Rough Sets and Sorites Paradox

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Abstract. We discuss the rough set approach to approximation of vague concepts. There are already published several papers on rough sets and vague concepts staring from the seminal papers by Zdzisław Pawlak. However, only a few of them are discussing the relationships of rough sets with the sorites paradox. This paper contains a continuation of discussion on this issue.

Key words: vagueness, vague concept, sorites paradox, (adaptive) rough set.

1 Introduction

The rough set (RS) approach was proposed by Professor Zdzisław Pawlak in 1982 [35, 36, 40] as a tool for dealing with imperfect knowledge, in particular with vague concepts. Over the years many applications of methods based on rough set theory alone or in combination with other approaches have been developed.

The rough set approach seems to be of fundamental importance in artificial intelligence and cognitive sciences, especially in machine learning, data mining and knowledge discovery from databases, pattern recognition, decision support systems, expert systems, intelligent systems, multiagent systems, adaptive systems, autonomous systems, inductive reasoning, commonsense reasoning, adaptive judgement, conflict analysis.

Rough sets have established relationships with many other approaches such as fuzzy set theory, granular computing, evidence theory, formal concept analysis, (approximate) Boolean reasoning, multicriteria decision analysis, statistical methods, decision theory, matroids have been clarified. Despite the overlap with many other theories rough set theory may be considered as an independent discipline in its own right. There are reports on many hybrid methods obtained by combining rough sets with other approaches such as soft computing (fuzzy sets, neural networks, genetic algorithms), statistics, natural computing, mereology, principal component analysis, singular value decomposition or support vector machines.

In particular some relationships of the rough set approach with vague concepts were shown (see, *e.g.*, [2, 3, 5, 13, 29, 31, 32, 37, 38, 41, 45, 49–51, 55]). However, the relationships with sorites paradox are not explored well yet. In this paper, we extend a discussion on this topic, especially presented in [25].

Let us also note that the relationships with vague concepts of other approaches to uncertainty such as fuzzy sets or graded consequence are elaborated in the literature (see, *e.g.*, [7–13, 17, 28, 43, 44]).

This paper is structured as follows. In Sect. 2 we present some preliminaries on vague sets. Rudiments of rough sets, in particular approximations of concepts, are discusses in Sect. 3. The rough set approach to sorites paradox is presented in Sect. 4. The issue of higher order vagueness in rough sets is covered in Sect. 5. In Sect. 6, we present some constraints on induced classifiers for vague concepts related to the sorites paradox. They are making it possible to eliminate the contradiction characteristic to sorites paradox which is related to behaviour of the classifier when it passes through different approximation regions. Sect. 7 emphasizes the need for development the adaptive rough set approach.

2 Vague Sets

Mathematics requires that all mathematical notions (including set) must be exact, otherwise precise reasoning would be impossible. However, philosophers (see, *e.g.*, [26] and recently computer scientists as well as other researchers have become interested in *vague* (imprecise) concepts. Moreover, in the XX century one can observe the drift paradigms in modern science from dealing with precise concepts to vague concepts, especially in the case of complex systems (*e.g.*, in economy, biology, psychology, sociology, quantum mechanics).

Almost all concepts we are using in natural language are vague [1, 6]. Therefore, common sense reasoning based on natural language must be based on vague concepts and not on classical logic. Interesting discussion of this issue can be found in [45]. The idea of vagueness can be traced back to the ancient Greek philosopher Eubulides of Megara (ca. 400BC) who first formulated so called "sorites" (heap) and "falakros" (bald man) paradox (see, *e.g.*, [26]). There is a huge literature on issues related to vagueness and vague concepts in philosophy (see, *e.g.*, [4, 14, 19, 26, 27, 46–48]).

Vagueness is often associated with the boundary region approach (*i.e.*, existence of objects which cannot be uniquely classified relative to a set or its complement) which was first formulated in 1893 by the father of modern logic, German logician, Gottlob Frege (1848-1925) (see [15]). According to Frege (see Grundgesetze der Arithmetik, vol. ii, Sect.56 [15, 16]) the concept must have a sharp boundary:

> *To the concept without a sharp boundary there would correspond an area that would not have any sharp boundary – line all around.*

It means that mathematics must use crisp, not vague concepts, otherwise it would be impossible to reason precisely. However, vagueness in opinion of Ludwig Wittgenstein is an essential feature of language with semantics specified by 'language games'. A language is not a calculus with rigid rules that provide for all possible circumstances. There are many vague concepts in natural languages [1, 6]. One should also note that vagueness also relates to insufficient specificity, as the result of lack of feasible searching methods for sets of features adequately describing concepts.

Discussion on vague (imprecise) concepts in philosophy includes the following characteristic features of them [26]: (i) the presence of borderline cases, (ii) boundary regions of vague concepts are not crisp, (iii) vague concepts are susceptible to sorites paradox. In the sequel we discuss these issues in the RS framework. The reader can find the discussion on application of the RS approach to vagueness in [45].

3 Rough Set Based Concept Approximation

The starting point of rough set theory is the indiscernibility relation, which is generated by information about objects of interest (defined later in this section as signatures of objects). The indiscernibility relation expresses the fact that due to a lack of information (or knowledge) we are unable to discern some objects employing available information (or knowledge). This means that, in general, we are unable to deal with each particular object but we have to consider granules (clusters) of indiscernible objects as a fundamental basis for our theory.

¿From a practical point of view, it is better to define basic concepts of this theory in terms of data. Therefore we will start our considerations from a data set called an *information system*.

Suppose we are given a pair $\mathbb{A} = (U, A)$ of non-empty, finite sets U and A, where U is the *universe* of *objects*, and A – a set consisting of *attributes*, i.e. functions $a : U \longrightarrow V_a$, where V_a is the set of values of attribute a, called the *domain* of a. The pair $\mathbb{A} = (U, A)$ is called an *information system* (see, *e.g.*, [34]). Any information system can be represented by a data table with rows labeled by objects and columns labeled by attributes. Any pair (x, a), where $x \in U$ and $a \in A$ defines the table entry consisting of the value $a(x)$.

Any subset B of A determines a binary relation IND_B on U, called an *indiscernibility relation*, defined by

$$x \; IND_B \; y \text{ if and only if } a(x) = a(y) \text{ for every } a \in B, \tag{1}$$

where $a(x)$ denotes the value of attribute a for object x.

Obviously, IND_B is an equivalence relation. The family of all equivalence classes of IND_B, i.e., the partition determined by B, will be denoted by U/IND_B, or simply U/B; an equivalence class of IND_B, i.e., the block of the partition U/B, containing x will be denoted by $B(x)$ (other notation used: $[x]_B$ or more precisely $[x]_{IND_B}$). Thus in view of the data we are unable, in general, to observe individual objects but we are forced to reason only about the accessible granules of knowledge (see, *e.g.*, [33, 36, 42]).

If $(x, y) \in IND_B$ we will say that x and y are *B-indiscernible*. Equivalence classes of the relation IND_B (or blocks of the partition U/B) are referred to as *B-elementary sets* or *B-elementary granules*. In the rough set approach the elementary sets are the basic building blocks (concepts) of our knowledge about reality. The unions of *B-elementary sets* are called *B-definable sets*.

For $B \subseteq A$ we denote by $Inf_B(x)$ the *B-signature* of $x \in U$, i.e., the set $\{(a, a(s)) : a \in B\}$. Let $INF(B) = \{Inf_B(s) : s \in U\}$. Then for any objects $x, y \in U$ the following equivalence holds: $x \mathcal{IND}_B y$ if and only if $Inf_B(x) = Inf_B(y)$.

The indiscernibility relation will be further used to define basic concepts of rough set theory. Let us define now the following two operations on sets $X \subseteq U$

$$\mathsf{LOW}_B(X) = \{x \in U : B(x) \subseteq X\}, \tag{2}$$

$$\mathsf{UPP}_B(X) = \{x \in U : B(x) \cap X \neq \varnothing\}, \tag{3}$$

assigning to every subset X of the universe U two sets $\mathsf{LOW}_B(X)$ and $\mathsf{UPP}_B(X)$ called the *B-lower* and the *B-upper approximation* of X, respectively. The set

$$\mathsf{BN}_B(X) = \mathsf{UPP}_B(X) - \mathsf{LOW}_B(X), \tag{4}$$

will be referred to as the *B-boundary region* of X.

If the boundary region of X is the empty set, i.e., $\mathsf{BN}_B(X) = \varnothing$, then the set X is *crisp* (*exact*) with respect to B; in the opposite case, i.e., if $\mathsf{BN}_B(X) \neq \varnothing$, the set X is referred to as *rough* (*inexact*) with respect to B. Thus any rough set, in contrast to a *crisp set*, has a non-empty boundary region.

Thus a set is *rough* (imprecise) if it has nonempty boundary region; otherwise the set is crisp (precise). This is exactly the idea of vagueness proposed by Frege.

Let us observe that the definition of rough sets refers to data (knowledge), and is *subjective*, in contrast to the definition of classical sets, which is in some sense an *objective* one.

Due to the granularity of knowledge, rough sets cannot be characterized by using available knowledge. Therefore with every rough set we associate two *crisp* sets, called *lower* and *upper approximation*. Intuitively, the lower approximation of a set consists of all elements that *surely* belong to the set, whereas the upper approximation of the set constitutes of all elements that *possibly* belong to the set, and the *boundary region* of the set consists of all elements that cannot be classified uniquely to the set or its complement, by employing available knowledge.

4 Approximations of Concepts and Sorites Paradox

Let us consider the *heap paradox*.

1. 10,000 grains of sand is a heap of sand.
2. 10,000 grains of sand is a heap of sand, then 9999 grains of sand is a heap of sand.
3. 9999 grains of sand is a heap of sand, then 9998 grains of sand is a heap of sand.

4. ...
5. *Conclusion.* 1 grain of sand is a heap of sand.

For a given set X by $card(X)$ we denote the cardinality of X. Let us consider the sequence of collections of grains of sand: $x_1, \ldots, x_i, x_{i+1}, \ldots, x_N$, such that $card(x_i) - card(x_{i+1}) = 1$ for $i = 1, \ldots, N-1$.

It is worthwhile mentioning that the concept of *heap* is vague. This concept may be perceived differently by different agents. Now let us consider an agent ag having a decision system $\mathbb{A} = (U, A, d)$, where $U \supseteq \{x_1, \ldots, x_i, x_{i+1}, \ldots, x_N\}$ is a family of collections of grains and A is a set of conditional attributes over U. The decision d assigns to each $x \in U$ the decision $d(x)$ equal to 1 if x is a heap and 0, otherwise. This decision is made, *e.g.*, by another agent ag_{dec} on the basis of some attributes (usually different from attributes from A). We denote by H the decision class $\{x \in U : d(x) = 1\}$ and by $-H$ the decision class $\{x \in U : d(x) = 0\}$. In particular, the decision d is assigned to each x_i from the considered sequence. The agent ag defines a partition of U using the the lower approximation of H, *i.e.*, $\mathsf{LOW}_A(H)$, the boundary region of H, *i.e.*, $\mathsf{BN}_A(H)$, and the lower approximation of $-H$, *i.e.*, $\mathsf{LOW}_A(-H)$.

In our example, we assume that $x_1 \in \mathsf{LOW}_A(H)$ and $x_N \in \mathsf{LOW}_A(-H)$.

By a *bounce* we understand any i such that one of the following conditions is satisfied: (i) $x_i \in \mathsf{LOW}_A(H)$ & $x_{i+1} \in \mathsf{BN}_A(H)$, (ii) $x_i \in \mathsf{LOW}_A(H)$ & $x_{i+1} \in \mathsf{LOW}_A(-H)$, (iii) $x_i \in \mathsf{BN}_A(H)$ & $x_{i+1} \in \mathsf{LOW}_A(-H)$.

Now, we explain why such *bounces* may occur.

Let us consider the first case. The two remaining cases are analogous. One could argue that we have a problem because there exists i such that $x_i \in \mathsf{LOW}_A(H)$ and $x_{i+1} \in \mathsf{BN}_A(H)$ but the difference between cardinalities x_i and x_{i+1} is negligible ($card(x_i) - card(x_{i+1}) = 1$) from the point of view of the concept *heap*. Observe that in the rough set approach the agent ag using the decision system $\mathbb{A}$ is perceiving objects (*i.e.*, in our example collections of grains) by means of attributes from A. Let us assume that $A = \{card\}$ and the conditional attribute $card$ assigns to any collection of grains $x \in U$ its cardinality. The decision d is taken by another agent ag_{dec} and it may be based, *e.g.*, on the basis of a *shape* of collection of grains. For example, $d(x) = 1$ if the shape of x is trapezoidal with sufficiently large ratio of the trapezoid hight to the length of the longest parallel sides of the trapezoid, and 0, otherwise. It may happen in U that the decision made by the agent ag_{dec} for all collections of grains from the elementary granule $A(x_i)$ (with the same cardinality, say n) are equal to 1, *i.e.*, all collections of grains from $A(x_i)$ have the relevant trapezoidal shape accepted by d as the positive examples of the concept *heap*. However, in case of $A(x_{i+1})$ (consisting of collections of grains with the same cardinality equal to $n-1$) there are in U collections x, y of grains such that $d(x) = 1$ (*i.e.*, $x \in H$) and $d(y) = 0$ (*i.e.*, $y \in -H$). This explains that the considered case of bounce is possible despite the fact that the difference between $card(x_i)$ and $card(x_{i+1})$ looks negligible from the point of view of the concept *heap*.

¿From the above considerations, we conclude that in general one can assume that for any i:

$$x_i \in \mathsf{LOW}_A(H) \text{ implies } x_{i+1} \in \mathsf{LOW}_A(H) \cup \mathsf{BN}_A(H) \cup \mathsf{LOW}_A(-H), \quad (5)$$

instead of

$$x_i \in \mathsf{LOW}_A(H) \text{ implies } x_{i+1} \in \mathsf{LOW}_A(H). \quad (6)$$

Analogous conditions may be formulated when we change the condition in the predecessor from the lower approximation of H, to the boundary region of H, or to the lower approximation of the complement of H.

Of course, in some cases some arguments of the alternative on the right hand side may be eliminated. For example, in some cases of decision table $\mathbb{A}$ of agent ag in Eq. 5 may be eliminated on the right had side of implication the third argument of the alternative.

5 Higher Order Vagueness and Rough Sets: Toward Adaptive Rough Sets

In [26], it is stressed that boundaries of vague concepts are not crisp. In the definition presented in this chapter, the notion of boundary region is defined as a crisp set $\mathsf{BN}_B(X)$. However, let us observe that this definition is relative to the subjective knowledge expressed by attributes from B. Different sources of information may use different sets of attributes for concept approximation. Hence, the boundary region can change when we consider these different views. Another reason for boundary change may be related to incomplete information about concepts. They are known only on samples of objects [18]. Hence, when new objects appear again the boundary region may change. ¿From the discussion in the literature it follows that vague concepts cannot be approximated with satisfactory quality by *static* constructs such as induced membership inclusion functions, approximations or models derived, *e.g.*, from a sample. Understanding of vague concepts can be only realized in a process in which the induced models are adaptively matching the concepts in dynamically changing environments. This conclusion seems to have important consequences for further development of rough set theory in combination with fuzzy sets and other soft computing paradigms for adaptive approximate reasoning. For further details the reader is referred, *e.g.*, to [49, 50, 56].

¿From the above considerations it follows that for dealing with higher order vagueness one should consider an extension of the rough set approach to the adaptive rough set approach. In this approach, approximations of a vague concept are considered over a family of decision systems $\{\mathbb{A}_t\}_{t \in T}$, where T is a set of indices, *e.g.*, time points. Hence, we obtain a family of the lower approximations, upper approximations and boundary regions of the considered vague concept which are changing, *e.g.*, over time (see Figure 1).

It is worthwhile mentioning that the elements of this family are obtained through interaction with the environment what is pointing to the necessity of

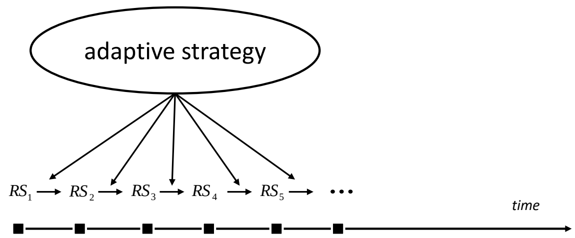


Fig. 1. Adaptive rough sets.

embedding the adaptive rough set approach in the framework of interactive granular computing and WisTech program (see, *e.g.*, [22–24, 21]).

6 Constraints on Bouncing Between Different Approximation Regions

If one would like to obtain some constraints on bouncing collections of sand grains between different approximation regions of the concept 'to be a heap of sand' assuming that succeeding collections are obtained by individually removing one grain from the preceding ones, then more details on rough set based approximations should be considered. For example, one may require that the changes of membership functions on consecutive collections of sand grains are below a given threshold. Let us consider an illustrative example to explain this issue in more detail.

First of all, one should note that usually information about approximated concept is partial, *e.g.*, provided by a sample of cases 'for' and 'against' a given concepts. Hence, in the rough set approach were developed methods for inductive extensions of approximation spaces from samples U of objects represented by decision systems on the universe U^* of all objects [39, 30].

In Figure 2 is presented a simple example of classifier for a concept $C \subseteq U^*$. The classifier is induced from a given partial information about C represented by a decision system $\mathbb{A}_d = (U, A, d)$ with the set of objects $U \subseteq U^*$ and the decision d equal (or almost equal) on U to the restriction to U of the characteristic function of C. The classifier represents an approximation of the characteristic function of the concept $C \subseteq U^*$.

The procedure of conflict resolution shown in Figure 2 between induced decision rules matching a given new case x (belonging to an extension U^* of U, *i.e.*, $U \subseteq U^*$) (and perceived as a signature of x, *i.e.*, $Inf_A(x)$) may be realized using arguments 'for' and 'against' membership in C determined by these rules. The arguments are aggregated using weights w_k as it is presented in Figure 3 what finally leads to the classifier computing the membership function μ_C for a given concept C.

Now, one can consider the membership function μ_{H^*} as an approximation of the characteristic function of the vague concept 'to be a heap of sand' $H^* \subseteq U^*$

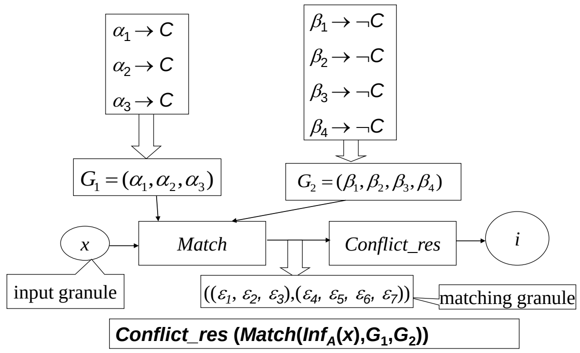


Fig. 2. Rough set-based rule classifier for a concept C (partially) specified by a decision system $\mathbb{A}_d = (U, A, d)$, where d is a characteristic function of a vague concept $C \subseteq U^*$ restricted to the sample $U \subseteq U^*$.

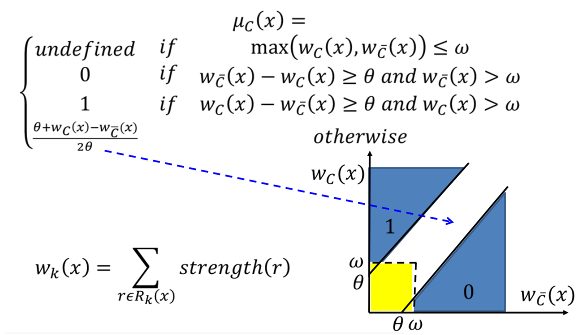
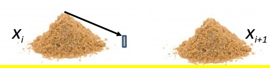


Fig. 3. Rough set based rule classifier for a concept C partially specified by a decision system, where Θ, ω are thresholds specified by the user, $strength(r)$ denotes the strength of the rule r (*e.g.*, defined by the support of the rule r [30]), and $R_k(x)$ denotes the set of decision rules induced from a given decision system for the decision $k \in \{C, \overline{C}\}$ ($\overline{C} = U^* \setminus C$) matching the case x.

induced (analogously as above) from a partial information represented by a decision system $\mathbb{A}_d = (U, A, d)$, where d is a characteristic function of a vague concept $H^* \subseteq U^*$ restricted to the sample $U \subseteq U^*$. We use μ_{H^*} in considerations concerning the paradox of heap of sand (Figure 4). Note that the induced approximations of the concept H^* are now defined as follows. The lower approximation of H^* is defined by $\mathsf{LOW}_A(H^*) = \{x \in U^* : \mu_{H^*}(x) = 1\}$, the upper approximation of H^* is defined by $\mathsf{UPP}_A(H^*) = \{x \in U^* : \mu_{H^*}(x) > 0 \vee \mu_{H^*}(x) = undefined\}$ and the boundary region of H^*: $\mathsf{BN}_A(H^*) = \mathsf{UPP}_A(H^*) \setminus \mathsf{LOW}_A(H^*)$.

In the considered example, we assume that the induced approximation of H^* represented by μ_{H^*} is consistent with a given sequence $x_1, \ldots, x_N$, *i.e.*, for any x_i and x_{i+1} representing consecutive collections of sand grains after individually removing one grain in each step, we have $\mu_{H^*}(x_i) = 1$ if $x_i \in H^*$ and

$\mu_{H^*}(x_i) = 0$ if $x_i \notin H^*$. In Figure 4 is presented a simple property of behavior of the model of H^* on elements of a sequence $x_1, \ldots, x_N$. If some additional constraints concerning weights are satisfied than one can see that the boundary region cannot be omitted. Moreover, using the assumption about a 'bounce size' of the membership values in passing from x_i to x_{i+1}, one can see that it can be necessary for the considered sequence to 'spend more time' in the boundary region before going out of it. One can specify such assumptions about 'bounce size' using the following constraints for induced classifiers: $w_C(x_i) - w_C(x_{i+1}) \leq \delta$ and $w_{\overline{C}}(x_{i+1}) - w_{\overline{C}}(x_i) \leq \delta$, where δ is a given threshold bounding bounces in degrees of memberships of x_i and x_{i+1}.



$$x_1, \ldots, x_i, x_{i+1}, \ldots, x_N;\ \mu_{H^*}(x_1) = 1,\ \mu_{H^*}(x_N) = 0$$

If

$$x_i \in \mathrm{LOW}_A(H^*) \rightarrow x_{i+1} \in \mathrm{UPP}_A(H^*)$$

$$w_C(x_{i+1}) \leq w_C(x_i),\ w_{\overline{C}}(x_{i+1}) \geq w_{\overline{C}}(x_i)$$

then

there exists $i_0 :\ x_{i_0} \in \mathrm{BN}_A(H^*)$

$$x_i \in \mathrm{BN}_A(H^*) \rightarrow x_{i+1} \in \mathrm{UPP}_A(U^* \setminus H^*)$$

$$x_i \in \mathrm{LOW}_A(U^* \setminus H^*) \rightarrow x_{i+1} \in \mathrm{LOW}_A(U^* \setminus H^*)$$

Fig. 4. Heap of sand - additional constraints on weights and their consequences.

7 Conclusions

We discussed the sorites paradox in the rough set approach. We have added a discussion on possible new constraints which should be added and preserved by approximations of vague concepts. These constraints are related to behavior of induced classifiers approximating vague concepts on sequences of objects considered in the sorites paradox for these vague concepts. We have also pointed the necessity of development of the adaptive rough set approach.

There are numerous logical approaches to vagueness (see, *e.g.*, [20, 52, 54]). However, from the above considerations it follows that adaptive logic based on rough sets can be relevant for the outlined approach. It is worthwhile mentioning here the following sentences from [53]:

> *Aristotle's man of practical wisdom, the phronimos, does not ignore rules and models, or dispense justice without criteria. He is observant of principles and, at the same time, open to their modification. He begins*

with nomoi established law and employs practical wisdom to determine how it should be applied in particular situations and when departures are warranted. Rules provide the guideposts for inquiry and critical reflection.

We plan to develop a logical approach to vagueness based on rough sets and adaptive judgement [21–24].

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