Nonlinear Dynamic Model of a Microeconomic System with Different Reciprocity and Expectations Types of Firms: Stability and Bifurcations

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Abstract. This paper analyzes the dynamic interaction between selfish and reciprocity firms in the market of homogeneous product. The decisions of both types of firms in respect of their output strategies are investigated under naive, adaptive and generalized expectations. The standard postulate for competitive firms' model has been extended by the assumption that there is a share of reciprocity firms which, unlike selfish firms, maximize both private and social benefits as consumer surplus. It has been proved that the unique Nash equilibrium is stable for all affordable values of parameters in the model with adaptive expectations, and is unstable for the model with naive expectations at sufficiently large number of firms in the market. A special desktop application has been created for animation of model trajectory and demonstration of stable quantity trajectories and bifurcation diagrams of firms' output. Naive expectations of two-thirds of firms result in a state of dynamic chaos in the market leading to degeneration of the existing competition model between the two types of firms. The crucial factor which ensures the stable equilibrium in the market and the ability to predict firms' output is the adaptive approach which takes into account the adaptive expectations of firms planning their product quantity.

Keywords: microeconomic system, reciprocity, naïve expectations, adaptive expectations, consumer surplus, stability, bifurcation.

Key Terms: DynamicSystem, DesktopApplication, NashEquilibrium, Expectations

1 Introduction

In recent years the researchers are renouncing the assumption of perfect rationality as unconditional basis of economic agents' behavior. The neoclassical 'rational man' does not exist in reality; economic agents act according to established rules, without being fully informed and maximizing their own utility [1].

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Karl Polanyi identified the alternative economic organization where social norms are not generated by economic self-interest of the individual. This network of reciprocal relations is based on mutual economic cooperation of efforts and resources between the members of non-economic network, dominated by cultural norms rather than market laws. Under reciprocity relations the exchange donor and recipient can be transposed. So this is a symmetrical relationship of gifts exchange between members of horizontal social networks [2].

This relationship is not regulated by formal institutions but based on informal commitments giving moral right to mutual help and reciprocal exchange on sustainable basis in the long run period. But this is a relationship with minimal risk for participants and the penalty of loss of social capital (reputation and trust) and social isolation. Society supports the stability of the exchange to ensure their survival during crises and wars. Reciprocity is not altruism which does not create reciprocal obligations in quantitative, qualitative or time respects, just vague commitment (e.g., you give, if you can).

Actually, reciprocity relations, commodity exchange and hierarchical subordination exist at the same time. But it is reciprocity that underlies most decentralized corrections of diverse shortcomings and failures of markets and firms. These relations are long-run factors of economic efficiency; they set most social obligations of firms towards individuals without government intervention. No society can exist without reciprocal relationship [3].

Reciprocity or social responsibility implies that the firms not only pursue their selfish goal of increasing profits, but are also ready to sacrifice some of their own profits for the benefit of consumers without direct compensation for it by the state [1]. Such targets can be stipulated by the firms' desire to get stable profits in the long run rather than maximal short-run profits. Such forward-thinking firms-reciprocators are considered in the model of this paper. Their objective function is a weighted average of the profits and consumer surplus of their market segment.

The real economic processes make a clear demonstration that neoclassical "rational man" is not their subject. In real economy "optimal imperfect decisions" are taken by simple and non-expensive calculations, well adapted to frequent repetitions, to evolution; it is more efficient for perfectly rational firm to perform multiple experiments with quantity to estimate the demand function rather than search for nonrecurrent, instantaneous achieving of equilibrium. New paradigm of nonlinear economics is a mix of qualitative theory of nonlinear dynamical system, optimal control theory, game theory, and theory of stochastic processes [4, 5].

The evolutionary approach and analysis of the dynamics allow to explain why one type of firm ousts another from the market, why sometimes the economic system is stable, but in other cases is unstable [6, 7]. If the system has multiple equilibria, the dynamics and evolution is the selection mechanism of best equilibrium according to certain criteria [8]. The evolutionary process is analogous to social learning. An example of its application is the pricing mechanisms for auctions that occur in agents social networks, e-commerce and trade through the Internet [9, 10].

The study of the evolution of the markets with the strategic interaction usually uses the following assumptions: (a) two firms or two types of similar firms operate in the market; (b) firms produce homogeneous goods in quantities of $x_1(t)$ and $x_2(t)$; (c) no firm knows the rivals' quantities; (d) the firms seek to predict the output of the competitors using the adaptive scheme.

Planning of quantity for the following period firms resolve optimization problem: $Max\Pi_i(x_i; x_j^e(t+1))$, where Π_i is the objective function of firm *i*, $x_j^e(t+1)$ - expected quantity of a competitor *j* ($i, j = 1, 2$).

Examples of bounded rationality of firms are: ignoring the impact of competitors' actions on their own output (local monopoly approximation LMA), naive expectation (assumption of unchangeable behavior of competitors for a long time and using $x_j(t)$ instead $x_j^e(t+1)$). [8] Of course both decision making approaches (adaptive and naive, bounded rational) coexist in the market with a certain probability.

Analysis of nonlinear oligopoly with heterogeneous players reveals that a higher degree of product differentiation may destabilize the Cournot-Nash equilibrium. Authors showed that a cascade of flip bifurcation may lead to periodic cycles and chaotic motions [11]. Stability conditions of Nash equilibrium and complex dynamics are also studied for heterogeneous duopoly with isoelastic demand function. For such heterogeneous players a cascade of flip bifurcation leads to periodic cycles and chaos and the Neimark-Sacker bifurcation generates attractive invariant closed curve [12].

Such scheme serves as the basis for mathematical model of this paper, which distinguishes from the other models in that: (a) firms use more than one way of decision-making, and combine different ones; (b) except their own selfish interests, firms take into account social ones.

The **paper goal** is to consider the impact of naive, adaptive and generalized expectations of egoist and reciprocator firms on stability of equilibrium and the conditions of transition to dynamic chaos in the numerical experiment using a specially designed desktop application.

The paper is organized as follows: part 2 describes two-dimensional market model with naïve and adaptive expectations; part 3 is devoted to dynamics model in general case; part 4 demonstrates C#-application model for numerical investigation; part 5 concludes.

2 Two-Dimensional Market Model

We consider the market of homogeneous product, which consists of n firms, including *k* identical firms-reciprocators, each of them producing *x* units of product and *n – k* identical selfish firms, each of them producing y units of product. Thus, the industry quantity of the two types of firms is $Q = k \cdot x + (n-k) \cdot y$. Product price P in the market is given by the inverse market demand function $P = P(Q) = \dfrac{b}{Q}$ ($b > 0$).

The objective function of a firm-egoist is profit $\pi_Y = (P - v) \cdot y$, where v is the firm's costs per unit in the market. Firm-reciprocator maximizes both its own profit $\pi_X = (P - v) \cdot x$ and consumer surplus CS of its own market segment:

$CS = \frac{\gamma}{k}\left(\int_{\varepsilon}^{Q} P(q)dq - PQ\right)$, where γ is the parameter defining the segment of the market, which the reciprocator firm believes its own and optimizes it ($0 < \gamma \le k$), ε is the minimal technologically possible quantity of product. Then

$$CS = \frac{\gamma}{k}\left(b \cdot \ln\left(\frac{Q}{\varepsilon}\right) - \frac{b}{Q}\cdot Q\right) = \frac{b\gamma}{k}\left(\ln\left(\frac{Q}{\varepsilon}\right) - 1\right) = \frac{b\gamma}{k}\ln\left(\frac{Q}{\hat{\varepsilon}}\right),$$

where $\hat{\varepsilon} = \varepsilon e$. The specific choice of ε does not affect the dynamics of the model because the objective function, as any potential, is set up to an arbitrary constant accuracy, so further we will write ε instead of $\hat{\varepsilon}$. Then the objective function of firm-reciprocator is:

$$\Pi_X = \alpha(P - v)x + (1 - \alpha)CS = \alpha(P - v)x + (1 - \alpha)\frac{b\gamma}{k}\ln(\frac{Q}{\varepsilon}),$$

where α is share of private interest PI (reciprocator's profit), $1-\alpha$ is share of social interest (responsibility) SR (consumer surplus from its own market segment) in the objective function.

2.1 Dynamics Model Equations with Naive Expectations

Consider the dynamics of this two-dimensional model in discrete time t = 0, 1,...; where x_t, y_t are the outputs at time t of reciprocator and egoist firm, respectively. On the basis of these values at time t each firm finds the optimal value for its own production quantity in the next moment $t+1$, maximizing its objective function. It distinguishes this model, in which the firm responds to changes in output of both their and other types of firms from traditional competition models, where one type of firm responds to changes in other types only. So each selfish firm is looking for such value of y_{t+1} at which it maximizes its own profits, suggesting that *SR* firms and the other $n-k-1$ *PI* firms leave their quantities unchanged:

$$\pi_Y = \left(\frac{b}{y_{t+1} + k\cdot x_t + (n-k-1)\cdot y_t} - v\right)\cdot y_{t+1}. \qquad (1)$$

Obviously, the maximum point for y_{t+1} is found from the condition $\frac{\partial \pi}{\partial y_{t+1}} = 0$, whence:

$$v(y_{t+1} + kx_t + (n-k-1)y_t)^2 = b(kx_t + (n-k-1)y_t). \qquad (2)$$

From equation (2) we obtain the response function of the *PI* firm:

$$y_{t+1} = \sqrt{\frac{b}{v}(kx_t + (n-k-1)y_t} - kx_t - (n-k-1)y_t. \qquad (3)$$

Similarly, firm-reciprocator finds such value of x_{t+1} at which the maximum value of its objective function is:

$$\Pi_X = \alpha\cdot\left(\frac{b}{x_{t+1} + (k-1)\cdot x_t + (n-k)\cdot y_t}\cdot x_{t+1} - vx_{t+1}\right) + (1-\alpha)\cdot\frac{b\gamma}{k}\ln\left(\frac{x_{t+1} + (k-1)x_t + (n-k)y_t}{\varepsilon}\right) \qquad (4)$$

.

Here the maximum point for x_{t+1} is found from the condition $\frac{\partial \Pi_X}{\partial x_{t+1}} = 0$:

$$\frac{\partial \Pi_X}{\partial x_{t+1}} = \alpha\left(\frac{b(x_{t+1}+(k-1)x_t+(n-k)y_t)-bx_{t+1}}{(x_{t+1}+(k-1)x_t+(n-k)y_t)^2}x_{t+1}-v\right)+(1-\alpha)\cdot\frac{b\gamma}{k}\cdot\frac{1}{x_{t+1}+(k-1)x_t+(n-k)y_t}=0$$

Further, without loss of generality, we assume here $\gamma = 1$, otherwise we redefine the share of profit as $\hat{\alpha} = \frac{\alpha}{\alpha+(1-\alpha)\gamma}$. Then

$$v(x_{t+1}+(k-1)x_t+(n-k)y_t)^2 = b((k-1)x_t+(n-k)y_t)+\frac{1-\alpha}{\alpha}\frac{b}{k}(x_{t+1}+(k-1)x_t+(n-k)y_t) \quad (5)$$

.

Assuming $z = x_{t+1}+(k-1)x_t+(n-k)y_t$, we present (5) as:

$$z^2 = \frac{b}{v}((k-1)x_t+(n-k)y_t)+\frac{1-\alpha}{\alpha}\frac{b}{vk}z\,.$$

Hence, in view of (3), we obtain a system of dynamics equations of this model:

$$\begin{cases} x_{t+1} = \sqrt{\frac{b}{v}((k-1)x_t+(n-k)y_t)+\left(\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b}{vk}\right)^2}-(k-1)x_t-(n-k)y_t+\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b}{vk}, \\ y_{t+1} = \sqrt{\frac{b}{v}(kx_t+(n-k-1)y_t}-kx_t-(n-k-1)y_t. \end{cases} \quad (6)$$

2.2 Equilibrium Conditions for the Model with Naive Expectations

In the Nash equilibrium point $x_{t+1}=x_t=x$, $y_{t+1}=y_t=y$ for all t = 0, 1, … . Therefore, at this point, by (2) and (5) we obtain:

$$\left(kx+(n-k)y\right)^2 = \frac{b}{v}\left(kx+(n-k-1)y\right) = \frac{b}{v}((k-1)x+(n-k)y)+\frac{1-\alpha}{\alpha}\frac{b}{vk}(kx+(n-k)y) \quad (7)$$

.

From the last equation we obtain $x-y=\frac{1-\alpha}{\alpha}(x+\frac{n-k}{k}y)$, whence it follows that $\frac{2\alpha-1}{\alpha}x = (1+\frac{1-\alpha}{\alpha}\frac{n-k}{k})\cdot y$, i.e. the response functions of both types of firms are respectively:

$$x = \frac{\alpha k+(1-\alpha)\cdot(n-k)}{(2\alpha-1)\cdot k}\cdot y, \qquad y = \frac{(2\alpha-1)\cdot k}{\alpha k+(1-\alpha)\cdot(n-k)}\cdot x \quad (8)$$

To calculate the coordinates of a fixed point, we substitute the expression of y through x in the first equation (7). Thus the following proposition is proved.

Proposition 1. There is unique Nash equilibrium point in a dynamic system (6):

$$\begin{cases} x^* = \frac{b}{vn}\left(1 - \frac{2\alpha - 1}{\alpha n}\right)\left(1 + \frac{1-\alpha}{\alpha}\frac{n-k}{k}\right), \\ y^* = \frac{b}{vn} \cdot \frac{2\alpha - 1}{\alpha} \cdot \left(1 - \frac{2\alpha - 1}{\alpha n}\right). \end{cases} \tag{9}$$

However, is this point stable?

Proposition 2. For any set b, $v > 0$ and α ($0 \le \alpha \le 1$) Nash equilibrium (9) is unstable for sufficiently large number of firms n if $\left|\frac{k}{n}\right| > \varepsilon$ and $\left|\frac{k}{n} - \frac{3}{4}\right| > \varepsilon$ for any $\varepsilon > 0$.

The destabilizing role of number of players n is well known for the evolution of firms' strategies in oligopoly games [8]. However, in this case, according to calculations, point (9) is unstable even at $n \ge 5$.

Proof. We show that in dynamic system (6) at Nash equilibrium point (9) modulus of Jacobian J is greater than 1: |det J|>1. This implies that at least one eigenvalue of the Jacobian is greater than 1 in absolute value, which means instability of the fixed point (9). Here, the Jacobian of the system (6):

$$J = \begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_{t+1}}{\partial x_t} & \frac{\partial x_{t+1}}{\partial y_t} \\ \frac{\partial y_{t+1}}{\partial x_t} & \frac{\partial y_{t+1}}{\partial y_t} \end{pmatrix}.$$

$$J_{xx} = \frac{\frac{b}{v}(k-1)}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - (k-1), \quad J_{xy} = \frac{\frac{b}{v}(n-k)}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - (n-k),$$

$$J_{yx} = \frac{\frac{b}{v}k}{2\sqrt{\frac{b}{v}(kx_t + (n-k)y_t}} - k, \quad J_{yy} = \frac{\frac{b}{v}(n-k-1)}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k-1)y_t}} - (n-k-1),$$

where $d = \frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b}{vk}$, then $\det J = J_{xx} \cdot J_{yy} - J_{xy} \cdot J_{yx} =$

$$= (1-n) \cdot \left(\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - 1\right) \cdot \left(\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k-1)y_t}} - 1\right).$$

But for point (9) in the denominator $\frac{b}{v}((k-1)x^* + (n-k-1)y^*) =$

$$=\left(\frac{b}{v}\right)^2\left[\frac{1-\alpha}{\alpha}(1-\frac{k}{n})+\frac{2\alpha-1}{\alpha}(1-\frac{k}{n})\right]+o(\frac{1}{n})=\left(\frac{b}{v}\right)^2\cdot(1-\frac{k}{n})+o(\frac{1}{n}),$$

where $o(\frac{1}{n})\to 0$ for $n\to\infty$. Similarly, we obtain for the second denominator:

$$\frac{b}{v}((k-1)x^*+(n-k)y^*)+d^2=\left(\frac{b}{v}\right)^2\cdot(1-\frac{k}{n})+o(\frac{1}{n}).$$

But by the data $\left|\frac{k}{n}-\frac{3}{4}\right|>\varepsilon$ at a certain $\varepsilon>0$, which guarantees that the factors $\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k)y_t+d^2}}-1$ and $\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k-1)y_t}}-1$ do not equal zero for all possible n, k, b, v> 0 and α (0≤α≤1), Q.E.D.

2.3 Dynamic Model Equations with Adaptive Expectations

Since all selfish firms are assumed as identical, it is natural to suggest that they have the same planning at moment t, so their production quantities y_{t+1} at moment $t+1$ will be equal too. Given these expectations, each selfish firm is looking for such value y_{t+1} at which it obtains the highest profit, suggesting that production quantity of SR firms will remain unchanged:

$$\pi_Y=\left(\frac{b}{kx_t+(n-k)y_{t+1}}-v\right)\cdot y_{t+1}. \tag{10}$$

Obviously, the maximum point for y_{t+1} is found from the condition $\frac{\partial\pi_Y}{\partial y_{t+1}}=0$, which gives us:

$$\left(kx_t+(n-k)y_{t+1}\right)^2=\frac{b}{v}kx_t. \tag{11}$$

Then $kx_t+(n-k)y_{t+1}=\sqrt{\frac{b}{v}kx_t}$, from here response function of PI firms is:

$$(n-k)y_{t+1}=\sqrt{\frac{b}{v}kx_t}-kx_t. \tag{12}$$

Similarly, firm-reciprocator naturally expects that the quantity of production of all these firms at moment $t+1$ would be the same. Based on this expectation, each firm-reciprocator finds the value of x_{t+1} at which the objective function is maximal, assuming that the output of PI firms does not change:

$$\Pi_X=\alpha\left(\frac{b}{kx_{t+1}+(n-k)y_t}x_{t+1}-vx_{t+1}\right)+(1-\alpha)\frac{b\gamma}{k}\ln\left(\frac{kx_{t+1}+(n-k)y_t}{\varepsilon}\right). \tag{13}$$

Here we can find the maximum point for x_{t+1} from the condition $\frac{\partial \Pi_X}{\partial x_{t+1}} = 0$, hereof:

$$(kx_{t+1} + (n-k)y_t)^2 = \frac{b}{v}(n-k)y_t + \frac{b\gamma}{v}\frac{1-\alpha}{\alpha}\cdot(kx_{t+1} + (n-k)\cdot y_t)\,. \tag{14}$$

Let $z = kx_{t+1} + (n-k)\cdot y_t$, represent (14) as:

$$\left(z - \frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha}\right)^2 = \frac{b}{v}(n-k)\cdot y_t + \left(\frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha}\right)^2.$$

Hence $z - \frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha} = \sqrt{\frac{b}{v}(n-k)y_t + (\frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha})^2}$. Thus, in view of (12), we obtain a system of dynamics equations of the model, taking into account the forecast:

$$\begin{cases} kx_{t+1} = \sqrt{\frac{b}{v}(n-k)y_t + (\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b\gamma}{v})^2} - (n-k)y_t + \frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b\gamma}{v}, \\ (n-k)y_{t+1} = \sqrt{\frac{b}{v}kx_t} - kx_t. \end{cases} \tag{15}$$

2.4 Equilibrium Conditions for the Model with Adaptive Expectations

In the Nash equilibrium point $x_{t+1}=x_t=x$, $y_{t+1}=y_t=y$ for all $t = 0,1,\ldots$. Therefore, at this point in view of (11) and (14) we get:

$$(kx + (n-k)y)^2 = \frac{b}{v}kx = \frac{b}{v}(n-k)y + \frac{b\gamma}{v}\frac{1-\alpha}{\alpha}\cdot(kx + (n-k)\cdot y)\,. \tag{16}$$

From the second equation we get $x - \frac{1-\alpha}{\alpha}\gamma x = \frac{n-k}{k}\cdot(1 + \frac{1-\alpha}{\alpha}\gamma)y$, whence response functions for selfish and reciprocator firms are, respectively:

$$y = \frac{k}{n-k}\frac{\alpha-(1-\alpha)\gamma}{\alpha+(1-\alpha)\gamma}\cdot x \qquad x = \frac{n-k}{k}\frac{\alpha+(1-\alpha)\gamma}{\alpha-(1-\alpha)\gamma}\cdot y \tag{17}$$

To calculate the coordinates of the fixed point, we substitute this expression y in terms of x at first equation (16):

$$\left(kx + (n-k)\frac{k}{n-k}\frac{\alpha-(1-\alpha)\gamma}{\alpha+(1-\alpha)\gamma}\cdot x\right)^2 = \frac{b}{v}kx \qquad (kx)^2\cdot\left(\frac{\alpha-(1-\alpha)\gamma}{\alpha+(1-\alpha)\gamma} + 1\right) = \frac{b}{v}kx$$

Hence, we obtain:

Proposition 3. There is unique Nash equilibrium point in the dynamic system (15) with adaptive expectations:

$$\begin{cases} x^* = \frac{b}{vk}(\frac{\alpha+(1-\alpha)\gamma}{2\alpha})^2, \\ y^* = \frac{b}{v(n-k)}\frac{\alpha^2-((1-\alpha)\gamma)^2}{(2\alpha)^2}. \end{cases} \tag{18a}$$

As before, without loss of generality, let $\gamma = 1$, otherwise we can override the share of profit as $\hat{\alpha} = \dfrac{\alpha}{\alpha + (1-\alpha)\gamma}$. At $\gamma = 1$ system (18) takes the form:

$$\begin{cases} x^* = \dfrac{b}{vk}\left(\dfrac{1}{2\alpha}\right)^2, \\ y^* = \dfrac{b}{v(n-k)} \dfrac{2\alpha - 1}{(2\alpha)^2} = \dfrac{b}{v(n-k)} \dfrac{1}{2\alpha}\left(1 - \dfrac{1}{2\alpha}\right). \end{cases} \quad (18b)$$

Proposition 4. The equilibrium point (18) is stable for all possible values of the parameters.

Proof. To prove the stability of dynamic system (15) in Nash equilibrium point (18) it is necessary and sufficient to demonstrate that for Jacobian *J* of this system in (18) the following conditions named after Shur were satisfied:

$$\begin{cases} 1 + tr\,J + \det J > 0, \\ 1 - tr\,J + \det J > 0, \\ 1 - \det J > 0. \end{cases}$$

Here, the Jacobian of system (15)

$$J = \begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_{t+1}}{\partial x_t} & \dfrac{\partial x_{t+1}}{\partial y_t} \\ \dfrac{\partial y_{t+1}}{\partial x_t} & \dfrac{\partial y_{t+1}}{\partial y_t} \end{pmatrix},$$

obviously, $J_{xx} = J_{yy} = 0$ where $trJ = J_{xx} + J_{yy} = 0$. Thus, to test Shur conditions it is sufficient to establish that $\det J < 1$. But at point (18) $y^* = \dfrac{b}{v(n-k)} \dfrac{\alpha^2 - ((1-\alpha)\gamma)^2}{(2\alpha)^2}$ and therefore

$$kJ_{xy} = \frac{\frac{b}{v}(n-k)}{2\sqrt{\frac{b}{v}(n-k)y^* + d^2}} - (n-k) = \frac{n-k}{2\sqrt{\frac{\alpha^2}{4\alpha^2}}} - (n-k) = 0$$

Consequently, $\det J = J_{xx} \cdot J_{yy} - J_{xy} \cdot J_{yx} = 0$, Q.E.D.

The price of product P in the market is given by the inverse market demand function $P = P(Q) = \dfrac{b}{Q}$ ($b > 0$), and the price is not less than a cent, i.e. $P \geq 0.01$. Therefore, the product quantity of each firm-reciprocator is $x \leq \dfrac{100b}{k}$. Similarly, the product quantity of each selfish firm is $y \leq \dfrac{100b}{n-k}$.

Corollary. The trajectories of the dynamical system (15) converge to a Nash equilibrium (18) for any initial values $x_0 \leq \dfrac{100b}{k}$, $y_0 \leq \dfrac{100b}{n-k}$.

3 Dynamic Model Equations in a General Case

Suppose that in planning under the given market model adaptive expectations are used with probability p, naïve ones - with probability $q = 1 - p$. Then the profit function for a typical (representative) firm-egoist has the form:

$$\pi_Y = \left(\frac{b}{y_{t+1} + kx_t + p(n-k-1)y_{t+1} + q(n-k-1)y_t} - v \right) \cdot y_{t+1}, \tag{19}$$

and the objective function for the representative firm-reciprocator

$$\Pi_X = \alpha \cdot \left(\frac{b}{x_{t+1} + p(k-1)x_{t+1} + q(k-1)x_t + (n-k)y_t} x_{t+1} - vx_{t+1} \right) + , \\ +(1-\alpha)\frac{b\gamma}{k} \ln\left(\frac{x_{t+1} + p(k-1)x_{t+1} + q(k-1)x_t + (n-k)y_t}{\varepsilon}\right). \tag{20}$$

Obviously, for $p = 0$ ($q = 1$) objective functions π_Y and Π_X are consistent with the results of naive model (1) and (4), for $p = 1$ ($q = 0$), they are consistent with the results of the adaptive model (10) and (13) respectively. Let us assume

$z_{\pi_X} = y_{t+1} + kx_t + (n-k-1)(py_{t+1} + qy_t)$ $\quad z_{\Pi_X} = x_{t+1} + (n-k)y_t + (k-1)(px_{t+1} + qx_t)$

In this notation $\pi_Y = \left(\frac{b}{z_{\pi_X}} - v \right) \cdot y_{t+1}$; $\Pi_X = \alpha \cdot \left(\frac{b}{z_{\Pi_X}} x_{t+1} - vx_{t+1} \right) + (1-\alpha)\frac{b\gamma}{k} \ln z_{\Pi_X}$.

Then the point y_{t+1} of maximum profit function π_X is found from the condition $\frac{\partial \pi_X}{\partial y_{t+1}} = 0$, here

$$z^2_{\pi_X} = \frac{b}{v}(kx_t + q(n-k-1)y_t \tag{21}$$

whence

$$y_{t+1} \cdot (1 + p(n-k-1)) = \sqrt{\frac{b}{v}(kx_t + q(n-k-1)y_t} - k \cdot x_{t+1} - (n-k-1) \cdot q \cdot y_t \tag{22}$$

The maximum point x_{t+1} for the objective function Π_X is found from the first order condition $\frac{\partial \Pi_X}{\partial x_{t+1}} = 0$. Forth without loss of generality we assume here $\gamma = 1$, otherwise as above we redefine the share of profit as $\tilde{\alpha} = \frac{\alpha}{\alpha + (1-\alpha)\gamma}$. Then

$$z^2_{\Pi_X} = \frac{b}{v} \cdot ((n-k) \cdot y_t + (k-1) \cdot qx_t) + \frac{1-\alpha}{\alpha} \frac{b(1 + p(k-1))}{vk} z_{\Pi_X} \tag{23}$$

Thus, in view of (22), we obtain the dynamics model of equations system of in the general case:

$$\begin{cases} (1+p(k-1))x_{t+1} = \sqrt{\frac{b}{v}w_x + d^2} - w_x + d, \\ (1+p(n-k-1))y_{t+1} = \sqrt{\frac{b}{v}w_y} - w_y, \end{cases} \quad (24)$$

where $\begin{cases} w_x = q(k-1)x_t + (n-k)y_t, \\ w_y = kx_t + q(n-k-1)y_t. \end{cases}$

3.1 Equilibrium Conditions in a General Case

Since Nash equilibrium point is $x_{t+1}=x_t=x$, $y_{t+1}=y_t=y$ for all $t = 0,1,...$, then at this point in view of (21) and (23) we obtain:

$$z_{\pi_Y} = z_{\Pi_X} = (kx+(n-k)y)^2 = \frac{b}{v}(kx+q(n-k-1)y) = \frac{b}{v}((k-1)qx+(n-k)y) + \\ + \frac{1-\alpha}{\alpha}\frac{b(1+p(k-1))}{vk}\cdot(kx+(n-k)\cdot y) \quad (25)$$

From the second equation we get:

$$y\left[(n-k)+\frac{1-\alpha}{\alpha}\frac{1+p(k-1)}{k}(n-k)-q(n-k-1)\right] = x\left[k-(k-1)q-\frac{1-\alpha}{\alpha}\frac{1+p(k-1)}{k}k\right]$$

Thus,

$$y\left[p\frac{n-k}{\alpha}+q(1+\frac{1-\alpha}{\alpha}\cdot\frac{n-k}{k})\right] = x\left[(1+p(k-1))\cdot\frac{2\alpha-1}{\alpha}\right],$$

where response function in this case

$$\frac{x}{y} = G = \frac{p(n-k)+q(\alpha+(1-\alpha)\frac{n-k}{k})}{(2\alpha-1)(1+p(k-1))}. \quad (26)$$

To calculate the coordinates of the fixed point we substitute from (26) expression for $\frac{x}{y}$ in the first equation of (25) $y^2(kG+(n-k))^2 = \frac{b}{v}y(kG+q(n-k-1))$. Hence

Proposition 5. There is unique Nash equilibrium point in a general dynamical system (24):

$$y^* = \frac{\frac{b}{v}(kG+q(n-k-1))}{(kG+(n-k))^2} \qquad x^* = Gy^* = \frac{\frac{b}{v}(k+q(1/G)(n-k-1))}{(k+(1/G)(n-k))^2} \quad (27)$$

where the function $G = G(p,q,n,k,\alpha)$ is given in (26).

Proposition 6. For $p=0$ ($q=1$) the equilibrium point $(x^*;y^*)$ coincides with point (9) of a dynamic system with naive expectations. When $p=1$ ($q=0$) the equilibrium point coincides with point (18b) of the dynamic system with adaptive expectations.

4 $C^{\#}$ - Application Model for Numerical Investigation

$C^{\#}$ window application *Model* has been created specifically for the numerical investigation of the model of this paper, using a graphical interface of $C^{\#}$ system libraries System.Drawing and System.Windows.Forms. Note that all the calculations associated with the model, are localized in the method *calc* of the application *Model* that makes it easy to modify the equations of the model and use the *Model* to study the other two-dimensional dynamical systems. Fig. 1 shows the application window.

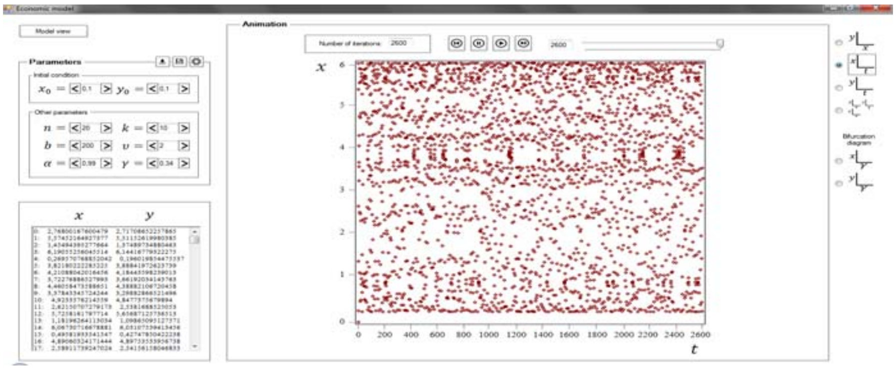

Fig. 1. Application *Model* for two-dimensional model

The right side presents 6 kinds of graphs displayed by the application; their examples are set forth in the paper. Selected switch indicates that here the graph of trajectory $x(t)$ is selected. On the left side counters allow us to specify the parameters of the model and the initial values of the trajectory. After their setting the calculation results of the iterations' coordinates below and their image in the center of the window. This displays an animation of a selected path, the number of iterations been set on the scroll bar above. Pressing the button *Model view* left displays information about the model, its equations and parameter information.

4.1 Numerical Experiment: from Stability to Chaos with Increasing of Naive Expectations

With the increasing probability of naive expectations q, that is with decreasing p, the market becomes unstable, evolving from simple dynamics (15) with a single stable equilibrium point to the unpredictable behavior of system (6). From the proof of Proposition 2 it follows that the market volatility is proportional to the number n of firms in the market. Therefore, for fixed q market instability increases with increasing n. Thus, model (24) has two parameters: the number of firms n and the probability of a naive approach q, whose growth leads to instability. The transition from stability to chaos is the same in both cases. Consider this transition for parameter q.

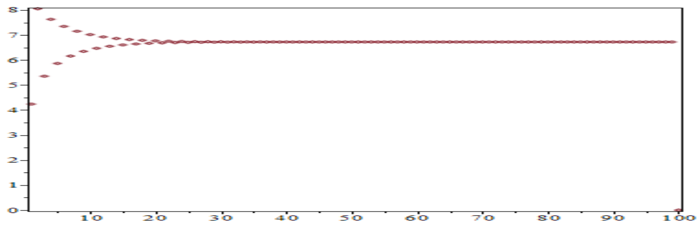

Fig. 2. Quantity trajectory of selfish firm under probability of naive expectations $q = 0.5$

Let n n=20, k=6, b=200, v=2, α=0.9, q=0.5. The trajectory of the dynamical system (24) with the following parameters and the initial point x_0=0.1, y_0=0.1 is shown in the following figures 2 and 3. In Fig. 2 on the x-axis of the system are given iterations of system (24) from m = 1 to m = 100, on the y-axis – corresponding quantity product of selfish firm y_m.

As we can see from the graph, the path quickly converges to the equilibrium value $y^*\approx 2.488$. The graph for the trajectory of firm-reciprocator x_m on y-axis is similar. The equilibrium value of x* is about 6.72. Let us consider the graph of the trajectory for the same parameters except q. Now q = 0.55 (Fig. 3).

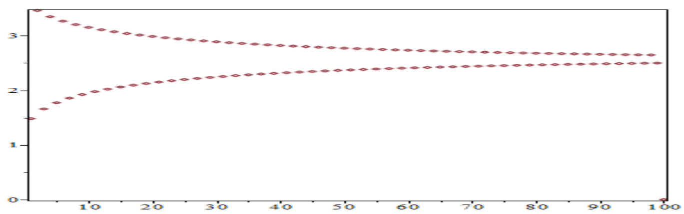

Fig. 3. Quantity trajectory of selfish firm under probability of naive expectations $q = 0.55$

It still has stable Nash equilibrium, but 100 iterations does not suffice for convergence. Further, let q = 0.6 (Fig. 4).

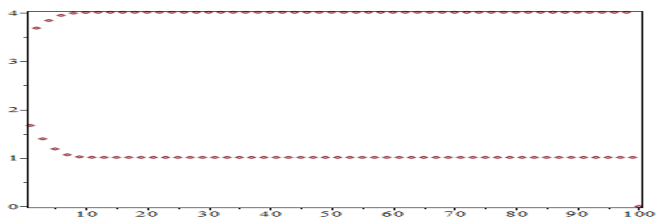

Fig. 4. Bifurcation quantity trajectory of selfish firm under probability of naive expectations $q = 0.6$

As we can see, bifurcation occurred, and instead of equilibrium point there was a steady cycle, where values of y_m are approaching the point of $y^*\approx 4$ for even m and the point of $y^*\approx 1$ for odd m. By doubling the lag between iterations only even or only odd iterations will be considered, and thus either point y*≈4, or y*≈1 respectively would be the equilibrium steady state.

Stable cycle has four cycles for q=0.64 (fig. 5). There was a new cycle doubling bifurcation. Calculations show that with increasing parameter q doubling bifurcation cycle continues, following Sharkovskii's scale. According to this scale, when q≈0.675 there is the state of dynamic chaos (fig. 6). Similarly, the graph of product x_m on y-axis by firm-reciprocator looks like trajectory of a selfish firm.

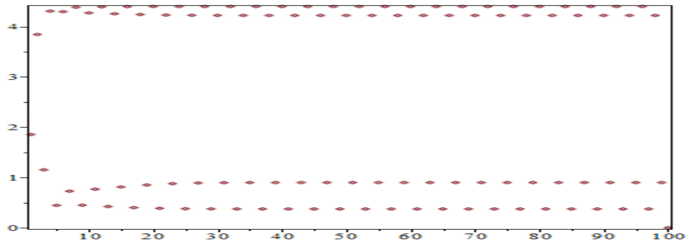

Fig. 5. Doubling bifurcation cycle of quantity by selfish firm under probability of naive expectations $q = 0.64$

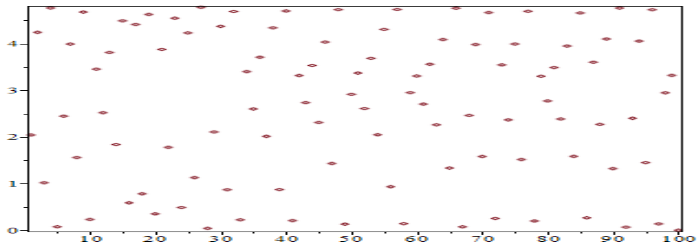

Fig. 6. The state of dynamic chaos of quantity by selfish firms under probability of naive expectations $q \approx 0.675$

Note that the ratio between the quantity of output by selfish firms and reciprocators remains almost unchanged. It is demonstrated in the graph of fig. 9, where each iteration on x-axis shows the value of output by firms-reciprocators x_m, and the vertical axis - the appropriate output of quantity y_m of selfish firms (fig. 7.).

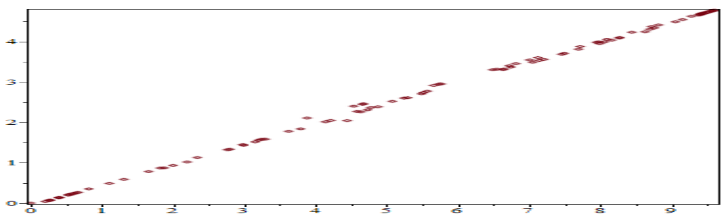

Fig. 7. The ratio between the quantity of product of selfish firms (horizontal axis) and reciprocator ones (vertical axis)

4.2 Bifurcation diagram

In detail the process of loss of stability and transition to chaos of dynamic system (24) can be presented in the following bifurcation diagram (fig. 8).

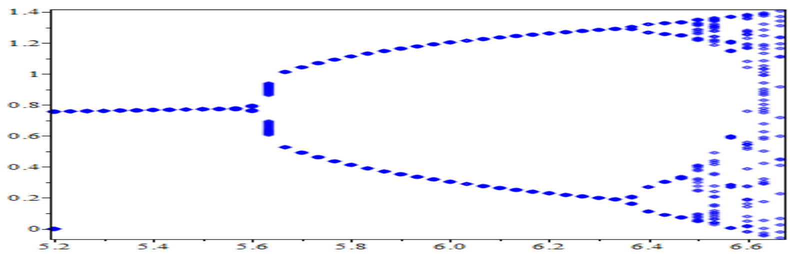

Fig. 8. The bifurcation diagram of dependence quantity product of selfish firm (y) on the probability of naive expectations (q) in a general dynamical system

Here the horizontal axis represents the parameter value of q multiplied by 10. The ordinate values quantity volumes of selfish firm on stable cycle, multiplied by 0.3. This rescaling is done for the sake of clarity. The values of the other parameters are the same as above. The bifurcation diagram, where on vertical axis are placed the values of output of firms-reciprocators x_m looks similar.

As noted in numerical simulations, the bifurcation may be interpreted as separation of equilibrium into several ways, one of which is selected by the market due to evolution of firms' strategies, such as repeated interactions and adaptations. Numerical experiments with n firms as the variable parameter are analogous to those described above.

5 Conclusion

Thus, we have designed the strategic model of cooperation between the two types of firms in the market of homogeneous product, where reciprocator and selfish firms plan their output using the adaptive approach with probability p and naïve (bounded rationality) one with a probability of $q = 1\text{-}p$, which distinguishes this model from existing analogues, where each type of firm adheres to one strategy rather than their combination and maximizes only its own profit rather than social welfare.

Desktop $C^{\#}$ application *Model* using a graphical interface to animate the model trajectories has been created specifically for the numerical investigation of the model.

It has been proved that in the model with adaptive expectations the unique Nash equilibrium in a dynamic system is stable for all possible values of the parameters. The trajectories of the dynamical system converge to the fixed point for any possible initial values. In the model with naive expectations the unique Nash equilibrium is unstable for sufficiently large values of n for all possible values of other parameters. According to the calculations, this point is unstable even at $n \geq 5$.

As a result of numerical experiment we have found that bifurcations of cycle doubling occur with an increase in naive expectations. This bifurcation can be interpreted as separation of equilibrium state into several ways, one of which is selected by the market in the evolution of firms' strategies. If two-thirds of firms use naive expectation ($q \approx 0.675$), then in accordance with the Sharkovskii scale there appears the state of dynamic chaos in the market, leading to degeneration of the existing competition model between two types of firms.

Thus, the crucial factor, which ensures sustainable equilibrium in the market and the ability to predict the product quantity of firms, is the adaptive approach, i.e. the one taking into account adaptive expectations of the firms when they plan their production.

Similar results are obtained if instead of q we use parameter n - number of firms in the market, where system also moves from stability to chaos if n increases.

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