Towards a sequent calculus for formal contexts

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Abstract. This work focuses on the definition of a consequence relation between contexts with which we can decide whether certain contextual information is a logical consequence from a set of contexts considered as underlying hypotheses.

1 Introduction

In real life, one often faces situations in which the underlying knowledge is given as a set of tables which can be interpreted as formal contexts, and we should decide on whether certain contextual information is a consequence from them.

This problem clearly resembles the notion of a formula being a logical consequence of a set of hypotheses, and suggests the possibility or convenience of defining a formal (mathematical) notion of logical consequence between contexts.

The only attempts to introduce logical content within the machinery of Formal Concept Analysis (FCA), apart from its ancient roots anchored in the Port-Royal logic, are the so-called logical information systems and the logical concept analysis [2, 9].

Of course, different links between FCA and logic have been studied but, to the best of our knowledge, the problem considered in this paper has not been explicitly studied in the literature. Nevertheless, it is worth to remark that the concluding section of [6] states that dual bonds could be given a proof-theoretical interpretation in terms of consequence relations.

In the present work, we consider the fact that the category ChuCors of contexts and Chu correspondences is *-autonomous, and hence a model of linear logic, in order to build some preliminary links with the conjunctive fragment of this logic.

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2 Preliminaries

In order to make the manuscript self-contained, the fundamental notions and their main properties are recalled in this section.

2.1 Context, concept and concept lattice

Definition 1. *A* formal context *is any triple $\mathcal{C} = \langle B, A, R\rangle$ where B and A are finite sets and $R \subseteq B \times A$ is a binary relation. It is customary to say that B is a set of* objects*, A is a set of* attributes *and R represents a relation between objects and attributes.*

Given a formal context $\langle B, A, R\rangle$, the derivation (or concept-forming) operators are a pair of mappings $\uparrow\colon 2^B \to 2^A$ and $\downarrow\colon 2^A \to 2^B$ such that if $X \subseteq B$, then $\uparrow X$ is the set of all attributes which are related to every object in X and, similarly, if $Y \subseteq A$, then $\downarrow Y$ is the set of all objects which are related to every attribute in Y.

Definition 2. *A* formal concept *of a formal context $\mathcal{C} = \langle B, A, R\rangle$ is a pair of sets $\langle X, Y\rangle \in 2^B \times 2^A$ which is a fixpoint of the pair of concept-forming operators, namely, $\uparrow X = Y$ and $\downarrow Y = X$. The object part X is called the* extent *and the attribute part Y is called the* intent*. The set of all formal concepts of a context $\mathcal{C}$ will be denoted by $\mathrm{CL}(\mathcal{C})$, set of all extents or intents of $\mathcal{C}$ will be denoted by $\mathrm{Ext}(\mathcal{C})$ or $\mathrm{Int}(\mathcal{C})$ respectively.*

2.2 Intercontextual structures

Two main constructions have been traditionally considered in order to relate two formal contexts: the bonds and the Chu correspondences.

Definition 3. *Let $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$ and $\mathcal{C}_2 = \langle B_2, A_2, R_2\rangle$ be two formal contexts. A* bond *between $\mathcal{C}_1$ and $\mathcal{C}_2$ is any relation $\beta \in 2^{B_1 \times A_2}$ such that, when interpreted as a table, its columns are extents of $\mathcal{C}_1$ and its rows are intents of $\mathcal{C}_2$. All bonds between such contexts will be denoted by* $\mathrm{Bonds}(\mathcal{C}_1, \mathcal{C}_2)$*.*

Another equivalent definition of bond between $\mathcal{C}_1$ and $\mathcal{C}_2$ defines it as any relation $\beta \in 2^{B_1 \times A_2}$ such that $\mathrm{Ext}(\langle B_1, A_2, \beta\rangle) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\langle B_1, A_2, \beta\rangle) \subseteq \mathrm{Int}(\mathcal{C}_2)$

Dual bonds *between $\mathcal{C}_1$ and $\mathcal{C}_2$ are bonds between $\mathcal{C}_1$ and transposition of $\mathcal{C}_2$. Transposition of any context $\mathcal{C} = \langle B, A, R\rangle$ is defined as a new context $\mathcal{C}^* = \langle A, B, R^t\rangle$ with $R^t(a, b)$ holds iff $R(b, a)$ holds.*

The notion of Chu correspondence between contexts can be seen as an alternative inter-contextual structure which, instead, links intents of $\mathcal{C}_1$ and extents of $\mathcal{C}_2$.

Definition 4. *Consider* $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$ *and* $\mathcal{C}_2 = \langle B_2, A_2, R_2\rangle$ *two formal contexts. A* Chu correspondence *between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is any pair* $\varphi = \langle \varphi_L, \varphi_R\rangle$ *of mappings* $\varphi_L\colon B_1 \to \mathrm{Ext}(\mathcal{C}_2)$ *and* $\varphi_R\colon A_2 \to \mathrm{Int}(\mathcal{C}_1)$ *such that for all* $(b_1, a_2) \in B_1 \times A_2$ *it holds that* $\uparrow_2\big(\varphi_L(b_1)\big)(a_2) = \downarrow_1 \big(\varphi_R(a_2)\big)(b_1)$.

All Chu correspondences between such contexts will be denoted by $\mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$.

The notions of bond and Chu correspondence are interchangeable; specifically, we can consider the bond β_φ associated to a Chu correspondence φ from $\mathcal{C}_1$ to $\mathcal{C}_2$ defined for $b_1 \in B_1, a_2 \in A_2$ as follows

$$\beta_\varphi(b_1, a_2) = \uparrow_2\big(\varphi_L(b_1)\big)(a_2) = \downarrow_1\big(\varphi_R(a_2)\big)(b_1)$$

Similarly, we can consider the Chu correspondence φ_β associated to a bond ρ defined by the following pair of mappings:

$$\varphi_{\beta L}(b_1) = \downarrow_2\big(\beta(b_1)\big) \ \varphi_{\beta R}(a_2) = \uparrow_1\big(\beta^t(a_2)\big) \text{ for all } a_2 \in A_2 \text{ and } o_1 \in B_1$$

The set of all bonds (resp. Chu correspondences) between two formal contexts endowed with the ordering given by set inclusion is a complete lattice. Moreover, both complete lattices are dually isomorphic.

2.3 Categorical products in ChuCors

Recall that it is possible to consider a category in which the objects are formal contexts and morphisms between two contexts are the Chu correspondences between them. This category, denoted ChuCors, has been proved to be *-autonomous and equivalent to the category of complete lattices and isotone Galois connections, more results on this category and its L-fuzzy extensions can be found in [4, 3, 5, 7].

Cartesian product in ChuCors The following definition provides a specific construction of the notion of (binary) cartesian product in the category ChuCors.

Definition 5. *Consider* $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$ *and* $\mathcal{C}_2 = \langle B_2, A_2, R_2\rangle$ *two formal contexts. The* product *of such contexts is a new formal context* $\mathcal{C}_1 \times \mathcal{C}_2 = \langle B_1 \uplus B_2, A_1 \uplus A_2, R_{1\times 2}\rangle$ *where the relation* $R_{1\times 2}$ *is given by*

$$((i, b), (j, a)) \in R_{1\times 2} \text{ if and only if } \big((i = j) \Rightarrow (b, a) \in R_i\big)$$

for any $(b, a) \in B_i \times A_j$ *and* $(i, j) \in \{1, 2\} \times \{1, 2\}$.

If we recall the well-known categorical theorem which states that if a category has a terminal object and binary product, then it has all finite products, we need to prove just the existence of a terminal object (namely, the nullary product) in order to prove the category ChuCors to be Cartesian.

Any formal context of the form $\langle B, A, B \times A \rangle$ where the incidence relation is the full Cartesian product of the sets of objects and attributes is (isomorphic to) the terminal object of ChuCors. Such a formal context has just one formal concept $\langle B, A \rangle$; hence, from any other formal context there is just one Chu correspondence to $\langle B, A, B \times A \rangle$.

The explicit construction of a general product (not necessarily either binary or nullary) is given below:

Definition 6. *Let* $\{\mathcal{C}_i\}_{i \in I}$ *be an indexed family of formal contexts* $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$, *the product* $\prod_{i \in I} \mathcal{C}_i$ *is the formal context given by*

$$\prod_{i \in I} \mathcal{C}_i = \left\langle \biguplus_{i \in I} B_i, \biguplus_{i \in I} A_i, R_{\times I} \right\rangle$$

where $\big((k, b), (m, a)\big) \in R_{\times I} \Leftrightarrow \big((k = m) \Rightarrow (b, a) \in R_k\big)$.

It is worth to note that the arbitrary product of contexts commutes with both the concept lattice construction and the bonds between contexts. These two results are explicitly stated below.

Lemma 1. *Let* $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ *be a formal context for* $i \in I$. *It holds that* $\mathrm{CL}(\prod_{i \in I} \mathcal{C}_i)$ *is isomorphic to* $\prod_{i \in I} \mathrm{CL}(\mathcal{C}_i)$.

Lemma 2. *Let* I *and* J *be two index sets, and consider the two sets of formal contexts* $\{\mathcal{C}_i\}_{i \in I}$ *and* $\{\mathcal{D}_j\}_{j \in J}$. *The following isomorphism holds*

$$\mathrm{Bonds}\Big(\prod_{i \in I} \mathcal{C}_i, \prod_{j \in J} \mathcal{D}_j\Big) \cong \prod_{(i,j) \in I \times J} \mathrm{Bonds}(\mathcal{C}_i, \mathcal{D}_j)\,.$$

Tensor product Another product-like construction can be given in the category ChuCors.

Note that if $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$, then we can consider $\varphi^* \in \mathrm{Chu}(C_2^*, C_1^*)$ defined by $\varphi_L^* = \varphi_R$ and $\varphi_R^* = \varphi_L$.

Definition 7. *The tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ of contexts $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ for $i \in \{1, 2\}$ is defined as the context $\langle B_1 \times B_2, \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*), R_\boxtimes \rangle$ where*

$$R_\boxtimes\big((b_1, b_2), \varphi\big) = \downarrow_2 \big(\varphi_L(b_1)\big)(b_2).$$

The properties of the tensor product were shown in [7], together with the result that ChuCors with $\boxtimes$ is symmetric and monoidal. Those results were later extended to the L-fuzzy case in [3]. In both papers, the structure of the formal concepts of a tensor product context was established as an ordered pair formed by a bond and a set of Chu correspondences.

Lemma 3. *Let (β, X) be a formal concept of the tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$, it holds that $\beta = \bigwedge_{\psi \in X} \beta_\psi$ and $X = \{\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*) \mid \beta \leq \beta_\psi\}$.*

Due to the monoidal properties of $\boxtimes$ on ChuCors we can add a notion of n-ary tensor product of n formal contexts $\boxtimes_{i=1}^n \mathcal{C}_i$ of any n formal contexts $\mathcal{C}_i$ for $i \in \{1, \dots, n\}$. Hence, it is possible to consider a notion of n-ary bond that we can imagine as any extent of n-ary tensor product.

Definition 8. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ be formal contexts for $i \in \{1, \dots, n\}$. A* dual n-ary bond *between $\{\mathcal{C}_i\}_{i=1}^n$ is an n-ary relation $\beta \subseteq \prod_{i=1}^n B_i$ such that for all $i \in \{1, 2, \dots, n\}$ and any $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n) \in \prod_{j=1, j \neq i}^n B_i$ it holds that*

$$\beta(b_1, \dots, b_{i-1}, (-), b_{i+1}, \dots, b_n) \in \mathrm{Ext}(\mathcal{C}_i)\,.$$

Lemma 4. *Let $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ be a set of n formal contexts and β be some n-ary bond between such contexts. Let $\mathcal{D}_i^\beta$ be a new formal context defined as $\langle B_i, \prod_{j=1, j \neq i}^n B_j, \mathcal{R}_i \rangle$ where $\mathcal{R}_i(b_i, (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)) = \beta(b_1, \dots, b_n)$ for any $i \in \{1, 2, \dots, n\}$. Then $\mathrm{Ext}(\mathcal{D}_i^\beta) \subseteq \mathrm{Ext}(\mathcal{C}_i)$.*

3 Conjunctive linear logic in FCA

One of the main differences between linear and classical logic is the coexistence of two different conjunctions in linear logic, in both cases the underlying semantics is that two actions are possible, or can be executed, but the difference relies on how these actions are actually performed: on the one hand, we have the multiplicative conjunction $\otimes$ (times) which expresses that both actions will be performed; on the other hand, the additive conjunction & (with) states that, although both actions are possible, actually just one will be performed.

3.1 Additive conjunction

The categorical product $\times$ on ChuCors plays the role of additive conjunction &. Recall that the product $\mathcal{C}_1 \times \mathcal{C}_2$ is defined as $\langle B_1 \uplus B_2, A_1 \uplus A_2, R_{1\times 2}\rangle$ where $R_{1\times 2}(b,a) = \big((i=j) \Rightarrow (b,a) \in R_i\big)$ for all $b \in B_i$ and $a \in A_j$ for all $i, j \in \{1, 2\}$.

The semantics of the additive conjunction is that, in order to perform the action of $\mathcal{C}_1 \times \mathcal{C}_2$, we have first to choose which among the two possible actions we want to perform, and then to do the one selected. And this is exactly what happens here. From the previous section about the categorical product of ChuCors, it is known that a concept lattice of a product of formal contexts is equal to a product of concept lattices of the input formal contexts. Hence no interaction or parallel action of input formal contexts occurs in case of the categorical product.

3.2 Multiplicative conjunction and dual bonds

Any dual bond between two formal contexts $\mathcal{C}_1$ and $\mathcal{C}_2$ plays a role of multiplicative conjunction $\otimes$. From the definition of bonds one can see the parallelism of use of input contexts, where every value in the bond, as a relation, is from *both* extents of input contexts.

The existing isomorphism between Chu correspondences and bonds, and monoidal properties of Chu correspondences which follows from the fact that Chu correspondences form a category, bonds satisfies all properties of conjunction.

In the sense of category theory, any dual bond can be seen as a Galois connection between concept lattices of their input contexts, because of the categorical equivalence between category ChuCors and the category of complete lattices and isotone Galois connections.

In [1, 8] one can find that the tensor product is used as multiplicative conjunction, due to its monoidal properties on category of Chu Spaces or on any monoidal category in general. Here, in FCA, the tensor product is a special formal context with nice properties that generates all bonds between the input formal contexts. Hence tensor product shows all possibilities how we can connect or use in parallel two formal contexts.

4 Sequent calculus

The idea here is to develop a logic between contexts and, specifically, a proof theory for this logic.

We will consider the problem of defining a consequence relation $\models$ which allows for developing formally a sequent calculus between contexts.

Definition 9. *Two contexts $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if their concept lattices are isomorphic.*

Definition 10. *Given formal contexts $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{C}$, we say that $\mathcal{C}$ is a* consequence *of contexts $\mathcal{C}_1, \ldots, \mathcal{C}_n$, denoted as $\mathcal{C}_1, \ldots, \mathcal{C}_n \models \mathcal{C}$, if $\mathcal{C}$ is isomorphic to a bond between all contexts in the left hand side. Specifically, $\mathcal{C}_1, \ldots, \mathcal{C}_n \models \mathcal{C}$ if and only if $\mathcal{C}$ is isomorphic to some n-ary bond between input formal contexts $\mathcal{C}_1, \ldots, \mathcal{C}_n$.*

The relation just defined satisfies the properties of closure operator and, hence, can be considered as a relation of logical consequence between contexts, since any consequence relation can be viewed as a finitary closure operator on a set (of sentences or formulas).

Lemma 5. *The relation $\models$ above is a consequence relation.*

Recall the notion of sequent of Gentzen calculus. Any sequent is of the form $\Gamma \vdash \Delta$ and has the following meaning: *from the conjunction of all hypothesis of Γ follows some formula of Δ*. Hence as a conjunction of all contexts we use some n-ary bond between input contexts.

Without entering into the details, the following sequent rules from conjunctive linear logic can be proved in terms of this definition.

Axiom rule: As a unary bond we use a context itself

$$\frac{}{\mathcal{C} \models \mathcal{C}}$$

Constants rule Due to isomorphism Bonds $(\mathcal{C}, \top) \cong \mathrm{Ext}(\mathcal{C})$ where $\top = \langle \{\diamond\}, \{\diamond\}, \neq \rangle$ we can write the following rule

$$\frac{\mathcal{C}_1, \ldots, \mathcal{C}_n \models \mathcal{C}}{\top, \mathcal{C}_1, \ldots, \mathcal{C}_n \models \mathcal{C}}$$

$\otimes$-left From the definition of $\models$ and associativity of tensor product, or of associativity inside of n-ary product, we can write

$$\frac{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n, \models \mathcal{C}}{\mathcal{C}_1, \ldots, \mathcal{C}_{n-2}, (\mathcal{C}_{n-1} \otimes \mathcal{C}_n) \models \mathcal{C}}$$

$\otimes$-right Any dual bond between n-ary and m-ary bonds is an $n+m$-ary bond between all input formal contexts

$$\frac{\mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{C} \qquad \mathcal{D}_1 \ldots, \mathcal{D}_m \models \mathcal{D}}{\mathcal{C}_1 \ldots, \mathcal{C}_n, \mathcal{D}_1 \ldots, \mathcal{D}_m \models \mathcal{C} \otimes \mathcal{D}}$$

×-left Due to the distributivity of tensor and categorical product on formal contexts, n-ary bond between $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n \times \mathcal{D}$ is equal to product of n-ary bonds between $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n$ and $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{D}$. Hence it is easy to use a full relation as the n-ary bond between $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{D}$ to obtain the following rule.

$$\frac{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n, \models \mathcal{C}}{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n \times \mathcal{D} \models \mathcal{C}}$$

×-right One of the possibilities here is to add to hypothesis a special context, product of two singletons $\top \times \top$.

$$\frac{\mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{D}_1 \qquad \mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{D}_2}{\mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{D}_1 \times \mathcal{D}_2}$$

5 Conclusion

We have obtained a preliminary notion of logical consequence relation between contexts which, together with the interpretation of the multiplicative (resp. additive) conjunction as the cartesian product (resp. bond) of contexts, enable to prove the correctness of the corresponding rules of the sequent calculus of the conjunctive fragment of linear logic.

Of course, we are just scratching the surface of the problem of providing a full calculus since we still need to find the adequate context-related constructions to interpret the rest of connectives. This is future work.

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