Optimal Control for Radiative Heat Transfer Model with Monotonic Cost Functionals

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Abstract. A boundary optimal control problem for a nonlinear nonstationary heat transfer model is considered. The model describes coupled conduction and radiation within the P_1 approximation. The control parameter is related to the emissivity of the boundary and varies with time. The optimal control problem is to minimize or maximize a cost functional which is assumed to be monotonic. Sufficient conditions of optimality are derived and the convergence of a simple iterative method is shown.

Keywords: optimal control; radiative heat transfer; conductive heat transfer, sufficient optimality conditions, simple iterative method, bang-bang

1 Introduction

Radiative heat transfer models can be used for describing various engineering processes. These models contain parameters related to some properties of a medium or a boundary surface. Optimal control problems for such models consist in determination of some parameters values in order to minimize (or maximize) a given cost functional. Papers [1–5] deal with problems of boundary temperature control for radiative heat transfer models including SP_N approximations of the radiative transfer equation (RTE). Note that approximations of RTE are employed to simplify numerical solution of governing equations, and SP_1 (P_1) approximation is valid mainly for optically thick and highly scattering media at large optical distances from the boundary [6, 7].

In this paper, we consider a diffusion model (P_1 approximation of RTE) including a nonstationary heat equation combined with a stationary equation for the mean intensity of thermal radiation. The control parameter depends on the emissivity of the boundary. We will assume that the cost functional is monotonic. Optimality systems for such functionals become simpler and do not contain an adjoint equation. Moreover, the

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monotonicity condition allows to obtain sufficient conditions of optimality and prove the convergence of a simple iterative method.

Paper [8] deals with the analogous optimal control problem of obtaining a desired temperature distribution by controlling the emissivity of the boundary. Note that the cost functional, representing L^2-deviation of the temperature from the desired field, turns monotonic if the desired temperarure equals 0. A similar optimal control problem was investigated in [9, 10], where the emissivity does not vary with time, and the work [11] is devoted to an analogous problem for a steady-state P_1 model. In the mentioned papers, an analog of the bang-bang principle for the optimal control was proven. Based on this principle, it is possible to construct efficient numerical algorithms for solving optimal control problems. In a general case, a simple iterative method fails to converge, that is why a generalized algorithm was applied in [9, 10]. However, if the cost functional is monotonic, the convergence of a simple iterative method can be proven.

2 Problem formulation

The nonstationary normalized P_1 model of radiative-conductive heat transfer in a bounded domain $\Omega \subset \mathbb{R}^3$ has the following form [9]:

$$\partial\theta/\partial t - a\Delta\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = 0, \quad -\alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = 0, \tag{1}$$

$$a\partial_n\theta + \beta(\theta - \theta_b)|_\Gamma = 0, \quad \alpha\partial_n\varphi + u(\varphi - \theta_b^4)|_\Gamma = 0, \tag{2}$$

$$\theta|_{t=0} = \theta_0. \tag{3}$$

Here, θ is the normalized temperature, φ the normalized radiation intensity averaged over all directions, κ_a the absorption coefficient, and θ_b the boundary temperature taken in Newton's law of cooling. The parameters a, b, and α are positive constants, and $\beta = \beta(x)$, $u = u(x,t)$, $x \in \Gamma$, $t \in (0,T)$ are positive functions. The control parameter u depends on the emissivity ε of the boundary surface as follows: $u = \varepsilon/2(2-\varepsilon)$. The symbol ∂_n denotes the derivative in the outward normal direction $\mathbf{n}$ on the boundary $\Gamma := \partial\Omega$.

Define the set of admissible controls U_{ad} of functions $u(x,t)$ such that $u_1 \le u \le u_2$, where $u_1(x,t)$ and $u_2(x,t)$ are positive functions. The problem of optimal control consists in the determination of functions $u \in U_{ad}$, θ, and φ which satisfy the conditions (1)-(3) and minimize (or maximize) an objective functional $J(\theta,\varphi)$ which is assumed to be monotonic. The precise definition of monotonicity will be given in the next section.

3 Formalization of the optimal control problem

Suppose that Ω is a Lipschitz bounded domain, $\Gamma = \partial\Omega$, $\Sigma = \Gamma\times(0,T)$, $Q = \Omega\times(0,T)$, and the model data satisfy the following conditions:
(i) $\beta \in L^\infty(\Gamma)$, $u_1, u_2, \theta_b \in L^\infty(\Sigma)$, $0 < \beta_0 \le \beta$, $0 < u_0 \le u_1 \le u_2$, $\beta_0, u_0 = \text{const}$, $\theta_b \ge 0$;
(ii) $0 \le \theta_0, \varphi_0 \in L^\infty(\Omega)$.

Denote $H = L^2(\Omega)$, $V = H^1(\Omega)$. Note that $V \subset H = H' \subset V'$. Let the value of a functional $f \in V'$ on an element $v \in V$ be denoted by (f, v), and (f, v) is the inner product in H if f and v are elements of H. Define the space $W = \{y \in L^2(0,T;V) \colon y' \in L^2(0,T;V')\}$, $y' = dy/dt$, as well as the space of states $Y = W \times L^2(0,T;V)$ and the space of controls $U = L^2(\Sigma)$, $U_{ad} = \{u \in U \colon u_1 \leq u \leq u_2\}$.

Definition 1. *A pair $\{\theta, \varphi\} \in Y$ is called weak solution of the problem (1)–(3), that corresponds to the control $u \in U_{ad}$, if the following equalities are fulfilled for any $v, w \in V$ a.e. on $(0,T)$:*

$$(\theta', v) + a(\nabla\theta, \nabla v) + \int_\Gamma \beta(\theta - \theta_b) v d\Gamma + b\kappa_a(|\theta|\theta^3 - \varphi, v) = 0, \tag{4}$$

$$\alpha(\nabla\varphi, \nabla w) + \int_\Gamma u(\varphi - \theta_b^4) w d\Gamma + \kappa_a(\varphi - |\theta|\theta^3, w) = 0, \tag{5}$$

and $\theta|_{t=0} = \theta_0$.

Theorem 1. (cf. [9]) *Let the conditions (i), (ii) be satisfied. For any $u \in U_{ad}$ the problem (1)–(3) has a unique weak solution $\{\theta, \varphi\}$, and the following inequalities are fulfilled: $0 \leq \theta \leq M$, $0 \leq \varphi \leq M^4$, where $M = \max\{\|\theta_b\|_{L^\infty(\Sigma)}, \|\theta_0\|_{L^\infty(\Omega)}\}$.*

Definition 2. *The cost functional $J \colon Y \cap [L^\infty(Q)]^2 \to \mathbb{R}$ is called monotonic, if, given any $0 \leq \theta_1 \leq \theta_2$, $0 \leq \varphi_1 \leq \varphi_2$ a.e. in Q, we have $J(\theta_1, \varphi_1) \leq J(\theta_2, \varphi_2)$.*

Next we state two optimization problems not depending on a specific monotonic cost functional.

Problem 1. Find $\widehat{u} \in U_{ad}$ such that for any $u \in U_{ad}$ we have $\widehat{\theta} \leq \theta$, $\widehat{\varphi} \leq \varphi$ a.e. in Q.

Problem 2. Find $\widehat{u} \in U_{ad}$ such that for any $u \in U_{ad}$ we have $\widehat{\theta} \geq \theta$, $\widehat{\varphi} \geq \varphi$ a.e. in Q.

Here, $\widehat{\theta} = \theta(\widehat{u})$, $\widehat{\varphi} = \varphi(\widehat{u})$, $\theta = \theta(u)$, $\varphi = \varphi(u)$. A weak solution of the problem (1)–(3), corresponding to the control $u \in U_{ad}$, is denoted by $\{\theta(u), \varphi(u)\}$.

Remark 1. It is readily seen that solutions of problems 1 and 2 are solutions of optimal control problems $J(\theta, \varphi) \to \inf$ and $J(\theta, \varphi) \to \sup$, respectively, where J is monotonic.

Definition 3. *Solutions of the problems 1 and 2 are called strong optimal controls.*

Let us give an example of a monotonic cost functional. Suppose that $\Gamma_1 \subset \Gamma$ is a part of the boundary, on which u is given that is $u = u_1 = u_2$ on Γ_1. The functional represents the energy outflow through Γ_1:

$$J(\theta, \varphi) = \int_0^T \int_{\Gamma_1} \left(\beta(\theta - \theta_b) + bu_1(\varphi - \theta_b^4)\right) d\Gamma dt.$$

Note that our goal is to minimize (or maximize) the temperature and radiative intensity fields in the entire domain and time interval. Therefore, the answer will be the same for any monotonic cost functional.

4 Optimality conditions

Lemma 1. *Let $u, \widetilde{u} \in U_{ad}$, $\theta = \theta(u)$, $\varphi = \varphi(u)$, $\widetilde{\theta} = \theta(\widetilde{u})$, $\widetilde{\varphi} = \varphi(\widetilde{u})$, and one of the following conditions is satisfied:*

a) $u = \begin{cases} u_1, & \text{if } \varphi - \theta_b^4 < 0, \\ u_2, & \text{if } \varphi - \theta_b^4 > 0; \end{cases}$ *b)* $u = \begin{cases} u_1, & \text{if } \widetilde{\varphi} - \theta_b^4 < 0, \\ u_2, & \text{if } \widetilde{\varphi} - \theta_b^4 > 0. \end{cases}$

Then $\varphi \leq \widetilde{\varphi}$, $\theta \leq \widetilde{\theta}$ a.e. in Q.

Proof. Set $\overline{\theta} = \theta - \widetilde{\theta}$, $\overline{\varphi} = \varphi - \widetilde{\varphi}$ and define the functions $\eta = \max\{\overline{\theta}, 0\}$, $\psi = \max\{\overline{\varphi}, 0\}$. Set $v = \eta$, $w = \psi$ in (4), (5) and integrate in t. We obtain

$$\frac{1}{2}\|\eta(t)\|^2 + \int_0^t \left[a\|\nabla\eta\|^2 + \int_\Gamma \beta\eta^2 d\Gamma + b\kappa_a \Big((\theta + \widetilde{\theta})(\theta^2 + \widetilde{\theta}^2)\eta, \eta \Big) \right] d\tau = \\ = b\kappa_a \int_0^t (\overline{\varphi}, \eta) d\tau \leq b\kappa_a \int_0^t (\psi, \eta) d\tau, \tag{6}$$

$$\int_0^t \left[\alpha\|\nabla\psi\|^2 + \int_\Gamma \widetilde{u}\psi^2 d\Gamma + \int_\Gamma (u - \widetilde{u})(\varphi - \theta_b^4)\psi d\Gamma + \kappa_a\|\psi\|^2 \right] d\tau = \\ = \kappa_a \int_0^t \Big((\theta + \widetilde{\theta})(\theta^2 + \widetilde{\theta}^2)\overline{\theta}, \psi \Big) d\tau \leq \kappa_a \int_0^t \Big((\theta + \widetilde{\theta})(\theta^2 + \widetilde{\theta}^2)\eta, \psi \Big) d\tau. \tag{7}$$

Condition a) implies that the third term in the left-hand side of (7) is nonnegative. Thus, we obtain the estimate $\int_0^t \|\psi(\tau)\|^2 d\tau \leq C_1 \int_0^t \|\eta(\tau)\|^2 d\tau$. Then (6) yields the estimate

$$\|\eta(t)\|^2 \leq C_2 \int_0^t \|\eta(\tau)\|^2 d\tau.$$

It follows from Gronwall lemma that $\eta = \psi = 0$, and so $\theta \leq \widetilde{\theta}$, $\varphi \leq \widetilde{\varphi}$ a.e. in Q.

The statement for condition b) can be proven similarly. □

The following theorem follows from Lemma 1 and establishes a sufficient condition of optimality.

Theorem 2. *Let $u \in U_{ad}$, $\varphi = \varphi(u)$, and*

$$u = \begin{cases} u_1, & \text{if } \varphi - \theta_b^4 < 0, \\ u_2, & \text{if } \varphi - \theta_b^4 > 0. \end{cases}$$

Then u is a solution of the problem 1.

Similar arguments lead to

Theorem 3. *Let $u \in U_{ad}$, $\varphi = \varphi(u)$, and*

$$u = \begin{cases} u_1, & \text{if } \varphi - \theta_b^4 > 0, \\ u_2, & \text{if } \varphi - \theta_b^4 < 0. \end{cases}$$

Then u is a solution of the problem 2.

Next prove the uniqueness of the strong optimal control.

Theorem 4. *If u and $\widetilde{u}$ are strong optimal controls, then $u = \widetilde{u}$ a.e. in $\{(x,t) \in \Sigma\colon \varphi(x,t) \neq \theta_b^4(x,t)\}$.*

Proof. By definition, $\varphi(u) = \varphi(\widetilde{u}) = \varphi$, $\theta(u) = \theta(\widetilde{u}) = \theta$ a.e. in Q. It follows from (5) that

$$\int\limits_\Gamma \left[u(\varphi - \theta_b^4) - \widetilde{u}(\varphi - \theta_b^4)\right] v d\Gamma = \int\limits_\Gamma (u - \widetilde{u})(\varphi - \theta_b^4) v d\Gamma = 0 \;\; \forall v \in V \text{ a.e. on } (0,T).$$

Hence $(u - \widetilde{u})(\varphi - \theta_b^4) = 0$ a.e. on Σ, therefore, $u = \widetilde{u}$ a.e. in $\{(x,t) \in \Sigma\colon \varphi(x,t) \neq \theta_b^4(x,t)\}$. □

Remark 2. It follows from (4), (5) that an arbitrary modification of a strong optimal control u in the set $\{(x,t) \in \Sigma\colon \varphi(x,t) = \theta_b^4(x,t)\}$ keeps the optimality of the control u, because such modification does not influence on the second term in (5).

5 Iterative algorithm

Describe a simple iterative method converging to a strong optimal control. Discuss the problem 1, considerations for problem 2 are similar.

Define the operator $U\colon L^\infty(\Sigma) \to L^\infty(\Sigma)$:

$$U(\varphi) = \begin{cases} u_1, & \text{if } \varphi - \theta_b^4 < 0, \\ u_2, & \text{if } \varphi - \theta_b^4 \geq 0. \end{cases}$$

If $u \in U_{ad}$ and

$$U(\varphi(u)) = u, \tag{8}$$

then, by Theorem 2, u is a strong optimal control.

Consider the simple iterative method for solving the equation (8). Choose an arbitrary initial guess $u^0 \in U_{ad}$. The iterative algorithm is as follows: $u^{k+1} = U(\varphi^k)$ where $\varphi^k = \varphi(u^k)$, $k = 0, 1, \ldots$.

It follows from Lemma 1 that $\varphi^{k+1} \leq \varphi^k$ $(k = 0, 1, \ldots)$ a.e. in Q. Thus, $u^{k+1} \leq u^k$ $(k = 1, 2, \ldots)$ a.e. on Σ. Taking into account that these sequences are bounded, we obtain that $u^k \to u^*$ a.e. on Σ, $\varphi^k \to \varphi^*$ a.e. in Q.

Lemma 2. *$\varphi^* = \varphi(u^*)$ a.e. in Q.*

Proof. Applying Lebesgue theorem, we obtain that $u^k \to u^*$ in $L^2(\Sigma)$, $\varphi^k \to \varphi^*$ in $L^2(Q)$.

It is easy to prove that the operator $\varphi\colon L^2(\Sigma) \to L^2(Q)$, defined on the set U_{ad}, is continuous. Therefore, $\varphi^* = \varphi(u^*)$ a.e. in Q. □

Lemma 3. *$u^* = U(\varphi^*)$ a.e. on Σ.*

Proof. Because $\varphi^{k+1} \leq \varphi^k$ a.e. in Q, we have $u^{k+1} = U(\varphi^k) \to U(\varphi^*)$ a.e. on Σ, and so $u^* = U(\varphi^*)$ a.e. on Σ. □

It follows from Lemmas 2, 3 that the simple iterative method converges to a solution of (8), therefore, u^* is a strong optimal control.

Theorem 5. *Problem 1 (or 2) is solvable.*

Remark 3. Let the solution $\{\theta, \varphi\}$ of problem (1)–(3) be computed with absolute error ε that is $|\varphi - \widetilde{\varphi}| \le \varepsilon$ in Q, where $\widetilde{\varphi}$ is a component of the approximate solution. Then the strong optimal control is determined ambiguously in the set $\{(x,t) \in \Sigma\colon |\varphi(x,t) - \theta_b^4(x,t)| \le \varepsilon\}$.

6 Numerical example

As an example, consider a one-dimensional model describing the radiative heat transfer problem in a slab of thickness $L = 50$ [cm]. The physical parameters are taken from [12]. The maximum temperature is chosen as $T_{\max} = 500\,^\circ$C. Notice that the absolute temperature is related to the normalized temperature as follows: $T = T_{\max}\theta$. Set $\theta_b = 0.4$ at $x = 0$, and $\theta_b = 0.7$ at $x = L$. The thermodynamical characteristics of the medium inside the slab correspond to air at the normal atmospheric pressure and the temperature $400\,^\circ$C, namely $a = 0.92$ [cm^2/s], $b = 18.7$ [cm/s], $\alpha = 3.3\ldots$ [cm], $\kappa_a = 0.01$ [cm^{-1}], and $\beta = 10$ [cm/s]. The initial function is $\theta_0(x) = 0.3 + 0.7x/L$. The time interval length is chosen as $T = 60$ [s]. The bounds of the control are $u_1 = 0.01$ and $u_2 = 0.5$.

The boundary-value problem (1)–(3) was solved by the finite difference method with Newton's linearization. Namely, we use the implicit time discretization (10001 grid points) that leads to a nonlinear algebraic system at each time step after the discretization in space (2501 grid points). After applying Newton's method to this system, one requires to solve a block-tridiagonal linear system with two blocks that is possible by using standard solvers.

It is worth noting that the simple iterative method for solving problem (8) does not require storing the solution $\{\theta, \varphi\}$ for all time grid lines, because the optimality conditions do not contain the adjoint system. The simple iterative method is applied to each individual time step and needs approximately 3 iterations. It follows from the statement of the algorithm that the resulting control will always be bang-bang one.

The solution of the problem 1 at $x = L$ is presented in Fig. 1. The strong optimal control in the problem 1 equals $u_2 = 0.5$ at $x = 0$. The solution of the problem 2 at $x = L$ is depicted in Fig. 2. The strong optimal control in the problem 2 equals $u_1 = 0.01$ at $x = 0$.

Figure 3 indicates the minimum and maximum temperatures at several time instants, and the minimum and maximum intensities of radiation are shown in Fig. 4. Notice that the maximum and minimum fields at large t are close to the corresponding optimal states in the steady-state optimal control problem due to the stabilization of the radiative heat transfer process. The strong optimal controls at large t are equal to the respective steady-state strong optimal controls as well.

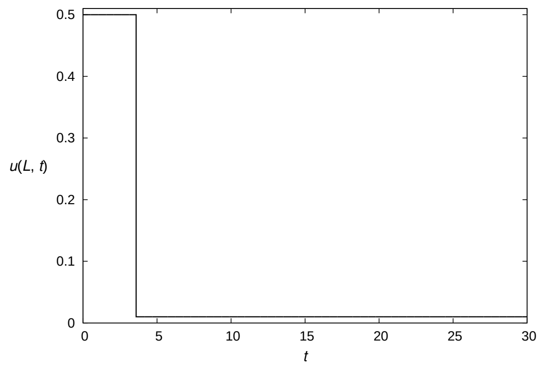

Fig. 1. Strong optimal control in problem 1

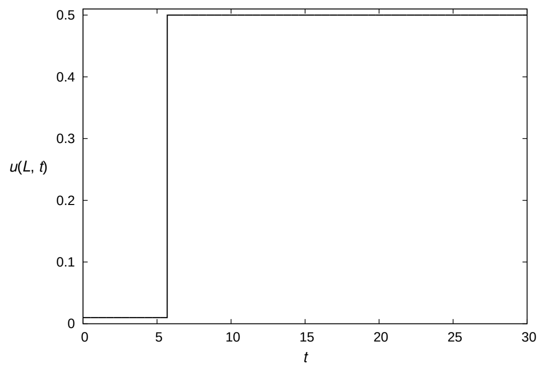

Fig. 2. Strong optimal control in problem 2

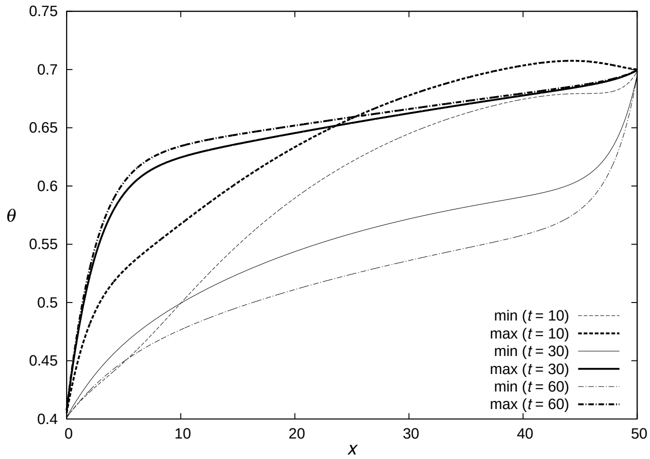

Fig. 3. Minimum and maximum temperatures at $t = 10,\ 30,$ and 60

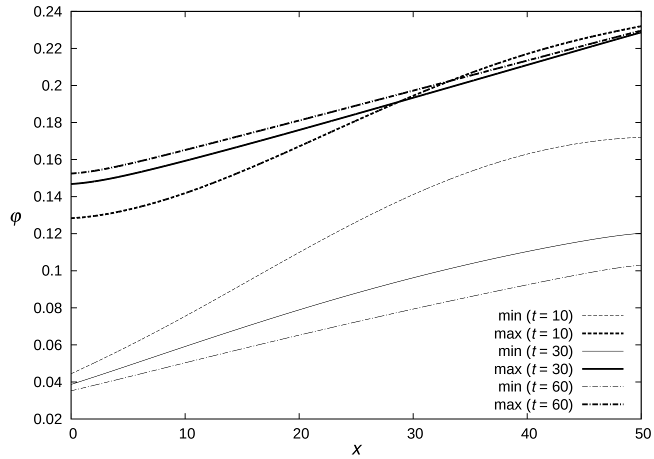

Fig. 4. Minimum and maximum radiative intensities at $t = 10,\ 30,$ and 60

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