Optimization Iterative Procedure for Radiative-Conductive Heat Transfer Model

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Abstract. A boundary optimal control problem for radiative-conductive heat transfer model in a layered medium is considered. The problem consists in minimization of a given cost functional by controlling the boundary temperature. The solvability of this control problem is proved, and optimality conditions are derived. An iteration algorithm is proposed, and numerical experiments are performed.

Keywords: optimal control; radiative heat transfer; conductive heat transfer

1 Introduction

Radiative-conductive heat transfer in a scattering and absorbing medium bounded by two reflecting and radiating planes is examined. The study of the coupled heat transfer [1] where the radiative and conductive contributions are simultaneously taken into account is important for many engineering applications. So, in [2–5], the radiative-conductive heat transfer model is studied in context of glass manufacturing. Modeling of thermal processes in the presence of radiation effects in nanofluids is performed in [6,7]. Notice that nanofluids have numerous applications in engineering and biomedicine (e.g., design of cooling systems, cancer therapy, etc.).

Usually, the process of radiative-conductive heat transfer is described by a nonlinear system of two differential equations: an equation of the radiative heat transfer and an equation of the conductive heat exchange. In general, the problem is characterized by anisotropic scattering and by specularly and diffusely reflecting boundaries.

A considerable number of works devoted to optimal control problems for radiative-conductive heat transfer models consider evolutionary systems (see e.g. [3–5, 8, 9]). Optimal control problems for steady-state radiative-conductive heat transfer model were studied weaker. Here, we can mention the works [10, 11] where optimal boundary control problems are studied. In [10], the optimal problem is formulated as the

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maximization of the energy outflow from the model domain by controlling reflection properties of the boundary. In [11], the problem of constructing the desired temperature and(or) intensity of radiation in part of the model domain is solved.

In this paper, an optimal control problem of minimization of a given cost functional by controlling the boundary temperature is studied. Particularly, it can be interpreted as a problem of obtaining a desired temperature in whole layer. Similar problems appear in many engineering applications and draw attention of many researchers. In the current work, the solvability of this problem is proved, an optimality system is derived, and the numerical algorithm is implemented.

2 Formulation of the optimal control problem

Let us consider the boundary-value problem for radiative-conductive heat transfer model in a layered medium [12]:

$$\begin{aligned} &-\theta''(x) + \alpha\sigma(|\theta(x)|\theta^3(x) - \varphi(x)) = 0, \\ &\qquad -\varphi(x)'' + \alpha(\varphi(x) - |\theta(x)|\theta^3(x)) = 0, \quad x \in (0,1), \end{aligned} \tag{1}$$

$$\begin{aligned} &\theta(0) = u_1, \ \ \theta(1) = u_2, \\ &\qquad B_1\varphi := \varphi(0) - \beta_1\varphi'(0) = u_1^4, \ \ B_2\varphi := \varphi(1) + \beta_2\varphi'(1) = u_2^4. \end{aligned} \tag{2}$$

Here, θ is the normalized temperature, and φ the normalized intensity of radiation averaged over all direction. The given positive constants α, σ, β_1, β_2 describe properties of the medium and boundaries. Specifically, $\alpha = 3\tau_0^2(1-\omega)$, $\sigma = 1/3N_c$, where ω is the albedo of single scattering, N_c the conduction-to-radiation parameter, and τ_0 the optical depth of the layer. The coefficients

$$\beta_i = \frac{2\,(2-\varepsilon_i)}{3\tau_0\varepsilon_i}, \quad i = 1,2$$

describe the reflection properties of the boundaries. Here, ε_1 and ε_2 are the emissivity coefficients for the boundary surfaces.

We will consider a vector $u = (u_1, u_2) \in \mathbb{R}^2$ as the boundary control. The optimal control problem is to find functions θ, φ and vector $u \in U_{ad} \subset \mathbb{R}^2$ that satisfy (1), (2) and minimize a cost functional:

$$J_\mu(\theta, \varphi, u) = J(\theta, \varphi) + \frac{1}{2}\mu|u|^2 \to \inf\,. \tag{3}$$

Here, $\mu \geq 0$, $|u|^2 = u_1^2 + u_2^2$, and U_{ad} is a nonempty set of admissible controls. Particularly, the functional J can describe the mean square deviation between the temperature θ and a desired temperature $\theta_d \in L^2(0,1)$, that is

$$J = \frac{1}{2}||\theta - \theta_d||^2_{L^2(0,1)}.$$

Taking into account that the temperature (and the control) is normalized, we can assume that $U_{ad} = [0,1]\times[0,1]$. But this condition is not necessary from a mathematical point of view.

3 Formalization of the optimal control problem

Let $H = L^2(0,1)$, and $W = W_2^2(0,1)$ be a Sobolev space. By $Y = W \times W$, we denote the state space of the controlled system, and $V = H \times H \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ the space of constraints.

Let us determine an operator $F : Y \times \mathbb{R}^2 \to V$,

$$F(\theta, \varphi, u) = \{-\theta'' + \alpha\sigma(|\theta|\theta^3 - \varphi),\ -\varphi'' + \alpha(\varphi - |\theta|\theta^3), \\ \theta(0) - u_1,\ \theta(1) - u_2,\ B_1\varphi - u_1^4,\ B_2\varphi - u_2^4\}.$$

Then the problem (1)-(3) can be written as follows:

$$J_\mu(\theta, \varphi, u) = J(\theta, \varphi) + \frac{1}{2}\mu|u|^2 \to \inf, \quad F(\theta, \varphi, u) = 0, \quad u \in U_{ad}. \tag{4}$$

Theorem 1. *Let*
(i) U_{ad} is a closed convex set; U_{ad} is a bounded set, if $\mu = 0$.
(ii) $J : Y \to \mathbb{R}$ is weakly lower semicontinuous.
Then there exists a solution of problem (4).

Proof. Notice that for a solution of the problem (1), (2) the following estimates hold [12]:

$$m \le \theta \le M,\ |m|m^3 \le \varphi \le |M|M^3, \tag{5}$$

where $m = \min\{u_1, u_2\}$, $M = \max\{u_1, u_2\}$.Therefore,

$$||\theta||_W + ||\varphi||_W \le C,$$

where C depends only on m, M, α, and σ.

Let $\{\theta_k, \varphi_k, u^k\}$ be a minimizing sequence of the problem (4),

$$u^k \in U_{ad}, \quad F(\theta_k, \varphi_k, u^k) = 0, \quad J_\mu(\theta_k, \varphi_k, u^k) \to \inf J_\mu. \tag{6}$$

It is obvious that the sequence $\{u^k\} \subset \mathbb{R}^2$ is bounded if $\mu > 0$ and the condition (i) guarantees the boundedness if $\mu = 0$. Therefore, by (5), sequences $\{\theta_k\}, \{\varphi_k\}$ are bounded in W. Thus, we can assume that

$$u_k \to \widehat{u} \text{ in } \mathbb{R}^2, \quad \theta_k \to \widehat{\theta}, \quad \varphi_k \to \widehat{\varphi} \text{ weakly in } W,$$

and in addition $\widehat{u} \in U_{ad}$. The convergence allows to pass to limit in (6), i.e. a triple $\{\widehat{\theta}, \widehat{\varphi}, \widehat{u}\}$ is admissible for the problem (4) and, by the condition (ii), it is a solution. $\square$

4 Optimality conditions

To derive the optimality system, we apply the principle of Lagrange for smooth convex extremal problems [13]. This principle requires only the convexity of the functional J_μ with respect to control. Let $\widehat{u} \in U_{ad}$ be the optimal control, and $\widehat{y} = \{\widehat{\theta}, \widehat{\varphi}\}$ be the

optimal state. We suppose that
(iii) $J : Y \to \mathbb{R}$ is Frechet differentiable in $\{\widehat{\theta}, \widehat{\varphi}\}$.

Let us prove that $\operatorname{Im} F'_y(\widehat{y}, \widehat{u}) = V$. Here, $F'_y(\widehat{y}, \widehat{u}) : Y \to V$ is a derivative of the constraint operator with respect to state.

The equation

$$F'_y(\widehat{y}, \widehat{u})\langle h\rangle = z,\ h = \{h_1, h_2\} \in Y,\ z = \{z_1, z_2, z_3, z_4, z_5, z_6\} \in V \tag{7}$$

is equivalent to the boundary-value problem

$$-h_1'' + \alpha\sigma(4|\widehat{\theta}|^3 h_1 - h_2) = z_1,\ -h_2'' + \alpha(h_2 - 4|\widehat{\theta}|^3 h_1) = z_2,\ x \in (0,1), \tag{8}$$

$$h_1(0) = z_3,\ h_1(1) = z_4,\ B_1 h_2 = 4\widehat{\theta}^3(0) z_3 + z_5,\ B_2 h_2 = 4\widehat{\theta}^3(1) z_4 + z_6. \tag{9}$$

Lemma 1. *The boundary-value problem (8), (9) is the unique solvable for all* $z_1, z_2 \in H$, $z_k \in \mathbb{R}$, $k = \overline{2,6}$

Proof. Due to Fredholm property of the problem (8), (9), to prove the lemma, it is sufficient to show that the homogeneous problem has only the zero solution. Set $z = 0$ in (8), (9). Let

$$r_\varepsilon(s) = \begin{cases} s/|s|, & |s| \ge \varepsilon, \\ s/\varepsilon, & |s| < \varepsilon. \end{cases}$$

Multiplying the first equation in (8) by $r_\varepsilon(h_1)$, the second by $\sigma r_\varepsilon(h_2)$, then integrating the result over $(0,1)$ and adding, we obtain

$$(h_1', r_\varepsilon'(h_1)h_1')\sigma(h_2', r_\varepsilon'(h_2)h_2') + \frac{\sigma}{\beta_1} h_2(0) r_\varepsilon(h_2(0)) \\ + \frac{\sigma}{\beta_2} h_2(1) r_\varepsilon(h_2(1)) + \alpha\sigma(4|\widehat{\theta}|^3 h_1 - h_2, r_\varepsilon(h_1) - r_\varepsilon(h_2)) = 0.$$

Notice that $r_\varepsilon'(s) \ge 0$, $s \in \mathbb{R}$. Dropping the first two terms and passing to limit as $\varepsilon \to +0$, we obtain

$$\sigma(\beta_1^{-1}|h_2(0)| + \beta_2^{-1}|h_2(1)|) + \alpha\sigma(4|\widehat{\theta}|^3 h_1 - h_2, \operatorname{sign} h_1 - \operatorname{sign} h_2) \le 0.$$

Therefore, $h_2(0) = h_2(1) = 0$, and

$$(h_1 + \sigma h_2)'' = 0,\ \ x \in (0,1);\ \ (h_1 + \sigma h_2)|_{x=0;1} = 0.$$

Thus, $h_1 + \sigma h_2 = 0$, and hence

$$-h_1'' + \alpha(1 + 4\sigma|\widehat{\theta}|^3) h_1 = 0,\ \ x \in (0;1);\ \ h_1(0) = h_1(1) = 0.$$

As a result $h_1 = 0$, and consequently $h_2 = 0$. This proves the lemma. □

Since, the derivative of the constraint operator with respect to state is epimorphism, then we can apply the Lagrange principle [14, Cor.2, Th. 1.5].

Let

$$L = J_\mu + (-\theta'' + \alpha\sigma(|\theta|\theta^3 - \varphi), p_1) + (-\varphi'' + \alpha(\varphi - |\theta|\theta^3), p_2) + (\theta(0) - u_1)q_1 + (\theta(1) - u_2)q_2 + (B_1\varphi - u_1^4)q_3 + (B_2\varphi - u_2^4)q_4.$$

Here, $p_{1,2} \in H$, and $q_k \in \mathbb{R}$, $k = \overline{1,4}$ are Lagrange multipliers.

Equating to zero derivatives of Lagrange function L with respect to θ and φ, we obtain:

$$-p_1'' + 4\alpha|\widehat{\theta}|^3(\sigma p_1 - p_2) = -J'_\theta(\widehat{\theta}, \widehat{\varphi}), \quad -p_2'' + \alpha(p_2 - \sigma p_1) = -J'_\varphi(\widehat{\theta}, \widehat{\varphi}), \tag{10}$$

$$p_1(0) = p_1(1) = 0, \quad B_1 p_2 = B_2 p_2 = 0, \tag{11}$$

$$q = \{q_1, q_2\} = \left\{p_1'(0) + 4\beta^{-1} p_2(0)\widehat{u}_1^3, \ -p_1'(1) + 4\beta_2^{-1} p_2(1)\widehat{u}_2^3\right\}. \tag{12}$$

From condition

$$(L'_u, \widehat{u} - v)_{\mathbb{R}^2} \le 0 \ \ \forall v \in U_{ad},$$

we obtain

$$(\mu\widehat{u} - q, \widehat{u} - v)_{\mathbb{R}^2} \le 0 \ \ \forall v \in U_{ad}. \tag{13}$$

Thus, we obtain the following optimality conditions of the first order.

Theorem 2. *Let $\{\widehat{\theta}, \widehat{\varphi}\}$ be an optimal state, $\widehat{u}$ an optimal control, and condition (iii) holds. Then there exists a unique adjoint state $p = \{p_1, p_2\} \in Y$ satisfying (10),(11), and the variational inequality (13) holds.*

5 Numerical algorithm

The algorithm is based on solving the optimality system consisting from boundary-value problem (1), (2), where $\theta = \widehat{\theta}$, $\varphi = \widehat{\varphi}$ and conditions (10)-(13). The system is solved by an iterative procedure based on method of the gradient projection of the original extremal problem:

$$u^{k+1} = P_{U_{ad}}\left(u^k - \lambda(\mu u^k - q^k)\right), \ k = 0, 1, 2, ...$$

Here, u^0 is a given initial approximation, λ is an iterative parameter, $P_{U_{ad}}$ the projection operator to U_{ad}. To find q_k, at first, the problem (1),(2) for $u = u^k$ is solved. Further, we solve the adjoint system (10),(11), and then we find q^k from (12), where $\widehat{u} = u^k$.

In conclusion, let us consider the numerical experiment. We took the following parameters of the model (see [15], Problem 2): $\omega = 0.9$, $\tau_0 = 3$, $\varepsilon_1 = 0.7$, $\varepsilon_2 = 0.6$, and $N_c = 0.05$. To determine the cost functional, we set $\theta_d = 0.8 - 0.4x$ and $\mu = 0.01$. In figure 1, the optimal temperature is shown (solid curve). The small value of μ practically means neglecting the second term in the cost functional. In this case, the optimal temperature approximates the given function θ_d (dashed line). Further, we consider the same model data as in the first experiment with the exception of $N_c = 0.00001$ (see [16], Problem 2). This corresponds to the case of a high temperature. To determine the cost functional, we set $\theta_d = 0.8-0.2x$ and $\mu = 0.01$. In figure 2, the obtained optimal temperature (solid curve) and given function θ_d (dashed line) are shown.

We did not study theoretically the rate of the convergence. Nevertheless, it was sufficient 10 iterations for convergence of the iterative procedure. The numerical experiments demonstrate the efficiency of the proposed algorithm.

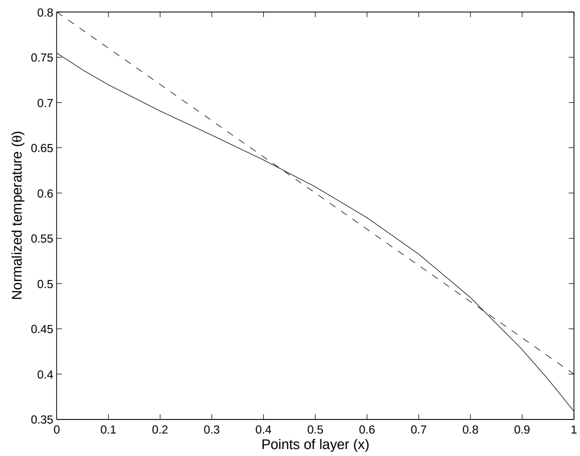

Fig. 1. The optimal normalized temperature θ (solid curve) and given temperature $\theta_d = 0.8 - 0.4x$ (dashed line) for $N_c = 0.05$.

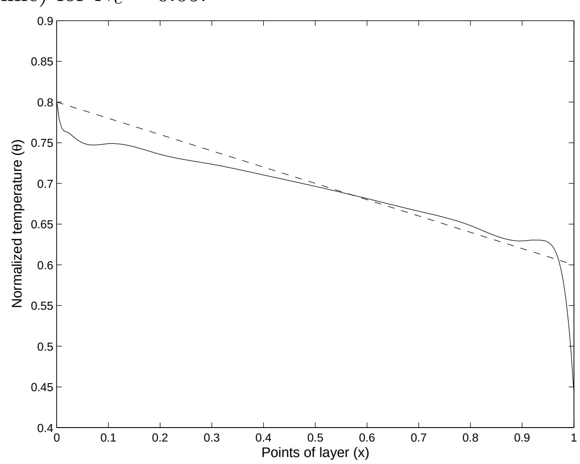

Fig. 2. The optimal normalized temperature θ (solid curve) and given temperature $\theta_d = 0.8 - 0.2x$ (dashed line) for $N_c = 0.00001$.

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