. Let us first show that the revised possibility distribution π' computed using the formula of Equation 13 satisfies condition **(C3)**.

$\Pi(\eta|\lambda_1)$:..: $\Pi(\eta|\lambda_n) = \Pi(\lambda_1,\eta)$:..: $\Pi(\lambda_n|\eta)$

=$\min(\Pi(\lambda_1|\eta),\Pi(\eta))$:..: $\min(\Pi(\lambda_n|\eta),\Pi(\eta))$

=$\min(\max_i(\Pi(\lambda_1|\lambda_i),\gamma_i*\Pi(\lambda_i)))$:..:$\min(\max_i(\Pi(\lambda_n|\lambda_i),\gamma_i*\Pi(\lambda_i)))$

=$\min(\Pi(\lambda_1|\lambda_1),\gamma_1*\Pi(\lambda_1))$:..:$\min(\Pi(\lambda_n|\lambda_n),\gamma_n*\Pi(\lambda_n))$

=$\gamma_1*\Pi(\lambda_1)$:..:$\gamma_n*\Pi(\lambda_n)$. □

The proof that π' satisfies condition C4 is similar to the proof of Proposition 1. Let us now provide the proof that if a distribution π' satisfies the conditions **(C3)** and **(C4)** then π' is computed using Equation 13.

$\forall\phi\subseteq\Omega$, $\Pi'(\phi)=\Pi(\phi|\eta)=\max_{\lambda_i}(\Pi(\phi|\eta,\lambda_i))$

$= \max_{\lambda_i}(\min(\Pi(\phi|\lambda_i),\Pi(\lambda_i|\eta))$

$= \max_{\lambda_i}(\min(\Pi(\phi|\lambda_i),\gamma_i*\Pi(\lambda_i))$ □

Relating the virtual evidence method with Jeffrey's rule in the qualitative possibilistic setting

As in the quantitative setting, it is straightforward to move from Jeffrey's rule to the virtual evidence method and vice versa (because of space limitation, we provide only basic results, the propositions and their proofs are similar to the corresponding transformations in the quantitative setting).

1. *From the virtual evidence method to Jeffrey's rule*: Just set the inputs α_i=γ_i*$\Pi(\lambda_i)$ for i=1..n and use Jeffrey's rule of Equation 12 will give exactly the same revised distribution π' as the distribution $\pi(.|\eta)$ obtained by the min-based possibilistic counterpart of the virtual evidence method of Equation 13.
2. *From Jeffrey's rule to the virtual evidence method* : Here, it is enough to set the inputs γ_i=$\alpha_i/\Pi(\lambda_i)$ for i=1..n then use the virtual evidence method of Equation 13 to obtain $\pi(.|\eta)$ which is equivalent to π' obtained using Jeffrey's rule of Equation 12.

Discussions and concluding remarks

In order to revise the beliefs encoded by means of a possibility distribution one can either use Jeffrey's rule or the virtual evidence method which are shown equivalent in both the quantitative and qualitative settings. However, revising a whole distribution is very costly while Pearl's method of virtual evidence allows to integrate the inputs and compute any possibility degree of interest directly from the network without revising the whole distribution. Moreover, the existing inference algorithms in graphical models (e.g. Junction tree) can be used directly to compute the revised beliefs.

This paper addressed reasoning with uncertain inputs in possibilistic networks. We provided possibilistic counterparts for Pearl's methods and compared them with the well-known Jeffrey's rule of conditioning. In Jeffrey's rule, the inputs (α_i, λ_i) are seen as constraints that should be satisfied leading to fully accepting the new beliefs. The way Jeffrey's method revises the old distribution π in order to fully accept the inputs (α_i, λ_i) complies with the probability kinematics principle (see condition C2) aiming to minimize belief change. In spite of the fact that Pearl's methods specify the inputs differently, the way the new inputs are graphically taken into account (see condition C4) complies with the probability kinematics principle hence minimizing also belief change. Regarding accepting the inputs, it is clear that even specified differently, the inputs to both methods are fully accepted (see conditions C1 and C3).

Regarding iterative revisions, it is well-known that Jeffrey's rule is no commutative (since the new inputs are fully accepted, then revising first with (λ_i, α_i) then with (λ_i, α'_i) will be different from first revising with (λ_i, α'_i) then with (λ_i, α_i)). However in the virtual evidence method, due to the commutativity of multiplication and the definition of the quantitative counterpart of this revision rule, it is easy to show that revising with a set of likelihood ratios γ_1,..,γ_n then revising the resulted beliefs with other new inputs γ'_1,..,γ'_n will give exactly the same results as revising first with γ'_1,..,γ'_n followed by revision with γ_1,..,γ_n. Revision using the qualitative virtual evidence method is not commutative because the inputs are no more likelihood ratios γ_1,..,γ_n but the new beliefs which are fully accepted as in Jeffrey's rule.

To sum up, the contributions of the paper are:

1. Providing counterparts to Pearl's method of virtual evidence and its generalization in the quantitative and qualitative settings. We showed that contrary to the probabilistic and quantitative possibilistic settings, the inputs for the qualitative counterparts of Pearl's methods should be possibility degrees satisfying condition C3. This is due to the fact that in possibilistic networks, we deal with local tables combined with the min-based operator which is idempotent and because of the definition of the qualitative conditioning. We showed also that it is impossible to enhance the possibility degree of an event using Pearl's method of virtual evidence in qualitative networks.
2. Analyzing the existence and uniqueness of the solutions using the proposed possibilistic counterparts of Pearl's methods. In the quantitative setting, the paper showed that the solution always exists and it is unique. In the min-based setting however, depending on the inputs, the solution is not guaranteed to satisfy conditions C3 using Pearl's method of virtual evidence while using the virtual evidence method the solution always exists and it is unique and satisfies conditions C3 and C4.
3. Providing transformations from Jeffrey's rule to the virtual evidence method and comparisons of these methods in both the quantitative and qualitative settings. We provided precise conditions where the methods are equivalent. Finally, we tried to relate the criteria underlying Jeffrey's rule and Pearl's methods and highlighted many related issues like iterated revisions in these formalisms.

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