Inverted fuzzy implications in approximate reasoning

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Abstract. In this paper, we propose a new method for choosing implications. Our method allows to compare two fuzzy implications. If the truth value of the antecedent and the truth value of the implication are given, by means of inverse fuzzy implications we can easily optimize the truth value of the implication consequent.

Key words: fuzzy logic, fuzzy implications, inverted fuzzy implications

1 Introduction

In 1975 Lotfi Zadeh introduced the theory of fuzzy logic [13]. Fuzzy logic was an extension of Boolean logic so that it allowed using not only Boolean values to express reality. One of basic logical operations in fuzzy logic are so-called implications. From over eight decades a number of different fuzzy implications have been proposed [2]-[12]. In the family of basic fuzzy implications the partial order induced from [0,1] interval exists. Pairs of incomparable fuzzy implications can generate new fuzzy implications by using min (inf) and max (sup) operations. As a result the structure of lattice is created ([1], page 186). This leads to the following question: how to choose the relevant functions among basic fuzzy implications and other generated as described above. In our paper, we propose a new method for choosing implications. Our method allows to compare two fuzzy implications. If the truth value of the antecedent and the truth value of the implication are given, by means of inverse fuzzy implications we can easily optimize the truth value of the implication consequent. In other words, we can choose fuzzy implication, which has the greatest or the smallest truth value of the implication consequent or which has greater or smaller truth value than another implication. Primary results regarding this problem are included in the paper [10].

The rest of this paper is organized as follows. In Sect. 2 the main research problem is formulated. Sect. 3 presents the solution of the given research problem. An example illustrating our approach is given in Sect. 4. Sect. 5 includes remarks on directions for further research.

2 Problem Statement

There is given a basic fuzzy implication $z = I(x, y)$, where x, y belong to [0,1]. x is the truth value of the antecedent and is known. z is the truth value of the implication and is also known. In order to determine the value of the truth of the implication's consequent y it is needed to compute the inverse function $InvI(x, z)$. In other words, the inverse function $InvI(x, z)$ has to be determined. Not every basic implication can be inverted. The function can be inverted only when it is injective.

3 Results

There are a few examples of basic fuzzy implications in Table 1.

Name	Year	Formula of basic fuzzy implication
Lukasiewicz	1923, [7]	$I_{LK}(x, y) = min(1, 1 - x + y)$
Gödel	1932, [4]	$I_{GD}(x, y) = \begin{cases} 1 \; if \; x \leq y \\ y \; if \; x > y \end{cases}$
Reichenbach	1935, [8]	$I_{RC}(x, y) = 1 - x + xy$
Kleene-Dienes	1938, [6]; 1949, [2]	$I_{KD}(x, y) = max(1 - x, y)$
Goguen	1969, [5]	$I_{GG}(x, y) = \begin{cases} 1 \; if \; x \leq y \\ \frac{y}{x} \; if \; x > y \end{cases}$
Rescher	1969, [9]	$I_{RS}(x, y) = \begin{cases} 1 \; if \; x \leq y \\ 0 \; if \; x > y \end{cases}$
Yager	1980, [12]	$I_{YG}(x, y) = \begin{cases} 1 \; if \; x = 0 \; and \; y = 0 \\ y^x \; if \; x > 0 \; or \; y > 0 \end{cases}$
Weber	1983, [11]	$I_{WB}(x, y) = \begin{cases} 1 \; if \; x < 1 \\ y \; if \; x = 1 \end{cases}$
Fodor	1993, [3]	$I_{FD}(x, y) = \begin{cases} 1 & if \; x \leq y \\ max(1 - x, y) & if \; x > y \end{cases}$

Table 1. Examples of basic fuzzy implications

Table 2 lists inverse fuzzy implications and their domains.

The resulting inverse functions can be compared with each other so that it is possible to order them. However, some of those functions are incomparable in the whole domain. Nevertheless, by dividing the domain into separable areas (see Figure 1), we have obtained the below given inequalities for any x, z belonging to (0,1). The interval to which x and z belong to is open, since we did not manage to deduce any inequalities on the edges of the domain. From the definition of a fuzzy implication ([1], page 2) we can conclude that every fuzzy implication I is constant for $x = 0$ and also for $y = 1$, i.e., $I(0, y) = 1$ for $y \in [0, 1]$, and $I(x, 1) = 1$ for $x \in [0, 1]$. So, one can not infer any inequality on the edges $x = 0$ and $z = 1$. For the edge $z = 0$ the following functions exist: $InvI_{GD}$, $InvI_{GG}$, $InvI_{YG}$.

Formula of inverted fuzzy implication	Domain of inverted fuzzy implication
$InvI_{LK}(x, z) = z + x - 1$	$1 - x \leq z < 1, x \in (0, 1]$
$InvI_{GD}(x, z) = z$	$0 \leq z < x, x \in (0, 1]$
$InvI_{RC}(x, z) = \frac{z+x-1}{x}$	$1 - x \leq z \leq 1, x \in (0, 1]$
$InvI_{KD}(x, z) = z$	$1 - x < z \leq 1, x \in (0, 1]$
$InvI_{GG}(x, z) = xz$	$0 \leq z < 1, x \in (0, 1]$
$InvI_{YG}(x, z) = z^{\frac{1}{x}}$	$0 \leq z \leq 1, x \in (0, 1]$
$InvI_{FD}(x, z) = z$	$1 - x < z < x, x \in (0, 1]$

Table 2. Inverted fuzzy implications

Values of those functions on this edge are equal to zero, hence, no inequality can be inferred from them. Likewise, on the edge $x = 1$ the inverse functions are $InvI_{LK}$, $InvI_{GD}$, $InvI_{RC}$, $InvI_{KD}$, $InvI_{GG}$, $InvI_{YG}$, $InvI_{FD}$. Values of these functions are equal z on this edge. For this reason it is not possible to infer any inequality.

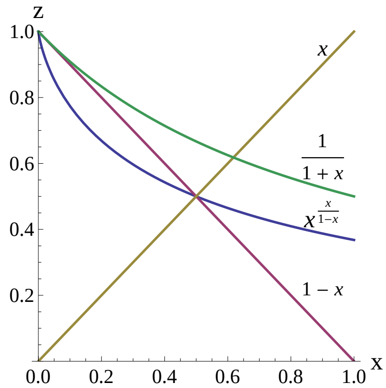

Fig. 1. The unit square $[0, 1]^2$ divided into separable areas

1. For $z \geq x$ and $z < x^{\frac{x}{1-x}}$
   $InvI_{YG} < InvI_{GG}$
2. For $z < x$ and $z < 1 - x$
   $InvI_{YG} < InvI_{GG} < InvI_{GD}$
3. For $z > 1 - x$ and $z < x^{\frac{x}{1-x}}$
   $InvI_{LK} < InvI_{RC} < InvI_{YG} < InvI_{GG} < InvI_{GD} = InvI_{KD} = InvI_{FD}$
4. For $z < 1 - x$ and $z > x^{\frac{x}{1-x}}$
   $InvI_{GG} < InvI_{YG}$

5. For $z > 1-x$ and $z \geq x$ and $z < \frac{1}{1+x}$
   $InvI_{LK} < InvI_{RC} < InvI_{GG} < InvI_{YG} < InvI_{KD}$
6. For $z > x^{\frac{x}{1-x}}$ and $z < 1-x$ and $z < \frac{1}{1+x}$
   $InvI_{LK} < InvI_{RC} < InvI_{GG} < InvI_{YG} < InvI_{KD} = InvI_{GD} = InvI_{FD}$
7. For $z > \frac{1}{1+x}$ and $z \geq x$
   $InvI_{LK} < InvI_{GG} < InvI_{RC} < InvI_{YG} < InvI_{KD}$
8. For $z > \frac{1}{1+x}$ and $z < x$
   $InvI_{LK} < InvI_{GG} < InvI_{RC} < InvI_{YG} < InvI_{KD} = InvI_{GD} = InvI_{FD}$
9. For $z = 1-x$ and $x \in (0, \frac{1}{2})$
   $InvI_{LK} = InvI_{RC} < InvI_{GG} < InvI_{YG}$
10. For $z = \frac{1}{2}$ and $x = \frac{1}{2}$
    $InvI_{LK} = InvI_{RC} < InvI_{GG} = InvI_{YG}$
11. For $z = 1-x$ and $x \in (\frac{1}{2}, 1)$
    $InvI_{LK} = InvI_{RC} < InvI_{YG} < InvI_{GG} < InvI_{GD}$
12. For $z = x^{\frac{x}{1-x}}$ and $x \in (0, \frac{1}{2})$
    $InvI_{YG} = InvI_{GG}$
13. For $z = x^{\frac{x}{1-x}}$ and $x \in (\frac{1}{2}, 1)$
    $InvI_{LK} < InvI_{RC} < InvI_{YG} = InvI_{GG} < InvI_{GD} = InvI_{KD} = InvI_{FD}$
14. For $z = \frac{1}{1+x}$ and $x \in (0, 1)$
    $InvI_{LK} < InvI_{RC} = InvI_{GG} < InvI_{YG} < InvI_{KD}$

By means of these inequalities it is possible to find a fuzzy implication which has, for example, the greatest or the smallest truth value of a consequent y whereas the truth value of the antecedent x and the truth value of the implication z are given. First, it should be checked to which area of the domain the point (x, z) belongs. It is one of the areas 1 - 14. Then, according to the inequality in the given area, the function of the smallest or the greatest truth value of the consequent $InvI(x, z)$ can be selected.

Then, wanting to expand the set of tested fuzzy implications we have started to study new fuzzy implications $H_1 - H_{15}$ generated from the basic fuzzy implications, which form an algebraic structure of lattice ([1], pages 184-185). Below given functions $H_1 - H_{15}$ are presented in one or two forms. The first form is taken from [1], while the second version is the transformation of the first to a form which is composed of pieces that are basic functions. It turned out that the inverses of fuzzy implications $H_1 - H_{15}$ are equal to the inverses of basic fuzzy implications in various areas of the unit square.

$$H_1(x, y) = \begin{cases} 1 & if\ x \leq y \\ max(\frac{y}{x}, 1 - x + xy) & if\ x > y \end{cases}$$

$$H_1(x, y) = \begin{cases} 1 & if\ x \leq y \\ \frac{y}{x} & if\ x > y \geq \frac{x}{1+x} \\ 1 - x + xy & if\ y < \frac{x}{1+x} \end{cases}$$

$$H_2(x,y) = \begin{cases} 1-x+xy & if\ x \le y \\ min(\frac{y}{x}, 1-x+xy) & if\ x > y \end{cases}$$

$$H_2(x,y) = \begin{cases} 1-x+xy & if\ y > \frac{x}{1+x} \\ \frac{y}{x} & if\ y \le \frac{x}{1+x} \end{cases}$$

$$H_3(x,y) = \begin{cases} 1 & if\ x \le y \\ 1-x+xy & if\ x > y \end{cases}$$

$$H_4(x,y) = \begin{cases} 1-x+xy & if\ x \le y \\ y & if\ x > y \end{cases}$$

$$H_5(x,y) = \begin{cases} 1 & if\ x \le y \\ max(\frac{y}{x}, 1-x) & if\ x > y \end{cases}$$

$$H_5(x,y) = \begin{cases} 1 & if\ x \le y \\ \frac{y}{x} & if\ x > y \ge x - x^2 \\ 1-x & if\ y < x - x^2 \end{cases}$$

$$H_6(x,y) = \begin{cases} max(1-x,y) & if\ x \le y \\ min(\frac{y}{x}, max(1-x,y)) & if\ x > y \end{cases}$$

$$H_6(x,y) = \begin{cases} \frac{y}{x} & if\ y < x - x^2 \\ 1-x & if\ x - x^2 < y \le 1-x \\ y & if\ 1-x < y \end{cases}$$

$$H_7(x,y) = \begin{cases} max(1-x,y) & if\ x \le y \\ y & if\ x > y \end{cases}$$

$$H_7(x,y) = \begin{cases} 1-x & if\ x \le y \le 1-x \\ y & if\ y < x\ or\ y \ge x \end{cases}$$

$$H_8(x,y) = \begin{cases} 1-x+xy & if\ x \le y \\ max(1-x, min(\frac{y}{x}, 1-x+xy)) & if\ x > y \end{cases}$$

$$H_8(x,y) = \begin{cases} 1-x & if\ y \le x - x^2 \\ \frac{y}{x} & if\ x - x^2 < y \frac{x}{1+x} \\ 1-x+xy & if\ \frac{x}{1+x} < y \end{cases}$$

$$H_9(x,y) = \begin{cases} 1-x+xy & if\ x \le y \\ max(1-x,y) & if\ x > y \end{cases}$$

$$H_9(x,y) = \begin{cases} 1-x & if\ y < x\ and\ y \le 1-x \\ y & if\ 1-x < y < x \\ 1-x+xy & if\ x \le y \le \end{cases}$$

$$H_{10}(x,y) = \begin{cases} 1 & if\ x \le y \\ min(\frac{y}{x}, 1-x+xy) & if\ x > y \end{cases}$$

$$H_{10}(x,y) = \begin{cases} \frac{y}{x} & if\ y \le \frac{x}{1+x} \\ 1-x+xy & if\ \frac{x}{1+x} < y < x \\ 1 & if\ x \le y \end{cases}$$

$$H_{11}(x,y) = \begin{cases} 1 & if\ x \le y \\ min(\frac{y}{x}, max(1-x,y)) & if\ x > y \end{cases}$$

$$H_{11}(x,y) = \begin{cases} \frac{y}{x} & if\ y \leq x - x^2 \\ 1-x & if\ x - x^2 < y < 1-x \\ y & if\ 1-x < y \leq x \\ 1 & if\ x \leq y \end{cases}$$

$$H_{12}(x,y) = \begin{cases} 1 & if\ x \leq y \\ min(1-x+xy, max(\frac{y}{x}, 1-x)) & if\ x > y \end{cases}$$

$$H_{12}(x,y) = \begin{cases} 1-x & if\ y < x - x^2 \\ \frac{y}{x} & if\ x - x^2 \leq y < \frac{x}{1+x} \\ 1-x+xy & if\ \frac{x}{1+x} \leq y < x \\ 1 & if\ x \leq y \end{cases}$$

$$H_{13}(x,y) = \begin{cases} 1-x+xy & if\ x \leq y \\ min(\frac{y}{x}, max(1-x, y)) & if\ x > y \end{cases}$$

$$H_{13}(x,y) = \begin{cases} \frac{y}{x} & if\ y \leq x - x^2 \\ 1-x & if\ x - x^2 < y \leq 1-x\ and\ y < x \\ y & if\ 1-x < y < x \\ 1-x+xy & if\ x \leq y \end{cases}$$

$$H_{14}(x,y) = \begin{cases} 1-x+xy & if\ x \leq y \\ 0 & if\ x > y \end{cases}$$

$$H_{15}(x,y) = \begin{cases} max(1-x, y) & if\ x \leq y \\ 0 & if\ x > y \end{cases}$$

$$H_{15}(x,y) = \begin{cases} 0 & if\ y < x \\ 1-x & if\ x \leq y \leq 1-x \\ y & if\ x \leq y\ and\ 1-x < y \end{cases}$$

4 Illustrating Example

In order to illustrate our approach, let us describe a simple example coming from the domain of train traffic control. We consider the following situation: a train B waits at a certain station for a train A to arrive in order to allow some passengers to change train A to train B. Now, a conflict arises when the train A is late. In this situation, the following alternatives can be taken into consideration:

1. Train B waits for train A to arrive. In this case, train B will depart with delay.
2. Train B departs in time. In this case, passengers disembarking train A have to wait for a later train.
3. Train B departs in time, and an additional train is employed for the train A passengers.

To make a decision, several inner conditions have to be taken into account such as the delay period, the number of passengers changing trains, etc. The discussion regarding an optimal solution to the problem of divergent aims such as: minimization of delays throughout the traffic network, warranty of connections for the customer satisfaction, efficient use of expensive resources, etc. is disregarded at this point. In order to describe the traffic conflict, we propose to consider the following three IF-THEN fuzzy rules:

- r_1: IF s_2 OR s_3 THEN s_6
- r_2: IF s_1 AND s_4 AND s_6 THEN s_7
- r_3: IF s_4 AND s_5 THEN s_8

where:

- s_1: 'Train B is the last train in this direction today',
- s_2: 'The delay of train A is huge',
- s_3: 'There is an urgent need for the track of train B',
- s_4: 'Many passengers would like to change for train B',
- s_5: 'The delay of train A is short',
- s_6: '(Let) train B depart according to schedule',
- s_7: 'Employ an additional train C (in the same direction as train B)', and
- s_8: 'Let train B wait for train A'.

In the further considerations we accept the following assumptions:

1. The logical operators OR, AND we interpret as *max* and *min* fuzzy operators, respectively.
2. The statements s_1, s_2, s_3, s_4 and s_5 we assign the fuzzy values 0.6, 0.4, 0.7, 0.5 and 1, respectively.
3. The truth-values of rules r_1, r_2 and r_3 are equal to 0.6, 0.7, 0.8, respectively.
4. The threshold values for three rules are equal to 0.1.
5. Each of rules r_1, r_2 and r_3, firstly, we interpret as the Łukasiewicz implication.

Assessing the statements from s_1 up to s_5, we observe that the rule r_1 and r_3 can be fired (activated). Firing these rules according to the above assumptions allows computation of the support for the alternatives in question. In this way, the possible alternatives are ordered with regard to the preference they achieve from the knowledge base. This order forms the basis for further examinations and simulations and, ultimately, for the dispatching proposal. If one chooses a sequence of rules r_1, r_2 then they obtain the final value, corresponding to the statement s_7, equal to 0. In the other case (i.e., for the rule r_3 only), the final value, this time corresponding to the statement s_8, equals 0.3. Secondly, if we interpret these three rules as the Reichenbach implications, and if we choose the same sequences of transitions as above we obtain the final values for the statements s_7 and s_8 equal to 0.3, 0.6, respectively. Thirdly, if we execute the similar simulation of approximate reasoning for three rules considered above and, if we interpret the rules as the Kleene-Dienes implications we obtain the final values for s_7 and s_8 equal to 0.7, 0.8, respectively. This example shows clearly that different interpretations for the rules may lead to quite different decision results. Nevertheless, they are conformable with the relationships between inverted fuzzy implication functions presented above (see, for example, items 3, 5-8, 13-14). Choosing a suitable interpretation for fuzzy implications we may try to obtain the optimal final values for decisions s_7 and s_8. The rest in this case certainly depends on the experience of the decision support system designer to a significant degree.

5 Conclusion and Further Research

In this paper, we presented an approach to finding the fuzzy implication which has for example the largest or the smallest truth value of the consequent when the truth value of the antecedent and the truth value of the implication are given.

Our next problem to be solved is how to choose the fuzzy implication so that the truth value of the antecedent is optimized at the given truth value of implication's conclusion and given value of truth of implication.

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