Analogical proportions and the factorization of information in distributive lattices

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Abstract. Analogical proportions are statements involving four entities, of the form 'A is to B as C is to D'. They play an important role in analogical reasoning. Their formalization has received much attention from different researchers in the last decade, in particular in a propositional logic setting. Analogical proportions have also been algebraically defined in terms of factorization, as a generalization of geometric numerical proportions (that equate ratios). In this paper, we define and study analogical proportions in the general setting of lattices, and more particularly of distributive lattices. The decomposition of analogical proportions in canonical proportions is discussed in details, as well as the resolution of analogical proportion equations, which plays a crucial role in reasoning. The case of Boolean lattices, which reflects the logical modeling, and the case corresponding to entities described in terms of gradual properties, are especially considered for illustration purposes.

Keywords: analogical proportion, lattice, factorization

1 Introduction

Analogical reasoning [1] plays an important role in human reasoning. It enables us to draw plausible conclusions by exploiting parallels between situations, and as such has been studied in AI for a long time, e.g., [2, 3] under various approaches [4]. A key pattern which is associated with the idea of analogical reasoning is the notion of analogical proportions, i. e. statements of the form 'A is to B as C is to D'. However, it is only in the last decade that researchers working in computational linguistics have started to study these proportions in a formal way [5–7]. More recently, analogical proportions have been shown as being of particular interest for classification tasks [8] or for solving IQ tests [9]. Moreover, in the last five years, there has been a number of works, e.g., [10, 11] studying the propositional logic modeling of analogical proportions. The logical view of an analogical proportion amounts to expressing that the difference between A and B (resp. B and A) is the same as the difference between C and D (resp. D and C). Although it can be proved that, beside symmetry, this view agrees with a crucial postulate of analogical proportions, namely that one can exchange B and C in the proportion (as well as A and D), it is not straightforward to see that it holds. In fact, a genuine parallel can be made between analogical proportions and numerical proportions. It suggests that since factorization plays a key role in geometric proportions (which equal two ratios of integers), factorization also makes sense for analogical proportions. This idea is investigated here in the abstract setting of lattices.

In order to do this, we go back to a factorization-based formalization of analogical proportions proposed in [12, 13]. On this basis, these authors proposed a definition of analogical proportions in different settings such as sets, sets of sequences, set of trees, and lattices. As shown in this paper their definition suggested for the lattice setting is incomplete. We then correct and complete this definition. We show that it encompasses the Boolean lattice case that corresponds to the propositional logic encoding of analogical proportions. We then study the more general setting of distributive lattices, identify canonical proportions, and show how analogical proportions can be decomposed into canonical ones, before discussing the solving of analogical proportion equations, a key issue for application to algorithms for analogical reasoning. We also illustrate the approach in the case of a distributive lattice induced by fuzzy sets.

The paper is organized as follows. The next section provides the necessary background on lattices and on analogical proportions. Section 3 establishes the basic form of analogical proportions in distributive lattices, which is illustrated on Boolean and on graded proportions, and then investigates their basic properties. Section 4 introduces the notion of canonical proportions and takes advantage of them for studying the composition and the decomposition of analogical proportions. Section 5 discusses the resolution of analogical equations, and briefly studies the transitivity of analogical proportions.

This paper is a preliminary investigation into the connexions between lattices and analogical proportion. In particular, we are interested in detecting analogical proportions in *concept lattices* (e.g. see [20]). However, since these lattices are generally non distributive, we will have to investigate which of the properties demonstrated here still hold true in concept lattices, and which of them have to be abandoned or weakened. A few hints are given in Sections 3 and 5.

2 Background: Lattices and analogical proportions

Lattices. They are mathematical structures commonly encountered in the semantics of representation and programming languages, in formal concept analysis, machine learning, data mining, and in other areas of computer sciences.

$(L, \vee, \wedge, \leq)$ is a *lattice* when [14]: i) L has at least two elements, ii) $\wedge$ and $\vee$ are two binary internal operations, both idempotent, commutative, associative, and satisfying the absorption laws. A lattice is *distributive* when $u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w)$, or equivalently $u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$ for all u, v and w in L. A *bounded* lattice has a greatest (or maximum) and least (or minimum) element, denoted $\top$ and $\bot$. A bounded lattice is *complemented* if each element x has a complementary y such that $x \wedge y = \bot$ and $x \vee y = \top$. A distributive, bounded and complemented lattice is called a *Boolean* lattice.

Duality theorem. If a theorem T is true in a lattice, then the dual of T is also true. This dual is obtained by replacing all occurrences of $\wedge$ (resp. $\vee$, $\leq$) by $\vee$ (resp. $\wedge$, $\geq$).

Examples. (a) $(2^{\Sigma}, \cap, \cup, \subseteq)$, where Σ is a finite set (alphabet), is a Boolean lattice. (b) $(\mathbb{N}^{+}, \mathrm{gcd}, \mathrm{lcm}, |)$ where $(x \mid y)$ iff x divides y is a distributive lattice, with the minimum element 1 but no maximum element. (c) The set $\mathcal{S}$ of closed intervals on $\mathbb{R}$, including $\emptyset$ and $\mathbb{R}$, is a non-distributive lattice when $\wedge$ is the intersection and $[a, b] \vee [c, d] = [\min(a, c), \max(b, d)]$, where min and max are defined according to the order in $\mathbb{R}$.

Analogical proportions. They are characterized by three axioms. They acknowledge the symmetrical role played by the pairs (A, B) and (C, D) in the proportion 'A is to B as C is to D', and enforce the idea that B and C can be interchanged if the proportion is valid, just as in the equality of two numerical ratios where means can be exchanged. This view dates back to Aristotle [15]. A third, optional, axiom insists on the unicity of the solution $x = B$ for completing the analogical proportion $A : B :: A : x$. These axioms are studied in [16].

Definition 1 (Analogical proportion) *An analogical proportion*[3] *(AP) on a set X is a quaternary relation on X, i.e. a subset of X^4. An element of this subset, written $A : B :: C : D$, which reads 'A is to B as C is to D', must obey the following two axioms:*

1) Symmetry of 'as'*:* $A : B :: C : D \;\Leftrightarrow\; C : D :: A : B$

2) Exchange of means*:* $A : B :: C : D \;\Leftrightarrow\; A : C :: B : D$

Then, thanks to symmetry, it can be easily seen that $A : B :: C : D \;\Leftrightarrow\; D : B :: C : A$ should also hold (exchange of the extremes). According to the first two axioms, five other formulations are equivalent to the canonical form $A : B :: C : D$, $B : A :: D : C$, $D : B :: C : A$, $C : A :: D : B$, $D : C :: B : A$ and $B : D :: A : C$.

Example. Let us take the lattice $(2^{\Sigma}, \cup, \cap, \subseteq)$, where Σ is a finite set $\{a, \ldots, n\}$. Σ may be for example a set of Boolean properties, and a subset of Σ can be used to characterize some object described by the corresponding properties. The four objects described by the subsets $x = \{a, b, e\}$, $y = \{b, c, e\}$, $z = \{a, d, e\}$ and $t = \{c, d, e\}$ are in analogical proportion in this order. Indeed, it suggests an intuitive meaning for 'is to': To transform x into y, one has to remove property a and to include property c; namely $x \setminus y = \{a\}$ and $y \setminus x = \{c\}$. z is transformed into t by exactly the same operations; namely $z \setminus t = \{a\}$ and $t \setminus z = \{c\}$. Such a view of the relation linking x, y, z, t is clearly symmetrical, and satisfies the exchange of the means: namely $x \setminus z = \{b\}$, $z \setminus x = \{d\}$ and $y \setminus t = \{b\}$, $t \setminus y = \{d\}$. This idea that x (resp. y) differs from y (resp. x) in the same way as z (resp. t) differs from t (resp. z) is at the core of the definition of the analogical proportion $x : y :: z : t$ in the Boolean setting [10], as further discussed in the following.

[3] When there is no ambiguity, an analogical proportion is also called a *proportion*.

Proportions in commutative semigroups. Stroppa and Yvon [12, 13] have given another definition of the analogical proportion, based on the notion of *factorization*, when the set of objects is a commutative semigroup $(X, \oplus)$.

Definition 2 *A 4-tuple (x, y, z, t) in a commutative semigroup $(X, \oplus)$ is an AP $x : y :: z : t$ when:*
1) either $(y, z) \in \{(x, t), (t, x)\}$,
2) or there exists $(x_1, x_2, t_1, t_2) \in X^4$ such that $x = x_1 \oplus x_2$, $y = x_1 \oplus t_2$, $z = t_1 \oplus x_2$ and $t = t_1 \oplus t_2$.

This definition satisfies the two basic axioms of the analogical proportion (Definition 1). For example, in $(X, \oplus) = (\mathbb{N}^+, \times)$, with $x_1 = 2$, $x_2 = 3$, $t_1 = 5$ and $t_2 = 7$, one has $(2\times3) : (2\times7) :: (5\times3) :: (5\times7)$, i.e. $6 : 14 :: 15 : 35$, a numerical geometric analogical proportion. Note that this particular proportion corresponds equivalently to the equality: $6 \times 35 = 14 \times 15$.

3 Analogical proportion in lattices

In this section, we are interested in studying how the definition of an analogical proportion by factorization applies to lattices. In particular we are wondering whether the equivalence of the two formulations in the preceding example can be transposed to this algebraic structure.

3.1 Definition

Considering that a lattice $(L, \vee, \wedge)$ is both a commutative semigroup $(L, \vee)$ and $(L, \wedge)$, we define an analogical proportion as follows.

Definition 3 *A 4-tuple (x, y, z, t) in $(L, \vee, \wedge)$ is an AP $(x : y :: z : t)$ when:*
1) there exists $(x_1, x_2, t_1, t_2) \in X^4$ such that $x = x_1 \vee x_2$, $y = x_1 \vee t_2$, $z = t_1 \vee x_2$ and $t = t_1 \vee t_2$,
2) and there exists $(x'_1, x'_2, t'_1, t'_2) \in X^4$ such that $x = x'_1 \wedge x'_2$, $y = x'_1 \wedge t'_2$, $z = t'_1 \wedge x'_2$ and $t = t'_1 \wedge t'_2$.

Note that when $x_2 = t_2$ then $y = x$ and $z = t$ and that when $x_1 = t_1$ then $y = t$ and $z = x$. Hence we can have $(y, z) = (x, t)$ or $(y, z) = (t, x)$.

Examples. (a) In $(\mathbb{N}^+, \mathrm{gcd}, \mathrm{lcm}, |)$, we have $(20 : 4 :: 60 : 12)$, with $x_1 = 20$, $x_2 = t_1 = 60$, $t_2 = 12$, $x'_1 = t'_2 = 4$, $x'_2 = 20$ and $t'_1 = 12$. (b) In the lattice $\mathcal{S}$ of closed intervals on $\mathbb{R}$, we have $([0, 3] : \{3\} :: [0, 4] : [3, 4])$ with $x_1 = \{3\}$, $x_2 = \{0\}$, $t_1 = [3, 4]$, $t_2 = \emptyset$, $x'_1 = [0, 3]$, $x'_2 = [0, 4]$, $t'_1 = [0, 4]$ and $t'_2 = [3, 4]$.

Proposition 1 *A 4-tuple (x, y, z, t) in $(L, \vee, \wedge)$ is an AP $(x : y :: z : t)$ iff:*

$$\begin{array}{ll} x = (x \wedge y) \vee (x \wedge z) & x = (x \vee y) \wedge (x \vee z) \\ y = (x \wedge y) \vee (y \wedge t) & y = (x \vee y) \wedge (y \vee t) \\ z = (z \wedge t) \vee (x \wedge z) & z = (z \vee t) \wedge (x \vee z) \\ t = (z \wedge t) \vee (y \wedge t) & t = (z \vee t) \wedge (y \vee t) \end{array}$$

Proof. ($\Rightarrow$). Taking $x_1 = x \wedge y$, $x_2 = x \wedge z$, $t_1 = z \wedge t$ and $t_2 = y \wedge t$ show directly that there exist factors satisfying Definition 3.
($\Leftarrow$). Let us show that $x = (x \wedge y) \vee (x \wedge z)$. Since $x = x_1 \vee x_2$ and $y = x_1 \vee t_2$, we have $x_1 \leq x$ and $x_1 \leq y$. Then $x_1 \leq x \wedge y$. Similarly, factor x_2 satisfies $x_2 \leq x \wedge z$. Hence, $x \leq (x \wedge y) \vee (x \wedge z)$. Besides, x being greater than $(x \wedge y)$ and $(x \wedge z)$, $(x \wedge y) \vee (x \wedge z) \leq x$. The antisymmetry of $\leq$ implies that $x = (x \wedge y) \vee (x \wedge z)$. We show the other equalities in the same manner. □

The above definition applies to general lattices. In this paper, we focus on distributive lattices, since most of the properties to come require this property.

Boolean lattices

Every finite Boolean lattice is isomorphic to the lattice $(X, \cup, \cap, \subseteq)$, where X is a finite set. When considering this lattice, the quantities involved in Definition 1 can be described more precisely (see [16, 17]), as explained below.

Proposition 2 *A 4-tuple (x, y, z, t) in the Boolean lattice $(2^{\Sigma}, \cup, \cap, \subseteq)$ is in the AP $(x : y :: z : t)$ iff there exists a partition of Σ composed of six subsets (a, b, c, d, e, f) such that $x = a \cup c \cup e$, $y = b \cup c \cup e$, $z = a \cup d \cup e$ and $t = b \cup d \cup e$.*

The link with Definition 3 is made by taking[4].: $x_1 = c \cup e$, $x_2 = a \cup e$, $t_1 = d \cup e$ and $t_2 = b \cup e$, and by duality: $x'_1 = \bar{d} \cap \bar{f}$, $x'_2 = \bar{b} \cap \bar{f}$, $t'_1 = \bar{c} \cap \bar{f}$ and $t'_2 = \bar{a} \cap \bar{f}$.

It is also easy to check that this definition is equivalent to Definition 3.

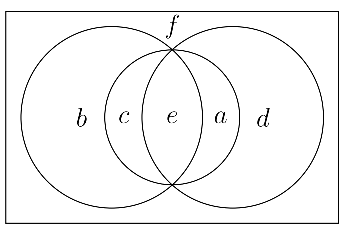

Fig. 1. An AP $(x : y :: z : t)$ in a Boolean lattice. $x = a \cup c \cup e$, $y = b \cup c \cup e$, $z = a \cup d \cup e$ and $t = b \cup d \cup e$.

It is worth noticing that the above result has a nice interpretation in practice. Let us view x, y, z, t as subsets of properties that hold true in four different situations. It is then clear that a is the subset of properties that are true in the first situation, but false in the second one, and again true in the third situation and false in the fourth one. Conversely, b is the subset of properties that are false in the first situation, true in the second one, and again false in the third situation and true in the fourth one. Besides, c (resp. d) is the set of properties that are true for both the first and the second situations and false for the third and the fourth ones (resp. false for the first and the second situations and true for the third and the fourth ones. In other words, the disjoint subsets a, b, c, d, e, f have the following interpretations $a = x \setminus y = z \setminus t$, $b = y \setminus x = t \setminus z$, $c \cup e = x \cap y$, $d \cup e = z \cap t$, where $e = x \cap y \cap z \cap t$ is the set of properties that are

[4] We denote the complement in 2^{Σ} with an overline

true in all situations (and f the set of properties that are false in all situations); see Figure 1. Thus, one can say that x, y, z, and t are respectively factorized under the form of pairs of disjoint subsets, namely $(a, c \cup e)$ for x, $(b, c \cup e)$ for y, $(a, d \cup e)$ for z, and $(b, d \cup e)$ for x, which perfectly parallels the equality of two numerical ratios of the form $\frac{\alpha\times\gamma}{\beta\times\gamma} = \frac{\alpha\times\delta}{\beta\times\delta}$.

Moreover, the above decomposition using the partition of the referential into six subsets exactly corresponds to the truth table of the analogical proportion in a propositional setting [10, 18] defined equivalently by

$x : y :: z : t \;\; = (x \wedge \neg y) \equiv (z \wedge \neg t) \;\wedge\; (y \wedge \neg x) \equiv (t \wedge \neg z)$

or $\quad x : y :: z : t \;\; = (x \wedge t) \equiv (y \wedge z) \;\wedge\; (x \vee t) \equiv (y \vee z)$.

Indeed, in the Boolean lattice associated to the two truth values $0, 1$, $x : y :: z : t$ is true (i.e., is equal to '1') for the six patterns $(x, y, z, t) = (1, 0, 1, 0)$, $(x, y, z, t) = (0, 1, 0, 1)$, $(x, y, z, t) = (1, 1, 0, 0)$, $(x, y, z, t) = (0, 0, 1, 1)$, $(x, y, z, t) = (1, 1, 1, 1)$ and $(x, y, z, t) = (0, 0, 0, 0)$, and false for the ten other possible patterns which are $(x, y, z, t) = (1, 0, 0, 1)$, $(x, y, z, t) = (0, 1, 1, 0)$ and the eight patterns having an odd number of '1' and '0' (e.g., $(x, y, z, t) = (0, 0, 1, 0)$ or $(x, y, z, t) = (0, 1, 1, 1)$). The six above patterns which make $x : y :: z : t$ true clearly correspond to the subsets a, b, c, d, e, f.

The case of graded properties

Analogical proportions have been also extended when properties are graded on a chain which is finite, or such as the unit interval $[0, 1]$ [19]. For instance, a property may be half-true. Then, in the case of a finite chain with three elements $\{0, \omega, 1\}$, two views make sense, for which the patterns having truth value '1' are respectively

- the 15 patterns, that includes the 6 of the binary case $(1, 0, 1, 0)$, $(0, 1, 0, 1)$, $(1, 1, 0, 0)$, $(0, 0, 1, 1)$, $(1, 1, 1, 1)$, $(0, 0, 0, 0)$, together with their 9 counterparts $(\omega, 0, \omega, 0)$, $(0, \omega, 0, \omega)$, $(1, \omega, 1, \omega)$, $(\omega, 1, \omega, 1)$, $(\omega, \omega, 0, 0)$, $(0, 0, \omega, \omega)$, $(1, 1, \omega, \omega)$, $(\omega, \omega, 1, 1)$, $(\omega, \omega, \omega, \omega)$
- the 15 above patterns together with the 4 additional ones $(1, \omega, \omega, 0)$, $(0, \omega, \omega, 1)$, $(\omega, 0, 1, \omega)$, $(\omega, 1, 0, \omega)$.

In the second view, we acknowledge the fact that when there is a change from x to y there is *the same* change from z to t, and otherwise there is no change between x and y, and between z and t, but also the fact that the proportion still holds when the change from x to y has the same direction and intensity as the change from z to t (considering that ω is exactly in the "middle" between 0 and 1). It is easy to see that the lattice-based definition proposed here agrees with the first view only, while the 4 additional patterns do not make analogical proportions.

In the case of the unit interval $[0, 1]$, this leads to the following graded view of the analogical proportion:

$x : y :: z : t \; = \min(1 - |\min(x, t) - \min(y, z)|, 1 - |\max(x, t) - \max(y, z)|)$.

It is easy to see that the above definition is a direct counterpart of the second form of the propositional expression of the analogical proportion given above. Moreover, it is equal to 1 only for the 15 patterns mentioned above.

3.2 Basic properties

We show here that in distributive lattices, a 4-tuple in analogical proportion is such that "the product of the means is equal to the product of the extremes".

Proposition 3 *In a distributive lattice,* $(x : y :: z : t)$ *is an AP iff:*

$$y \wedge z \leq x \leq y \vee z,\ x \wedge t \leq y \leq x \vee t,\ x \wedge t \leq z \leq x \vee t \text{ and } y \wedge z \leq t \leq y \vee z \tag{1}$$

Proof. ($\Rightarrow$). Using the derivations of x given in Proposition 1, we have $x = x \wedge (y \vee z)$ and $x = x \vee (y \wedge z)$ by distributivity and then $y \wedge z \leq x \leq y \vee z$. The other inequalities are similarly derived.
($\Leftarrow$). By distributivity, $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$. Moreover, $x \wedge (y \vee z) = x$ since $x \leq y \vee z$. The other equalities are obtained in the same way. □

The next property is a stronger result: the four values of the bounds in the preceding property are actually only two.

Proposition 4 *In a distributive lattice,* $(x : y :: z : t)$ *is an analogical proportion iff* $x \vee t = y \vee z$ *and* $x \wedge t = y \wedge z$*.*

Proof. ($\Rightarrow$). Using the expressions of x, y, z and t given by Proposition 1, we easily check that $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$.

($\Leftarrow$). By absorption law and distributivity, we have $x = x \wedge (x \vee t) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. The other equations of Proposition 1 can be obtained in a similar way. □

Comment 1. In [13, 7], an incomplete definition of a proportion in a lattice has been given. Actually, only four equalities of Definition 3 were given, and only four equalities of Proposition 1 were demonstrated (in a different manner than here). This definition was flawed, since for example in the lattice $(\{0, 1\}, \vee, \wedge)$ it would have given $(0 : 1 :: 1 : 1)$ as a proportion, although it does not satisfy the basic axioms. In the particular case of Boolean lattices, Proposition 4 has been shown in [10].

Comment 2. If the lattice is not distributive, Proposition 4 is not an equivalence, but an implication. For example, let us consider the elements $x = [2, 3]$, $y = [2, 6]$, $z = [8, 9]$ and t=[6, 9] of the lattice of closed intervals on $\mathbb{R}$. We have $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$ but the conditions of Definition 3 are not satisfied. We are currently studying in general lattices and concept lattices the properties of what can be called a *weak analogical proportion*, namely the fact that four elements are linked by the equalities $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$.

3.3 Determinism

The first and second axioms of Definition 1 are straightforwardly verified by Definition 3. What about the third axiom?

Proposition 5 (Determinism in a distributive lattice) *Let* x *and* y *be two elements of a distributive lattice, the equation in* z*:* $(x : x :: y : z)$ *has the unique solution* $z = y$*. This is also true for the equation* $(x : y :: x : z)$*.*

Proof. Let us consider a solution z of $(x : x :: y : z)$. From Proposition 4, we have

$$x \wedge z = x \wedge y \quad \text{and} \quad x \vee z = x \vee y\,. \tag{2}$$

Besides, using absorption law, $z = (x \vee z) \wedge z$. Consequently, using (2) and distributivity, $z = (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. Then, using (2), distributivity and absorption, we can conclude: $z = (x \wedge y) \vee (y \wedge z) = (x \vee z) \wedge y = (x \vee y) \wedge y = y$. □

4 Composition and decomposition of analogical equations

We present in this section a particular case of analogical proportion which will be shown later (see section 4) to be a "building block" of the general proportion.

Proposition 6 (Canonical proportions) *Let y and z be two arbitrary elements of a lattice. Then the following analogical proportion is true:*

$$y : y \vee z :: y \wedge z : z\,. \tag{3}$$

Proof. Equations of Proposition 4 are straightforwardly satisfied. □

In the following, we will call this particular analogical proportion a *canonical* analogical proportion (CAP). Note that the previous property holds in general lattices, not only distributive lattices.

In general, analogical proportions in a lattice are not canonical, such as $(14 : 21 :: 10 : 15)$ in $(\mathbb{N}^{+}, \mathrm{gcd}, \mathrm{lcm}, |)$.

Note that a canonical proportion can be written in eight different forms (see Definition 1), by applying the axioms of analogical proportion. We suppose in the following that one of the two particular following forms are used: $y : y \vee z :: y \wedge z : z$ or $z : y \vee z :: y \wedge z : y$. This form is called the $CAP1$ form, as opposed to the $CAP2$ one: $y : y \wedge z :: y \vee z : z$ or $z : y \wedge z :: y \vee z : y$.

We are interested in here in defining primitive proportions, that will be used as "building blocks" of the general proportion. This is done in particular to enlighten primitive chunks in a process of reasoning by analogy.

Definition 4 *Let $\mathtt{a} = (x, y, z, t)$ and $\mathtt{A} = (X, Y, Z, T)$ be two 4-tuples of a distributive lattice $(L, \vee, \wedge)$. We define the $\boxed{\vee}$-composition and the $\boxed{\wedge}$-composition of these two 4-tuples as the 4-tuples:*

$$\mathtt{a} \boxed{\vee} \mathtt{A} = (x \vee X, y \vee Y, z \vee Z, t \vee T) \quad \textit{and} \quad \mathtt{a} \boxed{\wedge} \mathtt{A} = (x \wedge X, y \wedge Y, z \wedge Z, t \wedge T)$$

Note that these operations are commutative and associative.

Definition 5 *A* degenerated *analogical proportion (DAP) is $(x : x :: x : x)$.*
A simple *analogical proportion (SAP) is $(x : y :: x : y)$ ($SAP1$)*
or $(x : x :: y : y)$ ($SAP2$).

The next results are all established in a distributive lattice.

Proposition 7 (Composition of an AP and a DAP) *The composition of an AP by a DAP is a AP.*

Proof. Using Proposition 4 and distributivity. $\square$

This property is a generalisation of a property in Boolean lattices, shown in [10]. Analogical proportions are not closed for general composition, as shown below.

Note that the composition of two AP's is not necessarily an AP (nor is the composition of an AP and a SAP) and that the composition of two CAP's is not necessarily an AP.

Proposition 8 *In a distributive lattice, for every AP $\mathtt{a}_1$ there exists a $SAP1$ $\mathtt{a}_2$ and a $SAP2$ $\mathtt{a}_3$ such that $\mathtt{a}_1 = \mathtt{a}_2 \boxed{\vee} \mathtt{a}_3$. There also exists a $SAP1$ $\mathtt{a}_3$ and a $SAP2$ $\mathtt{a}_4$ such that $\mathtt{a}_1 = \mathtt{a}_3 \boxed{\wedge} \mathtt{a}_4$.*

Proof. We check that $(x : y :: z : t)$ is the $\boxed{\wedge}$-composition of $(x \vee y) : (x \vee y) :: (z \vee t) : (z \vee t)$ and $(x \vee z) : (y \vee t) :: (x \vee z) : (y \vee t)$, by proposition 1. It is also the $\boxed{\vee}$-composition of $(x \wedge y) : (x \wedge y) :: (z \wedge t) : (z \wedge t)$ and $(x \wedge z) : (y \wedge t) :: (x \wedge z) : (y \wedge t)$. $\square$

Proposition 9 *In a distributive lattice, for every AP $\mathtt{a}_1$ there exists a $CAP1$ $\mathtt{a}_2$ and a $CAP2$ $\mathtt{a}_3$ such that $\mathtt{a}_1 = \mathtt{a}_2 \boxed{\vee} \mathtt{a}_3$. There exists also a $CAP1$ $\mathtt{a}_3$ and a $CAP2$ $\mathtt{a}_4$ such that $\mathtt{a}_1 = \mathtt{a}_3 \boxed{\wedge} \mathtt{a}_4$.*

Proof. We check that $(x : y :: z : t)$ is the $\boxed{\vee}$-composition of $(x \wedge y) : y :: (x \wedge y \wedge z \wedge t) : (y \wedge t)$ and $(x \wedge z) : (x \wedge y \wedge z \wedge t) :: z : (z \wedge t)$ and the $\boxed{\wedge}$-composition of $(x \vee y) : y :: (x \vee y \vee z \vee t) : (y \vee t)$ and $(x \vee z) : (x \vee y \vee z \vee t) :: z : (z \vee t)$. $\square$

Proposition 10 *In a distributive lattice, for every AP $\mathtt{a}_1$ there exists a $CAP1$ $\mathtt{a}_2$ such that $\mathtt{a}_1 \boxed{\vee} \mathtt{a}_2$ is a $CAP2$.*

Proof. Let $\mathtt{a}_1 = (x : y :: z : t)$, and take $\mathtt{a}_2 = ((z \wedge t) : z :: (x \wedge y \wedge z \wedge t) : (x \wedge z))$.

We have to show that $[x \vee (z \wedge t)] : (y \vee z) :: z : [t \vee (x \wedge z)]$. According to property 1, we show equivalently the two equalities: $[x \vee (z \wedge t)] \vee [t \vee (x \wedge z)] = (y \vee z) \vee z$ and $[x \vee (z \wedge t)] \wedge [t \vee (x \wedge z)] = (y \vee z) \wedge z$. For the second: $x \vee (z \wedge t)] \wedge [t \vee (x \wedge z) = [(x \vee z) \wedge (x \vee t)] \wedge [(t \vee x) \wedge (t \vee z)] = (x \vee z) \wedge (t \vee z) \wedge (x \vee t) = z \wedge (x \vee t) = z \wedge (y \vee z)$.

The first equality has a similar demonstration. $\square$

5 Resolution of analogical equations. Transitivity

In this section, we answer the following question: given three elements of an AP, can we find the fourth one? This is an important issue in analogical reasoning.

Let us suppose that in a distributive lattice we know three elements a, m and M. We are looking for an x satisfying the couple of equations:

$$a \vee x = M \quad \text{and} \quad a \wedge x = m \tag{4}$$

This is a more general question that wondering whether the analogical equation in a distributive lattice $(a : b :: c : x)$ has solutions, since we can take $M = b \vee c$ and $m = b \wedge c$.

Proposition 11 (Unicity of the solution) *When there is a solution to equations 4 in a distributive lattice, then it is unique. Consequently, if there exists a solution to the analogical equation* $(a : b :: c : x)$*, then it is unique.*

Proof. Supposing the equations have two solutions x_1 and x_2 leads to a contradiction with the distributivity between a, x_1 and x_2. □

Proposition 11 doesn't hold in general lattices: eq. 4 may have several solutions.

Proposition 12 *Let* a*,* m *and* M *be three elements of a distributive lattice such that* $m \leq a \leq M$*. If there exists* $\overset{\diamond}{a}$ *such that:* $(\overset{\diamond}{a} \vee a \geq M)$ *and* $(\overset{\diamond}{a} \wedge a \leq m)$ *then* $x = (M \wedge \overset{\diamond}{a}) \vee m = (m \vee \overset{\diamond}{a}) \wedge M$ *is the unique solution to equations (4).*

Proof. Firstly, we show that $x \wedge a = m$ with the equalities: $x \wedge a = [(M \wedge \overset{\diamond}{a}) \vee m] \wedge a = (M \wedge \overset{\diamond}{a} \wedge a) \vee (m \wedge a) = (\overset{\diamond}{a} \wedge a) \vee m = m$.

Secondly, the equality $x \vee a = M$ is demonstrated in the same manner, using $M \leq (\overset{\diamond}{a} \vee a)$ instead of $(\overset{\diamond}{a} \wedge a) \leq m$. Then $x = (M \wedge \overset{\diamond}{a}) \vee m$ is the solution.

Thirdly, we show in the same manner that $x = (m \vee \overset{\diamond}{a}) \wedge M$ is a solution to (4). Since the solution is unique, the property is demonstrated. □

When two or three elements are comparable, the solutions of the analogical equation are severely constrained.

Proposition 13 *Let* x*,* y*,* z *and* t *be four elements of a distributive lattice such as* $(x : y :: z : t)$*. If the three first elements are comparable then this AP is a SAP or a CAP. More precisely,* $(x : y :: z : t)$ *is*

1) $(y \wedge z : y :: z : y \vee z)$ *if* $x \leq y \wedge z$*, and* $(y \vee z : y :: z : y \wedge z)$ *if* $x \geq y \vee z$*. In particular, it is a* $SAP1$ *if* $y \leq z \leq x$ *or* $x \leq z \leq y$*, and a* $SAP2$ *if* $z \leq y \leq x$ *or* $x \leq y \leq z$

2) a $CAP1$ *(resp.* $CAP2$ *)if* $z \leq x \leq y$ *(resp.* $y \leq x \leq z$*).*

Proof.

1) We have from (1) $y \wedge z \leq x \leq y \vee z$. Let us consider the case where $x \leq y \wedge z$. We then have $x = y \wedge z$ and we can easily check that $t = y \vee z$ is solution of equations $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$. Then, using Propositions 4 and 11, $t = y \vee z$ is the unique solution of $(y \wedge z : y :: z : t)$. Moreover, if $x \leq z \leq y$, $t = y \vee z = y$ and then $x = z$. The other cases have a similar demonstration.

2) If $z \leq x \leq y$, $y = x \vee t$ and $z = x \wedge t$ using Proposition 4,

3) The reasoning is similar to the previous one. □

In the Boolean case, we recall a previous result.

Proposition 14 ([16]) *The analogical equation in* t*:* $(x : y :: z : t)$ *has a solution in a Boolean lattice if and only if* $y \cap z \subseteq x \subseteq y \cup z$*. In this case, the unique solution is* $t = ((y \cup z) \backslash x) \cup (y \cap z)$*.*

Finally, let us investigate transitivity, which propagates (dis)similarity. In the Boolean case, $(a : b) :: (c : d) :: (e : f)$ holds for general proportions [10]. In the distributive case, $CAP1$ (resp. $CAP2$) are transitive. Proof is omitted due to space limitation.

Proposition 15 (Transitivity of CAP) *If $(x : (x \vee t) :: (x \wedge t) : t)$ and $((x \wedge t) : t :: u : v)$ are two canonical proportions of form $CAP1$ (resp. $CAP2$), then $x : (x \vee t) :: u : v$ is a canonical proportion of form $CAP1$ (resp. $CAP2$).*

Conjecture 1 (Non transitivity of proportions) *If $(x : y :: z : t)$ and $(z : t :: u : v)$ are two analogical proportions in a distributive lattice, it does not necessarily imply that $(x : y :: u : v)$ is an analogical proportion.*

We have not found any example to show this property, albeit the transitivity seems impossible to prove. Therefore, the non transitivity in a general distributive lattice is a conjecture. However, we have found an example to show that transitivity doesn't hold in general in a non transitive lattice.

We have not found any counter example to show this property. We conjecture there is no transitivity in distributive lattices. Indeed, in a non distributive lattice, transitivity does not holds, as shown in the following example. In $\mathcal{S}$ (see section 2) we have $[0,3] : \{3\} :: \{0\} : \emptyset$ by considering Definition 3 and $x_1 = \{3\}$, $x_2 = \{0\}$, $t_1 = \emptyset$, $t_2 = \emptyset$, $x'_1 = x'_2 = [0,3]$, $t'_1 = \{0\}$ and $t'_2 = \{3\}$. Similarly, we have $\{0\} : \emptyset :: [0,4] : \{4\}$ using $x_1 = \emptyset$, $x_2 = \{0\}$, $t_1 = \{4\}$, $t_2 = \emptyset$, $x'_1 = \{0\}$, $x'_2 = [0,4]$, $t'_1 = [0,4]$ and $t'_2 = \{4\}$. However, $[0,3] : \{3\} :: [0,4] : \{4\}$ is not true because it is impossible to satisfy the second condition of Definition 3. Indeed, if there exists four elements x'_1, x'_2, t'_1 and t'_2 of $\mathcal{S}$ such that $[0,3] = x'_1 \wedge x'_2$, $\{0\} = x'_1 \wedge t'_2$, $[0,4] = t'_1 \wedge x'_2$ and $\{4\} = t'_1 \wedge t'_2$, the closed interval t'_2 contains 0 and 4 and then $[0,4] \subset t'_2$. Moreover, $[0,4] \subset t'_1$. Consequently, $t'_1 \wedge t'_2 \neq \{4\}$.

6 Conclusion

The results of this paper provide a better understanding of analogical proportions in the general setting of lattices structures. In particular, it relates a factorization-based view of analogical proportions to its propositional logical reading in the case of Boolean lattices. For graded proportions, where the underlying lattice of grades is a chain, it leads to consider that the only fully valid logical proportions are of the form $x : y :: x : y$ (and $x : x :: y : y$) where x and y are elements in the chain. It acknowledges the fact that the change should be exactly the same on both sides of the proportion in order to make it (completely) valid, an idea which is for instance (successfully) at work in [9]. The paper has also introduced canonical forms of analogical proportions that are instrumental in the decomposition of analogical proportions in distributive lattices. The unicity of the solution of an analogical proportion equation when it exists, is a important property that is preserved in distributive lattices, and which enables us to generate accurate conclusions.

Generally speaking, the results presented should be useful to design algorithms helping to propagate information in lattices, especially for purposes of reasoning and learning. Moreover, in [20] a first attempt has been provided for relating analogical proportions to formal concept analysis, and searching for analogical proportions that may hold in a formal context by exploiting the lattice structure of the set of formal concepts. This study of analogical proportions in lattice structures should contribute in the long range to a clearer view of the links between these formalizations of the two key cognitive processes that are conceptual categorization and analogical reasoning.

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