Concept of luminosity

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Abstract
The performance of particle colliders is usually quantified by the beam energy and the luminosity. We derive the expressions for the luminosity in case of bunched beams in terms of the beam parameters and the geometry. The implications of additional features such as crossing angle, offsets and hourglass effect on the luminosity are calculated. Important operational aspects like integrated luminosity, space and time structure of interactions etc. are discussed. The measurement of luminosity for e^+e^- as well as hadron colliders and the methods for the calibration of the absolute luminosity are described.

1 Introduction

In particle physics experiments the energy available for the production of new effects is the most important parameter. The required large centre of mass energy can only be provided with colliding beams where little or no energy is lost in the motion of the centre of mass system (cms). Besides the energy the number of useful interactions (events), is important. This is especially true when rare events with a small production cross section σ_p are studied. The quantity that measures the ability of a particle accelerator to produce the required number of interactions is called the luminosity and is the proportionality factor between the number of events per second dR/dt and the cross section σ_p:

$$\frac{dR}{dt} = \mathcal{L} \cdot \sigma_p . \tag{1}$$

The unit of the luminosity is therefore $\mathrm{cm}^{-2}\mathrm{s}^{-1}$.

In this lecture we shall first give the main arguments which lead to a general expression for the luminosity and derive the formula for basic cases. Additional complications such as crossing angle and offset collisions are added to the calculation. Special effects such as the hourglass effect and the consequences of different beam profiles are estimated from the generalized expression.

Besides the absolute value of the luminosity, other issues are important for physics experiments, such as the integrated luminosity and the space and time structure of the resulting interactions.

In the final section we shall discuss the measurement of luminosity in both, $e^+ e^-$ as well as hadron colliders.

2 Why colliding beams ?

The kinematics of a particle with mass m can be expressed by its momentum $\vec{p}$ and energy E which form a four-vector $\mathbf{p} = (E, \vec{p})$ whose square $\mathbf{p}^2$ is (with the appropriate norm):

$$\mathbf{p}^2 = E^2 - \vec{p}^{\,2} = m^2 . \tag{2}$$

In the collision of two particles of masses m_1 and m_2 the total centre of mass energy can be expressed in the form

$$(\mathbf{p}_1 + \mathbf{p}_2)^2 = E_{cm}^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 . \tag{3}$$

This is the available energy for physics experiments.

In the case of a collider where the collision point is at rest in the laboratory frame (i.e. $\vec{p_1} = -\vec{p_2}$), the centre of mass energy becomes:

$$E_{\text{cm}}^2 = (E_1 + E_2)^2 \, . \tag{4}$$

When one particle is at rest, i.e. in the case of so-called fixed target experiments, (i.e. $\vec{p_2} = 0$), we get:

$$E_{\text{cm}}^2 = (m_1^2 + m_2^2 + 2m_2 E_{1,\text{lab}}) \, . \tag{5}$$

A comparison for different types of collisions is made in Table 1. From this table it is rather obvious

Table 1: Centre of mass energy for different types of collisions.

	E_{cm} as collider (GeV)	E_{cm} with fixed target (GeV)
p on p (7000 on 7000 GeV)	14000	114.6
e on e (100 on 100 GeV)	200	0.32
e on p (30 on 920 GeV)	235	7.5

why colliding beams are necessary to get the high centre of mass energies required for particle physics experiments.

3 Computation of luminosity

3.1 Fixed target luminosity

In order to compute a luminosity for fixed target experiment, we have to take into account the properties of both, the incoming beam and the stationary target. The basic configuration is shown in Fig.1 The

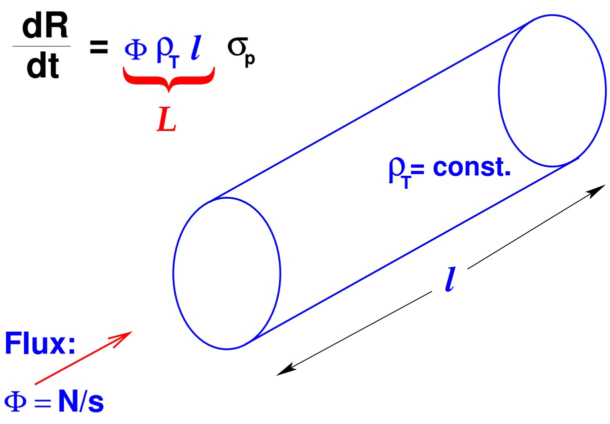

Fig. 1: Schematic view of a fixed target collision.

incoming beam is characterized by the flux Φ, i.e. the number of particles per second. When the target is homogeneous and larger than the incoming beam, the distribution of the latter is not important for the luminosity.

The target is described by its density ρ_{T} and its length l. With a definition of the luminosity like:

$$\mathcal{L}_{FT} = \Phi \rho_{\text{T}} l \tag{6}$$

we write the interaction rate

$$\frac{dR}{dt} = \Phi \rho_{\mathrm{T}} l \cdot \sigma_p = \mathcal{L}_{\mathrm{FT}} \cdot \sigma_p \tag{7}$$

as desired.

3.2 Colliding beams luminosity

In the case of two colliding beams, both beams serve as target and "incoming" beam at the same time. Obviously the beam density distribution is now very important and the generalization of the above expression leads to the convolution of the 3-D distribution functions. We treat the case of bunched beams,

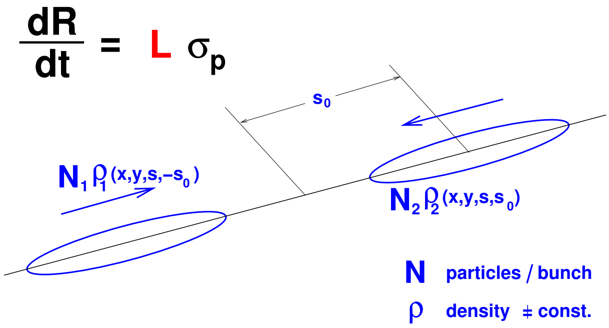

Fig. 2: Schematic view of a colliding beam interaction.

but it can easily be extended to unbunched beams or other concepts like very long bunches. A schematic picture is shown in Fig.2. Since the two beams are not stationary but moving through each other, the overlap integral depends on the longitudinal position of the bunches and therefore on the time as they move towards and through each other. For our integration we use the distance of the two beams to the central collision point $s_0 = c \cdot t$ as the "time" variable (see Fig. 2). A priori the two beams have different distribution functions and different number of particles in the beams.

The overlap integral which is proportional to the luminosity ($\mathcal{L}$) we can then write as: ¡

$$\mathcal{L} \propto K \cdot \iiint_{-\infty}^{+\infty} \rho_1(x, y, s, -s_0) \rho_2(x, y, s, s_0) dx dy ds ds_0 \; . \tag{8}$$

Here $\rho_1(x, y, s, s_0)$ and $\rho_2(x, y, s, s_0)$ are the time dependent beam density distribution functions. We assume, that the two bunches meet at $s_0 = 0$. Because the beams are moving against each other, we have to multiply this expression with a kinematic factor [1]:

$$K = \sqrt{(\vec{v_1} - \vec{v_2})^2 - (\vec{v_1} \times \vec{v_2})^2 / c^2} \; . \tag{9}$$

In the next step we assume head-on collisions ($\vec{v_1} = -\vec{v_2}$) and that all densities are uncorrelated in all planes. In that case we can factorize the density distributions and get for the overlap integral:

$$\mathcal{L} = 2 N_1 N_2 f N_b \iiiint_{-\infty}^{+\infty} \rho_{1x}(x) \rho_{1y}(y) \rho_{1s}(s - s_0) \rho_{2x}(x) \rho_{2y}(y) \rho_{2s}(s + s_0) \;\; dx dy ds ds_0 \; . \tag{10}$$

We have completed the formula with the beam properties necessary to calculate the value of the luminosity: N_1 and N_2 are the intensities of two colliding bunches, f is the revolution frequency and N_b is the number of bunches in one beam.

To evaluate this integral one should know all distributions. An analytical calculation is not always possible and a numerical integration may be required. However in many cases the beams follow "reasonable" profiles and we can obtain closed solutions.

4 Luminosity of Gaussian beams colliding head-on

Often it is fully justified to assume Gaussian distributions. The luminosity is determined by the overlap of the core of the distributions and the tails give practically no contribution to the luminosity. We shall come back to this point in a later section.

For the first calculation we assume Gaussian profiles in all dimensions of the form:

$$\rho_{iz}(z) = \frac{1}{\sigma_z\sqrt{2\pi}}\exp\left(-\frac{z^2}{2\sigma_z^2}\right) \text{ where } i = 1,2, \quad z = x, y \;, \tag{11}$$

$$\rho_s(s \pm s_0) = \frac{1}{\sigma_s\sqrt{2\pi}}\exp\left(-\frac{(s\pm s_0)^2}{2\sigma_s^2}\right) . \tag{12}$$

Furthermore we assume equal beams, i.e.: $\sigma_{1x} = \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \sigma_{1s} = \sigma_{2s}$

Next we assume the number of particles per bunch N_1 and N_2, a revolution frequency of f and the number of bunches we call N_b. In the case of exactly head-on collisions of bunches travelling almost at the speed of light, the kinematic factor becomes 2.

Using this in equation (10) we get the first integral:

$$\mathcal{L} = \frac{2\cdot N_1 N_2 f N_b}{(\sqrt{2\pi})^6\sigma_s^2\sigma_x^2\sigma_y^2}\iiiint e^{-\frac{x^2}{\sigma_x^2}} e^{-\frac{y^2}{\sigma_y^2}} e^{-\frac{s^2}{\sigma_s^2}} e^{-\frac{s_0^2}{\sigma_s^2}}\, dxdydsds_0 \tag{13}$$

integrating over s and s_0, using the well known formula:

$$\int_{-\infty}^{+\infty} e^{-at^2}dt \quad = \quad \sqrt{\pi/a} \tag{14}$$

we get a first intermediate result:

$$\mathcal{L} = \frac{2\cdot N_1 N_2 f N_b}{8(\sqrt{\pi})^4\sigma_x^2\sigma_y^2}\iint e^{-\frac{x^2}{\sigma_x^2}} e^{-\frac{y^2}{\sigma_y^2}}\, dxdy \;. \tag{15}$$

Finally, after integration over x and y:

$$\Longrightarrow \quad \mathcal{L} = \frac{N_1 N_2 f N_b}{4\pi\sigma_x\sigma_y} \;. \tag{16}$$

This is the well-known expression for the luminosity of two Gaussian beams colliding head-on. It shows how the luminosity depends on the number of particles per bunch and the beam sizes. This reflects the 2-dimensional target charge density we have seen in the evaluation of the fixed target luminosity.

For the more general case of: $\sigma_{1x} \neq \sigma_{2x}, \sigma_{1y} \neq \sigma_{2y}$, but still assuming approximately equal bunch lengths $\sigma_{1s} \approx \sigma_{2s}$ we get a modified formula:

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{2\pi\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}\sqrt{\sigma_{2y}^2+\sigma_{2y}^2}} \;. \tag{17}$$

This formula is easy to verify and also straightforward to extend to other cases. Here it is worth to mention that the luminosity does not depend on the bunch length σ_s. This is due to the assumption of uncorrelated density distributions.

5 Examples

In Table 2 we give some examples of different colliders and their luminosity and other relevant parameters. One may notice the very different interaction rate, in particular between hadron colliders and high energy lepton colliders. This is due to the small total cross section of e^+e^- interactions. Furthermore, since so-called B-factories such as PEP and KEKB operate near or on resonances, the interaction rate varies very strongly with the precise energy. Therefore we write the term NA in the table.

Table 2: Example of different colliders. We show the energy, luminosity, beam sizes and interaction rate for a comparison.

	Energy (GeV)	$\mathcal{L}$ $\mathrm{cm}^{-2}\mathrm{s}^{-1}$	rate s^{-1}	$\sigma x/\sigma_y$ μm/μm	Particles per bunch
SPS (p$\bar{p}$)	315x315	6 10^{30}	4 10^5	60/30	$\approx$ 10 10^{10}
Tevatron (p$\bar{p}$)	1000x1000	50 10^{30}	4 10^6	30/30	$\approx$ 30/8 10^{10}
HERA (e^+p)	30x920	40 10^{30}	40	250/50	$\approx$ 3/7 10^{10}
LHC (pp)	7000x7000	10000 10^{30}	10^9	17/17	11 10^{10}
LEP (e^+e^-)	105x105	100 10^{30}	≤ 1	200/2	$\approx$ 5 10^{11}
PEP (e^+e^-)	9x3	3000 10^{30}	NA	150/5	$\approx$ 2/6 10^{10}
KEKB (e^+e^-)	8x3.5	10000 10^{30}	NA	77/2	$\approx$ 1.3/1.6 10^{10}

6 Additional complications in real machines

So far we have assumed ideal head-on collisions of bunches where the particle densities in the three dimensions are uncorrelated. In practice, we have to include additional effects in our computations, some of the most important are:

- Crossing angle
- Collision offset (wanted or unwanted)
- Hour glass effect
- Non-Gaussian beam profiles
- Non-zero dispersion at collision point
- $\delta\beta^*/\delta s \; = \; \alpha^* \; \neq \; 0$

Crossing angles are often used to avoid unwanted collisions in machines with many bunches (e.g. LHC, CESR, KEKB). Such crossing angles can have important consequences for beam-beam effects [2] but also affect the luminosity. When beams do not collide exactly head-on but with a small offset, the luminosity is reduced. Such an offset can be wanted (e.g. to reduce luminosity or during measurements) or unwanted, for example as a result of beam-beam effects [2]. The so-called hourglass effect is a geometrical effect which includes a dependence of the transverse beam sizes on the longitudinal position and therefore violates our previous assumption of uncorrelated particles densities. When the beam profiles deviate from a Gaussian function, we may have to apply some correction factors and when the dispersion at the interaction point is not zero, the effective beam sizes are increased, leading to a smaller luminosity. In case of optical imperfections the collision point may not be at the minimum of the betatron function β^*, i.e. at the waist, but slightly displaced with implications for the effective beam sizes.

Some of the most important of these additional effects we shall investigate in the following sections.

6.1 Crossing angles

A very prominent collider with a crossing angle is the LHC presently under construction at CERN. In the LHC one has almost 3000 closely spaced bunches and to avoid numerous unwanted interactions, the two beams collide at a total crossing angle of around $\approx$ 300 μrad. The Fig.3 shows a schematic illustration of the collision region. Colliders with unbunched, i.e. coasting beams such as the ISR need a sizeable crossing angle to confine the interaction region (e.g. $\approx 18^o$ at the ISR).

In the following we shall assume without loss of generality that the crossing angle is in the horizontal plane. The overlap integrals are evaluated in the x and y coordinate system and therefore we have

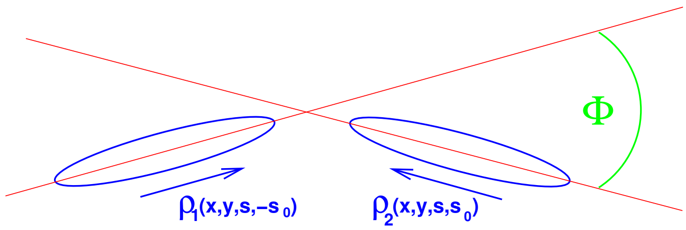

Fig. 3: Schematic view of two bunches colliding at a finite crossing angle.

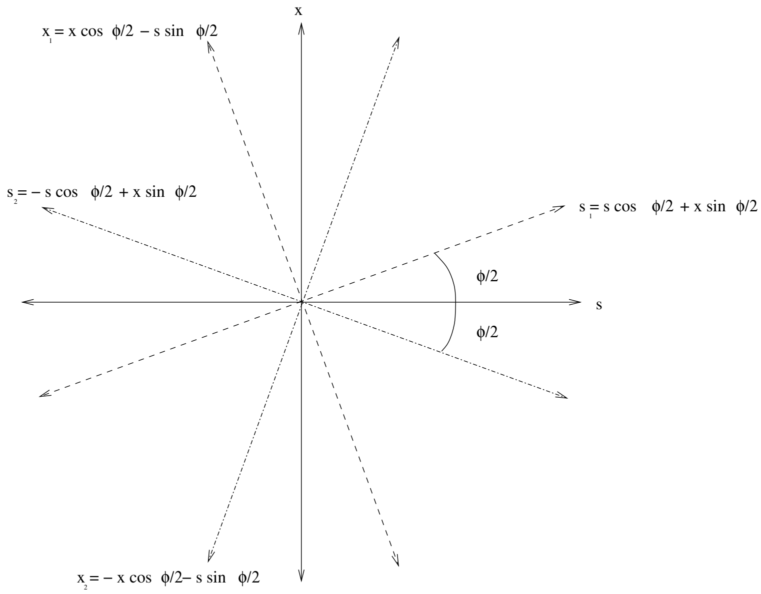

Fig. 4: Rotated reference system for collisions at a finite crossing angle.

to transform our bunches into the proper system. The geometry of a collision at a crossing angle Φ is shown in Fig.4. To make the treatment more symmetric, we have assumed that the total crossing angle is made up by two rotations $\Phi/2$ and $-\Phi/2$ each of the two beams in the x-s plane (see Fig.4).

To compute the integral we have to transform x and s to new, rotated coordinates which are now different for the two beams:

$$x_1 = x\cos\frac{\phi}{2} - s\sin\frac{\phi}{2}, \qquad s_1 = s\cos\frac{\phi}{2} + x\sin\frac{\phi}{2} \tag{18}$$

$$x_2 = x\cos\frac{\phi}{2} + s\sin\frac{\phi}{2}, \qquad s_2 = s\cos\frac{\phi}{2} - x\sin\frac{\phi}{2} \tag{19}$$

The overlap integral becomes:

$$\mathcal{L} = 2\cos^2\frac{\phi}{2}N_1N_2fN_b\iiiint_{-\infty}^{+\infty}\rho_{1x}(x_1)\rho_{1y}(y_1)\rho_{1s}(s_1-s_0)$$
$$\rho_{2x}(x_2)\rho_{2y}(y_2)\rho_{2s}(s_2+s_0)dxdydsds_0\ . \tag{20}$$

The factor $2\cos^2\frac{\phi}{2}$ is the kinematic factor when the two velocities of the bunches are not collinear (from Eq. (9)).

After the integration over y and s_0, using the formula:

$$\int_{-\infty}^{+\infty} e^{-(at^2+bt+c)}dt \quad = \quad \sqrt{\pi/a} \cdot e^{\frac{b^2-ac}{a}} \tag{21}$$

we get:

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{8\pi^2 \sigma_s \sigma_x^2 \sigma_y} 2\cos^2\frac{\phi}{2} \iint e^{-\frac{x^2\cos^2(\phi/2)+s^2\sin^2(\phi/2)}{\sigma_x^2}} e^{-\frac{x^2\sin^2(\phi/2)+s^2\cos^2(\phi/2)}{\sigma_s^2}} dxds\ . \tag{22}$$

We make the following approximations: since both x and sin($\phi/2$) are small, we drop all terms of the type $\sigma_x^k sin^l(\phi/2)$ or $x^k sin^l(\phi/2)$ for all k+l $\geq$ 4 and approximate sin($\phi/2$) $\approx$ tan($\phi/2$) by $\phi/2$. After the final integrations we get for the luminosity an expression of the form:

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{4\pi \sigma_x \sigma_y} \cdot S\ . \tag{23}$$

This looks exactly like the well known formula we have derived already, except for the additional factor S, the so-called luminosity reduction factor which can be written as:

$$\frac{1}{\sqrt{1+(\frac{\sigma_x}{\sigma_s}\tan\frac{\phi}{2})^2}}\frac{1}{\sqrt{1+(\frac{\sigma_s}{\sigma_x}\tan\frac{\phi}{2})^2}}\ . \tag{24}$$

For small crossing angles and $\sigma_s \gg \sigma_{x,y}$ we can simplify the formula to:

$$S = \frac{1}{\sqrt{1+(\frac{\sigma_s}{\sigma_x}\tan\frac{\phi}{2})^2}} \approx \frac{1}{\sqrt{1+(\frac{\sigma_s}{\sigma_x}\frac{\phi}{2})^2}}\ . \tag{25}$$

A popular interpretation of this result is to consider it a correction to the beam size and to introduce an "effective beam size" like:

$$\sigma_{eff} \ = \ \sigma \ \cdot \ \sqrt{1+(\frac{\sigma_s}{\sigma_x}\frac{\phi}{2})^2}\ . \tag{26}$$

The effective beam size can then be used in the standard formula for the beam size in the crossing plane. This concept of an effective beam size is interesting because it also applies to the calculation of beam-beam effects of bunched beams with a crossing angle [5].

As an example we use the parameters of the LHC. The number of particles per bunch is 1.15 10^{11}, the beam sizes in the two planes $\approx$ 16.7 μm, the bunch length σ_s = 7.7 cm and the total crossing angle $\Phi\ =\ 285\ \mu$rad. With the revolution frequency of 11.245 kHz and 2808 bunches, we get for the head-on luminosity 1.2 10^{34} cm^{-2}s^{-1}. For the luminosity reduction factor S we get 0.835 and the final LHC luminosity with a crossing angle becomes $\approx$ 1.0 10^{34} cm^{-2}s^{-1}.

6.2 Crossing angles and offset beams

A modification of the previous scheme is needed when the beams do not collide head-on, but with a small transverse offset. In order to be general, we shall treat the case with crossing angle and offsets. The Fig. 5 shows the modified geometry when we have the same crossing angle as before, but beam 1 is displaced by d_1 and beam 2 is displaced by d_2 with respect to their reference orbits.

The coordinate transformations are now:

$$x_1 = d_1 + x\cos\frac{\phi}{2} - s\sin\frac{\phi}{2}\ , \qquad s_1 = s\cos\frac{\phi}{2} + x\sin\frac{\phi}{2}\ , \tag{27}$$

$$x_2 = d_2 + x\cos\frac{\phi}{2} + s\sin\frac{\phi}{2}\ , \qquad s_2 = s\cos\frac{\phi}{2} - x\sin\frac{\phi}{2}\ . \tag{28}$$

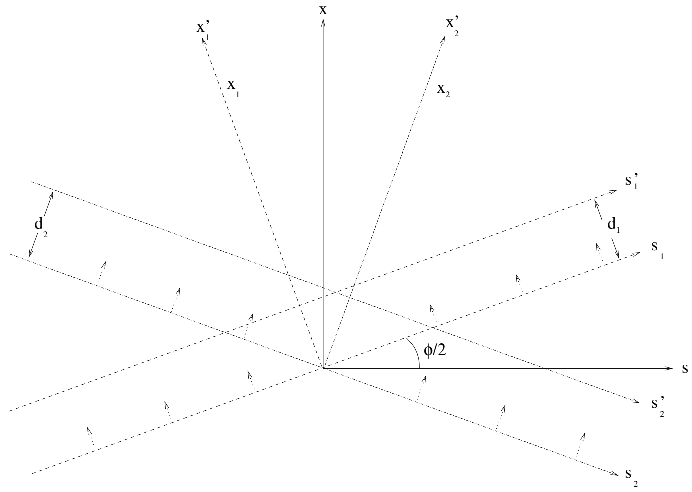

Fig. 5: Schematic view of two bunches colliding at a finite crossing angle and an offset between the two beams.

Following the previous strategy and approximations for the integration, we get after integrating y and s_0:

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{8\pi^2 \sigma_s \sigma_x^2 \sigma_y} 2\cos^2\frac{\phi}{2} \iint e^{-\frac{x^2\cos^2(\phi/2)+s^2\sin^2(\phi/2)}{\sigma_x^2}} e^{-\frac{x^2\sin^2(\phi/2)+s^2\cos^2(\phi/2)}{\sigma_s^2}} \times e^{-\frac{d_1^2+d_2^2+2(d_1+d_2)x\cos(\phi/2)-2(d_2-d_1)s\sin(\phi/2)}{2\sigma_x^2}} dxds \; . \tag{29}$$

After the integration over x we obtain:

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{8\pi^{\frac{3}{2}} \sigma_s} 2\cos\frac{\phi}{2} \int_{-\infty}^{+\infty} W \frac{e^{-(As^2+2Bs)}}{\sigma_x \sigma_y} ds \tag{30}$$

with:

$$A = \frac{\sin^2\frac{\phi}{2}}{\sigma_x^2} + \frac{\cos^2\frac{\phi}{2}}{\sigma_s^2}, \quad B = \frac{(d_2-d_1)\sin(\phi/2)}{2\sigma_x^2} \tag{31}$$

and

$$W = e^{-\frac{1}{4\sigma_x^2}(d_2-d_1)^2} \; . \tag{32}$$

We can re-write the luminosity with three correction factors:

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{4\pi \sigma_x \sigma_y} \cdot W \cdot e^{\frac{B^2}{A}} \cdot S \; . \tag{33}$$

This factorization enlightens the different contributions and allows straightforward calculations. The last factor S is the already calculated luminosity reduction factor for a crossing angle. One factor W reduces the luminosity in the presence of beam offsets and the factor $e^{\frac{B^2}{A}}$ is only present when we have a crossing angle and offsets simultaneously.

6.3 Hourglass effect

So far we have assumed uncorrelated beam density functions in the transverse and longitudinal planes. In particular, we have assumed that the transverse beam sizes are constant over the whole collision regions. However, since the β-functions have their minima at the collision point and increase with the distance this is not always a good approximation. In a low-β region the β-function varies with the distance s to

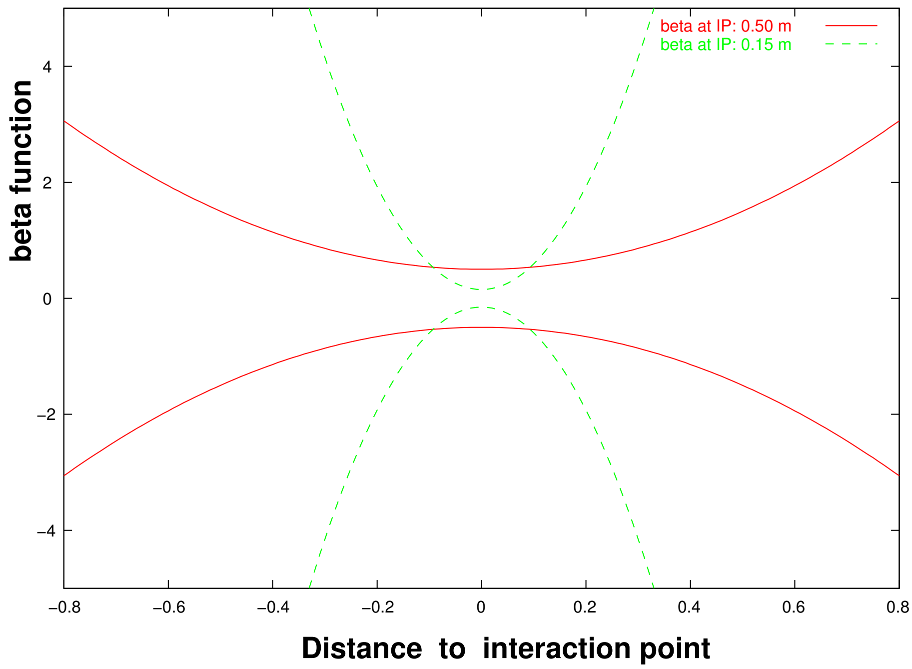

Fig. 6: Schematic illustration of the hourglass effect. β(s) is plotted for two different values of β^*.

the minimum like:

$$\beta(s) = \beta^*(1 + \left(\frac{s}{\beta^*}\right)^2) \tag{34}$$

and therefore the beam size $\sigma = \sqrt{\beta(s) \cdot \epsilon}$ increases approximately linearly with the distance to the interaction point. This is schematically shown in Fig. 6 where the functions β(s) are shown for two different values of β^* (0.50 m and 0.15 m). Because of the shape of the β(s) function this effect is called the hourglass effect. It is especially important when the β(s) function at the interaction point approaches the bunch length σ_s (Fig. 6) and not all particles collide at the minimum of the transverse beam size, therefore reducing the luminosity. Other effects such as a coupling between the transverse and longitudinal planes are ignored in this discussion.

In our formulae we have to replace σ by σ(s) and get a more general expression for the luminosity:

$$\mathcal{L} = \left(\frac{N_1 N_2 f N_b}{8\pi\sigma_x^*\sigma_y^*}\right) \frac{2\cos\frac{\phi}{2}}{\sqrt{\pi}\sigma_s} \int_{-\infty}^{+\infty} \frac{e^{-s^2 A}}{1 + (\frac{s}{\beta^*})^2} ds \tag{35}$$

with

$$A = \frac{\sin^2\frac{\phi}{2}}{(\sigma_x^*)^2[1 + (\frac{s}{\beta^*})^2]} + \frac{\cos^2\frac{\phi}{2}}{\sigma_s^2} \,. \tag{36}$$

Usually it is difficult to compute this integral analytically and it has to be evaluated by numerical integration.

To estimate the importance and relevance of this effect, we shall use the parameters of the LHC, i.e., $N_1 = N_2 = 1.15 \times 10^{11}$ particles/bunch, 2808 bunches per beam, a revolution frequency of f = 11.2455 kHz, and a crossing angle of $\phi = 285$ μrad. The nominal β-functions at the interaction point are $\beta_x^* = \beta_y^* = 0.55$ m, leading to beam sizes of $\sigma_x^* = \sigma_y^* = 16.7$ μm, and we use a r.m.s. bunch length of σ_s = 7.7 cm.

In the simplest case of a head on collision we get for the luminosity $\mathcal{L} = 1.200 \times 10^{34}$ cm^{-2}s^{-1}.

The effect of the crossing angle we can estimate by the evaluation of the factor S and get: $\mathcal{L} = 1.000 \times 10^{34}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$.

When we further include the hourglass effect we get: $\mathcal{L} = 0.993 \times 10^{34}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$.

While the effect of the crossing angle is very sizeable (S = 0.835), the further reduction by the hourglass effect is small, at least for the nominal LHC parameters. For smaller β-functions at the interaction point this may not be the case.

6.4 Sensitivity to beam profiles

So far we have assumed Gaussian distribution functions in all dimensions. However this is not always the case, in particular for hadron beams, and it is necessary to evaluate the validity of this assumption, i.e., the importance for the derived results. Since it is mainly the core of the beam distribution which contributes to the luminosity, we can hope that the r.m.s. as a measure of the beam size (and therefore implying a Gaussian profile) is a good approximation.

To make a quantitative study, we can compute the luminosity for a flat beam using the complete overlap integral and compare to the simplified calculation, i.e. compute the r.m.s. and use them in the standard formula.

We assume flat distributions of the form:

$$\rho_1(x,y) = \rho_2(x,y) = \frac{1}{2a}, \quad for[-a \le z \le a], \quad z = x, y \tag{37}$$

and calculate the r.m.s. in x and y:

$$< (x,y)^2 > = \int_{-\infty}^{+\infty} (x,y)^2 \cdot \rho(x,y)\ dxdy\ . \tag{38}$$

We compute the correct luminosity (without the constants which are equal for all cases) from the integral:

$$\mathcal{L} = \int_{-\infty}^{+\infty} \rho_1(x,y)\rho_2(x,y)\ dxdy\ . \tag{39}$$

The quantity $\mathcal{L}\sqrt{< x^2 >< y^2 >}$ gives a measure for the quality of the approximation when it is compared to the same expression for a Gaussian beam. The astonishing result is that the error one makes with this approximation is only a few % [3]. The same result holds for other "reasonable" distributions such as parabolic of cosine-like [3].

7 Other luminosity issues

There are further issues related to the luminosity which are important for the experiments, such as:

- Integrated luminosity
- Time structure of interactions
- Space structure of interactions

The geometry of the interaction regions as well as some basic parameters entering the standard luminosity formulae are very important for the above issues and may need reconsideration to trade off between the different requirements.

7.1 Integrated luminosity

7.1.1 Definition of integrated luminosity

The maximum luminosity, and therefore the instantaneous number of interactions per second, is very important, but the final figure of merit is the so-called integrated luminosity:

$$\mathcal{L}_{\text{int}} = \int_0^T \mathcal{L}(t')dt' \tag{40}$$

because it directly relates to the number of observed events:

$$\mathcal{L}_{\text{int}} \cdot \sigma_p = \text{number of events of interest} . \tag{41}$$

The integral is taken over the sensitive time, i.e., excluding possible dead time. For an evaluation one needs a realistic model for the decay of the luminosity with time. Different possibilities exist and usually one assumes some behaviour (e.g., exponential) with a given lifetime τ:

$$\mathcal{L}(t) \longrightarrow \mathcal{L}_0 \exp\left(-\frac{t}{\tau}\right) . \tag{42}$$

Contributions to this life time we have from the decay of beam intensity with time, the growth of the transverse emittance, increase of the bunch length etc. The advantage of assuming an exponential decay is that the contributions from different processes can be easily added. The differences between the different models is very small in practice.

7.1.2 Optimization of integrated luminosity

The aim of the operation of a collider must be to optimize the integrated luminosity. Two parts of the operation must be distinguished: the luminosity run with a lifetime τ and the preparation time between two luminosity runs t_p. The optimization problem is very similar to the challenge of a formula 1 racing team: the length of running with decreasing performance (slowing down with ageing tires) and the time needed to restore the performance (changing tires). The best strategy should minimize the overall time needed.

In principle, the knowledge of the preparation time allows an optimization of $\mathcal{L}_{\text{int}}$.

If we assume an exponential decay of the luminosity $\mathcal{L}(t) = \mathcal{L}_0 \cdot e^{t/\tau}$ we want to maximize the average luminosity $< \mathcal{L} >$:

$$< \mathcal{L} > = \frac{\int_0^{t_r} \mathcal{L}(t)dt}{t_r + t_p} = \mathcal{L}_0 \cdot \tau \cdot \frac{1 - e^{-t_r/\tau}}{t_r + t_p} . \tag{43}$$

Here t_r is the length of a luminosity run and t_p the preparation time between two runs. Since t_r is a "free" parameter, i.e. can be chosen by the operation crew, we can optimize this expression and get a (theoretical) maximum for:

$$t_r \approx \tau \cdot \ln(1 + \sqrt{2t_p/\tau} + t_p/\tau) . \tag{44}$$

Assuming some parameters for the LHC [4]: $t_p \approx$ 10h, $\tau \approx$ 15h, we get: $\Rightarrow t_r \approx$ 15h.

7.2 Luminous region and space structure of luminosity

In addition to the number of events, the space structure is important for the design and running of a particle physics experiment. The questions we asked are therefore:

- What is the density distribution of interaction vertices ?
- Which fraction of collisions occur $\pm$ s from the interaction point ?

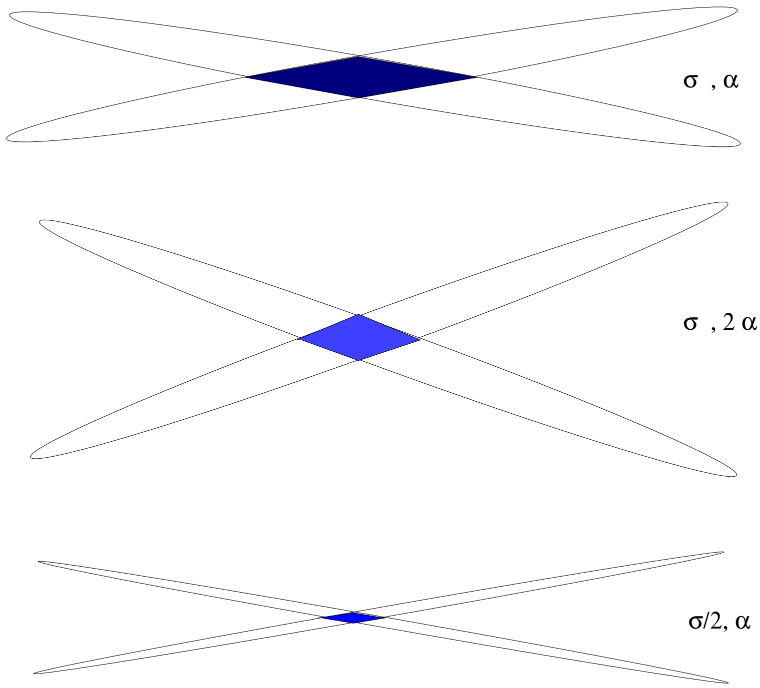

Fig. 7: Schematic illustration of luminous regions.

The answers depend on beam properties such as σ_x, σ_y, and σ_s but also on the crossing angle ϕ. This is very schematically indicated by the overlap regions in Fig. 7. Depending of the beam and machine parameters, this region can be very different, with important consequences for e.g., trigger system or pattern recognition. We evaluate:

$$\mathcal{L}_0 = \int_{-\infty}^{+\infty} \mathcal{L}(s')ds' \longrightarrow \mathcal{L}(s) = \int_{-s}^{+s} \mathcal{L}(s')ds' \tag{45}$$

and get:

$$\mathcal{L}(s) = \left(\frac{N_1 N_2 f N_b}{8\pi\sigma_x^*\sigma_y^*}\right) \frac{2\cos\frac{\phi}{2}}{\sqrt{\pi}\sigma_s} \sqrt{\frac{\pi}{A}} \,\mathrm{erf}\left(\sqrt{A}\, s\right) \ . \tag{46}$$

For the integrated luminosity this becomes:

$$\mathcal{L}_{\mathrm{int}}(s) = \int_0^T \int_{-s}^{+s} \mathcal{L}(s',t)ds'dt \ . \tag{47}$$

In order to evaluate this numerically, we use again LHC nominal parameters as above. The results of our calculations are shown in Tab.2. While practically all luminosity is seen at a distance of $\pm$ 12 cm from the interaction point, about 20% is lost when only a region of $\pm$ 5.5 cm is covered by the detector or the software. This does strongly depend on the crossing angle. A detailed examination of this property was done in [6].

7.3 Time structure of luminosity

In addtion to the space structure, the time structure of the interactions is an important input for the setup of an experiment and even on the possible physics that can be studied.

Table 3: Percentage of visible luminosity as a function of distance to interaction point.

integration range	percentage of luminosity
s = ± 12 cm	1.000
s = ± 8 cm	0.950
s = ± 7 cm	0.900
s = ± 6 cm	0.850
s = ± 5.5 cm	0.800

In the LHC the bunches cross every 25 ns and it can be calculated easily that for proton-proton collisions one has to expect $\approx$ 20 simultaneous interactions per bunch crossing. They must be digested by the detectors before the next bunch crossing occurs, a non trivial task for the experiments. Some physics studies that cannot be done with such an event pile up may require to run at a lower luminosity.

8 Luminosity measurement

The knowledge of the luminosity is of vital interest for both, the experiments and the accelerator physicists. The most important issues are:

- Determine the cross section from counting rates
- Optimize the luminosity
- Measure machine parameters from luminosity

This list could easily be extended.

8.1 Relative luminosity measurement

Since the luminosity is directly proportional to the interaction rate, luminosity measurement usually consists of fast counting devices which provide such a signal. However, some of the challenges for such an instrument are:

- Must cover a large dynamic range: 10^{27} cm^{-2}s^{-1} to 10^{34} cm^{-2}s^{-1}
- Very fast, if possible for individual bunches
- Reproducible from run to run
- Should run with different machine conditions (e.g. with and without crossing angle, different optics (β^*), etc.)
- Works for different types of particles (p, ions)
- Can be used in a feedback system to optimize the luminosity

When these requirements are fulfilled, the instrument will give a reliable signal that is proportional to the luminosity.

8.2 Absolute luminosity measurement

The relative signal has to be calibrated to deliver the absolute luminosity. We have already seen some effects that affect the absolute luminosity and therefore to a large extent the luminosity measurement. In particular the crossing angle and the luminous region are of importance since they have immediate implications for the geometrical acceptance of the instruments.

In principle one can determine the absolute luminosity when all relevant beam parameters are known, i.e. the bunch intensities, beam sizes (r.m.s. in case of unknown beam profiles) and the exact geometry. However the precise measurement of beam sizes is a challenge, in particular for hadron colliders when a non-destructive measurement is required. When the energy spread in the beams is large (e.g. some e^+e^- colliders, a residual dispersion at the interaction point increases significantly the beam size and must be included.

There exist other methods which relate the counting rate to well known processes which can be used for calibration. We shall discuss several methods for both, lepton and hadron colliders.

8.3 Absolute luminosity - lepton colliders

Once the relative luminosity is known, a very precise method is to compare the counting rate to well known and calculable processes. In case of e^+ e^- colliders these are electromagnetic processes such as elastic scattering (Bhabha scattering). The principle is shown in Fig. 8. Particle detectors are used

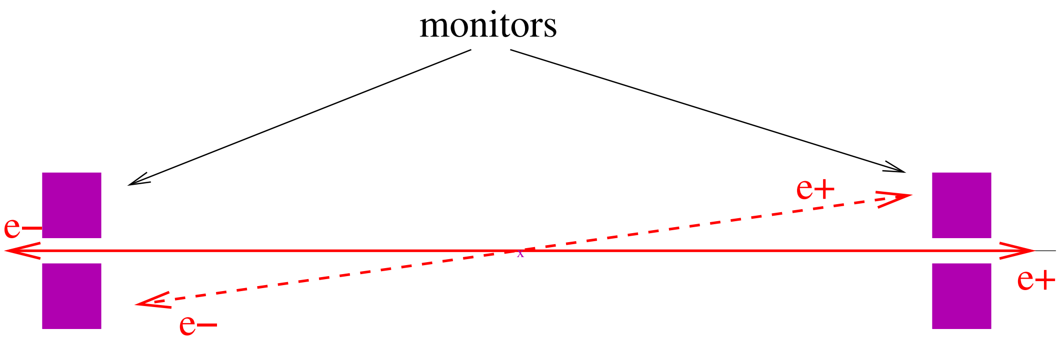

Fig. 8: Principle of luminosity measurement using Bhabha scattering for e^+ e^- colliders.

to measure the trajectories at very small angles and with a coincidence of particles on both sides of the interaction point. For a precise measurement one has to go to very small angles since the elastic cross section σ_{el} has a strong dependence on the scattering angle ($\sigma_{el} \propto \Theta^{-3}$).

Furthermore, the cross section diminishes rapidly with increasing energy ($\sigma_{el} \propto \frac{1}{E^2}$) and the result may be small counting rates. At LEP energies with $\mathcal{L} = 10^{30}\text{cm}^{-2}\text{s}^{-1}$ one can expect only about 25 Hz for the counting rate. Background from other processes can become problematic when the signal is small.

8.4 Absolute luminosity - hadron colliders

For hadron colliders two types of calibration have become part of regular operation, the measurement of the beam size by scanning the beam and the calibration with the cross section for small angle scattering. The determination of the bunch intensities is usually easier, although non-trivial in the case of a collider with several thousand bunches.

8.4.1 Measurement by profile monitors and beam displacement

Typical profile measurement devices are wire scanners where a thin wire is moved through the beam and the interaction of the beam with the wire gives the signal. For high intensity hadron beams this has however limitations. Non-destructive devices such as synchrotron light monitors are available but the emitted light from hadrons is often not sufficient for a precise measurement.

An alternative is to measure the beam size by displacing the two beams against each other. The relative luminosity reduction due to this offset can be measured and is described by the formula (32) developped earlier:

$$W = e^{-\frac{d^2}{4\sigma^2}} \tag{48}$$

where d is the separation between the beams and the measurement of the luminosity ratio

$$\mathcal{L}(d)/\mathcal{L}_0 \tag{49}$$

is a direct measurement of W.

This method was already used in the CERN Intersection Storage Rings (ISR) and known as "van der Meer scan".

The expected counting rate of such a scan is shown in Fig.9. A fit to the above formula gives

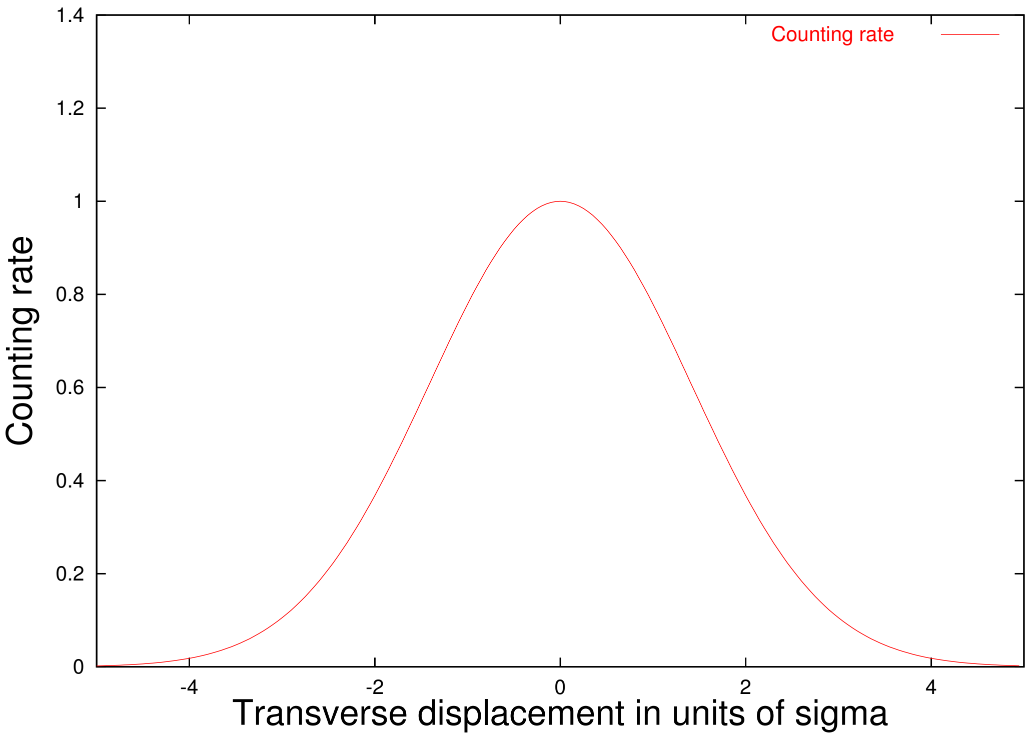

Fig. 9: Principle of luminosity measurement using transverse beam displacement.

the beam size. A drawback of this method is the distortion of the beam optics in case of very strong beam-beam interactions [2]. This effect has to be evaluated carefully.

A further alternative is a so-called "beam-beam deflection scan" where instead of the change of counting rate the effect on the closed orbit is measured. This method was largely used at LEP and is explained in [2].

8.4.2 *Absolute measurement with optical theorem*

This method is similar to the measurement of Bhabha scattering for $e^+ e^-$ colliders but requires dedicated experiments and often special machine conditions.

The total elastic and inelastic counting rate is related to the luminosity and the total cross section (elastic and inelastic) by the expression:

$$\sigma_{\text{tot}} \cdot \mathcal{L} = N_{\text{inel}} + N_{\text{el}} \text{ (Total counting rate)} . \tag{50}$$

The key to this method is that the total cross section is related to the elastic cross section for small values of the momentum transfer t by the so-called optical theorem [7]:

$$\lim_{t \to 0} \frac{d\sigma_{\text{el}}}{dt} = (1 + \rho^2) \frac{\sigma_{\text{tot}}^2}{16\pi} = \frac{1}{\mathcal{L}} \frac{dN_{\text{el}}}{dt}|_{t=0} . \tag{51}$$

Therefore the luminosity can in principle be calculated directly from experimental rates through:

$$\mathcal{L} = \frac{(1 + \rho^2)}{16\pi} \frac{(N_{\text{inel}} + N_{\text{el}})^2}{(dN_{\text{el}}/dt)_{t=0}} . \tag{52}$$

Both counting rates, the total number of events $N_{\text{inel}} + N_{\text{el}}$ and the differential elastic counting rate dN_{el}/dt at small t have to be measured with high precision. This requires a very good detector coverage of the whole space (4π) for the inelastic rate and the possibility to measure to very small values of t.

A slightly modified version of the above uses the Coulomb scattering amplitude which can be precisely calculated. The elastic scattering amplitude is a superposition of the strong (f_{s}) and Coulomb (f_{c}) amplitudes, the latter dominates at small t. We can re-write the differential elastic cross section $\frac{d\sigma_{\text{el}}}{dt}$:

$$\lim_{t \to 0} \frac{d\sigma_{\text{el}}}{dt} = \frac{1}{\mathcal{L}} \frac{dN_{\text{el}}}{dt}|_{t=0} = \pi \mid f_{\text{c}} + f_{\text{s}} \mid^2 \simeq \pi \mid \frac{2\alpha_{\text{em}}}{-t} + \frac{\sigma_{\text{tot}}}{4\pi}(\rho + i)e^{B\frac{t}{2}} \mid^2 \simeq \frac{4\pi\alpha_{\text{em}}^2}{t^2} \mid_{|t| \to 0} \tag{53}$$

If the differential cross section is measured over a large enough range, the unknown parameters $\sigma_{\text{tot}}, \rho, B$ and $\mathcal{L}$ can be determined by a fit. A measurement [8–10] together with some crude fits is shown in Fig. 10 to demonstrate the principle. The advantage of this method is that it can be performed measuring only elastic scattering without the need of a full coverage to measure N_{inel}. It is therefore a good way to measure the luminosity (and total cross section σ_{tot} and interference parameter ρ!) although the previous method is of more practical importance for regular use.

The measurement of the Coulomb amplitude usually requires dedicated experiments with detectors very close to the beam (e.g., with so-called Roman Pots) and therefore special parameters such as reduced intensity and zero crossing angle. Furthermore, in order to measure very small angle scattering, one has to reduce the divergence in the beam itself ($\sigma' = \sqrt{\epsilon/\beta}$). For that purpose special running conditions with a **high** β^* at the collision point are often needed ($\beta^* > 1000$ m) [9]. The precision of such a measurement is however as good as a few percent.

9 Not mentioned

At the end we would like to mention different types of colliders that were not treated explicitly, but the results can be extended easily. Examples of such machines are:

- Coasting beams (e.g., ISR).
- Asymmetric colliders (e.g., PEP, HERA).
- Linear colliders (SLC, TESLA etc.).

In this lecture we have tried to include some of the problems encountered in present and foreseen colliders which must be considered when a machine is built and the luminosity is optimized (not necessarily maximized!) To design a machine with the optimum luminosity, other limiting effects such as beam-beam effects [2] must be taken into account, but these are beyond the scope of this lecture.

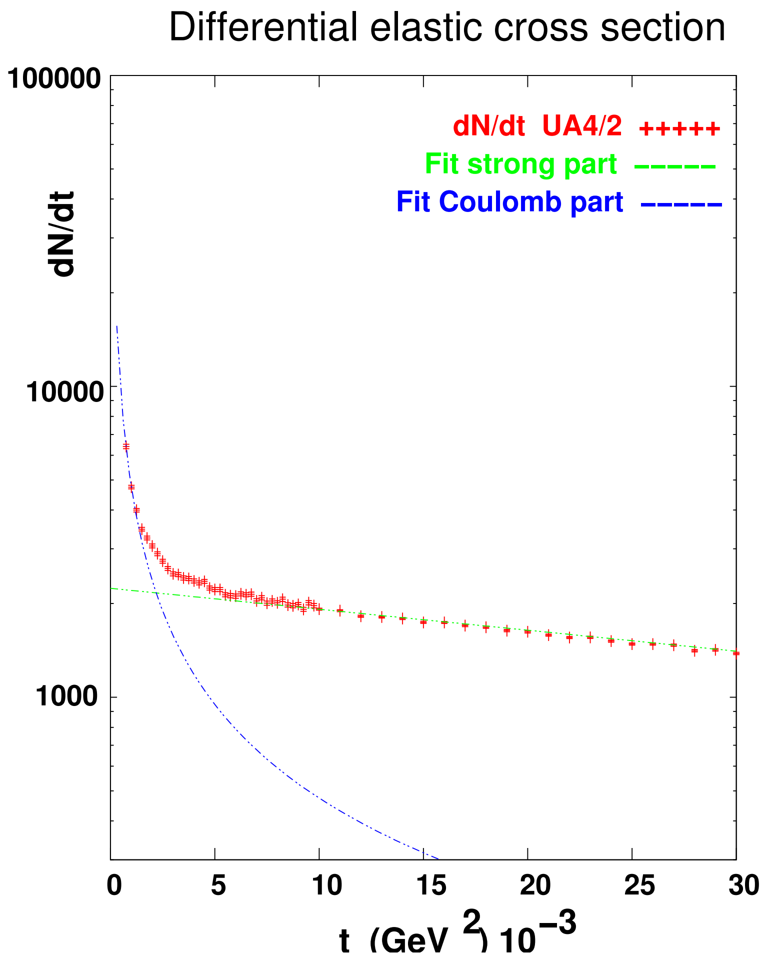

Fig. 10: Principle of luminosity measurement using optical theorem in proton proton (antiproton) collisions.

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