

# Conjunctive Query Answering via a Fragment of Set Theory<sup>\*</sup>

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**Abstract.** We address the problem of Conjunctive Query Answering (CQA) for the description logic  $\mathcal{DL}\langle 4\text{LQS}^{\text{R},\times}\rangle(\mathbf{D})$  ( $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ , for short) which extends the logic  $\mathcal{DL}\langle 4\text{LQS}^{\text{R}}\rangle(\mathbf{D})$  with Boolean operations on concrete roles and with the product of concepts. The result is obtained by formalizing  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge bases and  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive queries in terms of formulae of the four-level set-theoretic fragment  $4\text{LQS}^{\text{R}}$ , which admits a restricted form of quantification on variables of the first three levels and on pair terms. We solve the CQA problem for  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  through a decision procedure for the satisfiability problem of  $4\text{LQS}^{\text{R}}$ . We further define a KE-tableau based procedure for the same problem, more suitable for implementation purposes, and analyze its computational complexity.

## 1 Introduction

In the last few years, results from Computable Set Theory have been used as a means to represent and reason about description logics and rule languages for the semantic web. For instance, in [4–6], fragments of set theory with constructs related to *multi-valued maps* have been studied and applied to the realm of knowledge representation. In [8], an expressive description logic, called  $\mathcal{DL}\langle \text{MLSS}_{2,m}^{\times}\rangle$ , has been introduced and the consistency problem for  $\mathcal{DL}\langle \text{MLSS}_{2,m}^{\times}\rangle$ -knowledge bases has been proved **NP**-complete. The description logic  $\mathcal{DL}\langle \text{MLSS}_{2,m}^{\times}\rangle$  has been extended with additional constructs and SWRL rules in [6], proving that the decision problem for the resulting logic, called  $\mathcal{DL}\langle \forall_{0,2}^{\pi}\rangle$ , is still **NP**-complete under suitable conditions. The description logic  $\mathcal{DL}\langle \forall_{0,2}^{\pi}\rangle$  has been extended with some *metamodelling* features in [4]. In [7], the description logic  $\mathcal{DL}\langle 4\text{LQS}^{\text{R}}\rangle(\mathbf{D})$  (more simply referred to as  $\mathcal{DL}_{\mathbf{D}}^4$ ) has been introduced.  $\mathcal{DL}_{\mathbf{D}}^4$  can be represented

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in the decidable four-level stratified fragment of set theory  $4\text{LQS}^R$  involving a restricted form of quantification over variables of the first three levels and pair terms (cf. [2]). The logic  $\mathcal{DL}_{\mathbf{D}}^4$  admits concept constructs such as full negation, union and intersection of concepts, concept domain and range, existential quantification and min cardinality on the left-hand side of inclusion axioms. It also supports role constructs such as role chains on the left hand side of inclusion axioms, union, intersection, and complement of abstract roles, and properties on roles such as transitivity, symmetry, reflexivity, and irreflexivity. It admits datatypes, a simple form of concrete domains that are relevant in real world applications.

The consistency problem for  $\mathcal{DL}_{\mathbf{D}}^4$ -knowledge bases has been proved decidable in [7] by means of a reduction to the satisfiability problem for  $4\text{LQS}^R$ , proved decidable in [2]. It has also been proved, under not very restrictive constraints, that the consistency problem for  $\mathcal{DL}_{\mathbf{D}}^4$ -knowledge bases is **NP**-complete. The latter result has practical outcomes since, for example, the ontology *Ontoceramic* [9] can be expressed in such a restricted version of  $\mathcal{DL}_{\mathbf{D}}^4$ . Finally, we mention that the papers [4–8] are concerned with traditional research issues for description logics mainly focused on the parts of a knowledge base representing conceptual information, namely the TBox and the RBox, where the principal reasoning services are subsumption and satisfiability.

In this paper we exploit decidability results presented in [2, 7] to deal with reasoning services for knowledge bases involving ABoxes. The most basic service to query the instance data is *instance retrieval*, i.e., the task of retrieving all individuals that instantiate a class  $C$ , and, dually, all named classes  $C$  that an individual belongs to. In particular, a powerful way to query ABoxes is the *Conjunctive Query Answering* task (CQA). CQA is relevant in the context of description logics and, in particular, for real world applications based on semantic web technologies, since it provides a mechanism allowing users and applications to interact with ontologies and data. The task of CQA has been studied for several well-known description logics (cf. [1, 13, 15]).

In particular, we introduce the description logic  $\mathcal{DL}\langle 4\text{LQS}^{R,\times} \rangle(\mathbf{D})$  ( $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ , for short), extending  $\mathcal{DL}_{\mathbf{D}}^4$  with Boolean operations on concrete roles and with the product of concepts. Then we define the CQA problem for  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  and prove its decidability via a reduction to the CQA problem for  $4\text{LQS}^R$ , whose decidability follows from that of the satisfiability problem for  $4\text{LQS}^R$  (proved in [2]). Finally, we present a KE-tableau based procedure that, given a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -query  $Q$  and a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base  $\mathcal{KB}$  represented in set-theoretic terms, determines the answer set of  $Q$  with respect to  $\mathcal{KB}$ , providing also some complexity results. The choice of the KE-tableau system [10] is motivated by the fact that this variant of the tableau method allows one to construct trees whose distinct branches define mutually exclusive situations thus avoiding the proliferation of redundant branches, typical of semantic tableaux.

## 2 Preliminaries

### 2.1 The set-theoretic fragment 4LQS<sup>R</sup>

It is convenient to first introduce the syntax and semantics of a more general four-level quantified language, denoted 4LQS. Then we provide some restrictions on quantified formulae of 4LQS that characterize 4LQS<sup>R</sup>. We recall that the satisfiability problem for 4LQS<sup>R</sup> has been proved decidable in [2].

4LQS involves four collections,  $\mathcal{V}_i$ , of variables of sort  $i$ , for  $i = 0, 1, 2, 3$ . Variables of sort  $i$ , for  $i = 0, 1, 2, 3$ , will be denoted by  $X^i, Y^i, Z^i, \dots$  (in particular, variables of sort 0 will also be denoted by  $x, y, z, \dots$ ). In addition to variables, 4LQS involves also *pair terms* of the form  $\langle x, y \rangle$ , with  $x, y \in \mathcal{V}_0$ .

4LQS-*quantifier-free atomic formulae* are classified as:

- level 0:  $x = y, x \in X^1, \langle x, y \rangle = X^2, \langle x, y \rangle \in X^3$ ;
- level 1:  $X^1 = Y^1, X^1 \in X^2$ ;
- level 2:  $X^2 = Y^2, X^2 \in X^3$ .

4LQS *purely universal formulae* are classified as:

- level 1:  $(\forall z_1) \dots (\forall z_n) \varphi_0$ , where  $z_1, \dots, z_n \in \mathcal{V}_0$  and  $\varphi_0$  is any propositional combination of quantifier-free atomic formulae of level 0;
- level 2:  $(\forall Z_1^1) \dots (\forall Z_m^1) \varphi_1$ , where  $Z_1^1, \dots, Z_m^1 \in \mathcal{V}_1$  and  $\varphi_1$  is any propositional combination of quantifier-free atomic formulae of levels 0 and 1, and of purely universal formulae of level 1;
- level 3:  $(\forall Z_1^2) \dots (\forall Z_p^2) \varphi_2$ , where  $Z_1^2, \dots, Z_p^2 \in \mathcal{V}_2$  and  $\varphi_2$  is any propositional combination of quantifier-free atomic formulae and of purely universal formulae of levels 1 and 2.

4LQS-formulae are all the propositional combinations of quantifier-free atomic formulae of levels 0, 1, 2 and of purely universal formulae of levels 1, 2, 3.

Let  $\varphi$  be a 4LQS-formula. Without loss of generality, we can assume that  $\varphi$  contains only  $\neg, \wedge, \vee$  as propositional connectives. Further, let  $S_\varphi$  be the syntax tree for a 4LQS-formula  $\varphi$ ,<sup>1</sup> and let  $\nu$  be a node of  $S_\varphi$ . We say that a 4LQS-formula  $\psi$  occurs within  $\varphi$  at position  $\nu$  if the subtree of  $S_\varphi$  rooted at  $\nu$  is identical to  $S_\psi$ . In this case we refer to  $\nu$  as an occurrence of  $\psi$  in  $\varphi$  and to the path from the root of  $S_\varphi$  to  $\nu$  as its occurrence path. An occurrence of  $\psi$  within  $\varphi$  is *positive* if its occurrence path deprived by its last node contains an even number of nodes labelled by a 4LQS-formula of type  $\neg\chi$ . Otherwise, the occurrence is said to be *negative*.

The variables  $z_1, \dots, z_n$  are said to occur *quantified* in  $(\forall z_1) \dots (\forall z_n) \varphi_0$ . Likewise,  $Z_1^1, \dots, Z_m^1$  and  $Z_1^2, \dots, Z_p^2$  occur quantified in  $(\forall Z_1^1) \dots (\forall Z_m^1) \varphi_1$  and in  $(\forall Z_1^2) \dots (\forall Z_p^2) \varphi_2$ , respectively. A variable occurs *free* in a 4LQS-formula  $\varphi$  if it does not occur quantified in any subformula of  $\varphi$ . For  $i = 0, 1, 2, 3$ , we denote with  $\text{Var}_i(\varphi)$  the collections of variables of level  $i$  occurring free in  $\varphi$ .

<sup>1</sup> The notion of syntax tree for 4LQS-formulae is similar to the notion of syntax tree for formulae of first-order logic. A precise definition of the latter can be found in [11].

A (level 0) substitution  $\sigma := \{x_1/y_1, \dots, x_n/y_n\}$  is the mapping  $\varphi \mapsto \varphi\sigma$  such that, for any given 4LQS-formula  $\varphi$ ,  $\varphi\sigma$  is the 4LQS-formula obtained from  $\varphi$  by replacing the free occurrences of the variables  $x_1, \dots, x_n$  in  $\varphi$  with the variables  $y_1, \dots, y_n$ , respectively. We say that a substitution  $\sigma$  is free for  $\varphi$  if the formulae  $\varphi$  and  $\varphi\sigma$  have exactly the same occurrences of quantified variables.

A 4LQS-*interpretation* is a pair  $\mathcal{M} = (D, M)$  where  $D$  is a non-empty collection of objects (called *domain* or *universe* of  $\mathcal{M}$ ) and  $M$  is an assignment over the variables in  $\mathcal{V}_i$ , for  $i = 0, 1, 2, 3$ , such that:  $MX^0 \in D, MX^1 \in \mathcal{P}(D), MX^2 \in \mathcal{P}(\mathcal{P}(D)), MX^3 \in \mathcal{P}(\mathcal{P}(\mathcal{P}(D)))$ , where  $X^i \in \mathcal{V}_i$ , for  $i = 0, 1, 2, 3$ , and  $\mathcal{P}(s)$  denotes the powerset of  $s$ .

Pair terms are interpreted *à la* Kuratowski, and therefore we put

$$M\langle x, y \rangle := \{\{Mx\}, \{Mx, My\}\}.$$

Next, let

- $\mathcal{M} = (D, M)$  be a 4LQS-interpretation,
- $x_1, \dots, x_n \in \mathcal{V}_0, X_1^1, \dots, X_m^1 \in \mathcal{V}_1, X_1^2, \dots, X_p^2 \in \mathcal{V}_2$ , and
- $u_1, \dots, u_n \in D, U_1^1, \dots, U_m^1 \in \mathcal{P}(D), U_1^2, \dots, U_p^2 \in \mathcal{P}(\mathcal{P}(D))$ .

By  $\mathcal{M}[\vec{x}/\vec{u}, \vec{X}^1/\vec{U}^1, \vec{X}^2/\vec{U}^2]$ , we denote the interpretation  $\mathcal{M}' = (D, M')$  such that  $M'x_i = u_i$  (for  $i = 1, \dots, n$ ),  $M'X_j^1 = U_j^1$  (for  $j = 1, \dots, m$ ),  $M'X_k^2 = U_k^2$  (for  $k = 1, \dots, p$ ), and which otherwise coincides with  $M$  on all remaining variables. For a 4LQS-interpretation  $\mathcal{M} = (D, M)$  and a 4LQS-formula  $\varphi$ , the satisfiability relationship  $\mathcal{M} \models \varphi$  is defined inductively over the structure of  $\varphi$  as follows. Quantifier-free atomic formulae are evaluated in a standard way according to the usual meaning of the predicates ‘ $\in$ ’ and ‘ $=$ ’, and purely universal formulae are evaluated as follows:

- $\mathcal{M} \models (\forall z_1) \dots (\forall z_n) \varphi_0$  iff  $\mathcal{M}[\vec{z}/\vec{u}] \models \varphi_0$ , for all  $\vec{u} \in D^n$ ;
- $\mathcal{M} \models (\forall Z_1^1) \dots (\forall Z_m^1) \varphi_1$  iff  $\mathcal{M}[\vec{Z}^1/\vec{U}^1] \models \varphi_1$ , for all  $\vec{U}^1 \in (\mathcal{P}(D))^m$ ;
- $\mathcal{M} \models (\forall Z_1^2) \dots (\forall Z_p^2) \varphi_2$  iff  $\mathcal{M}[\vec{Z}^2/\vec{U}^2] \models \varphi_2$ , for all  $\vec{U}^2 \in (\mathcal{P}(\mathcal{P}(D)))^p$ .

Finally, compound formulae are interpreted according to the standard rules of propositional logic. If  $\mathcal{M} \models \varphi$ , then  $\mathcal{M}$  is said to be a 4LQS-model for  $\varphi$ . A 4LQS-formula is said to be *satisfiable* if it has a 4LQS-model. A 4LQS-formula is *valid* if it is satisfied by all 4LQS-interpretations.

We are now ready to present the fragment 4LQS<sup>R</sup> of 4LQS of our interest. This is the collection of the formulae  $\psi$  of 4LQS fulfilling the restrictions:

1. for every purely universal formula  $(\forall Z_1^1) \dots (\forall Z_m^1) \varphi_1$  of level 2 occurring in  $\psi$  and every purely universal formula  $(\forall z_1) \dots (\forall z_n) \varphi_0$  of level 1 occurring negatively in  $\varphi_1$ ,  $\varphi_0$  is a propositional combination of quantifier-free atomic formulae of level 0 and the condition

$$\neg \varphi_0 \rightarrow \bigwedge_{i=1}^n \bigwedge_{j=1}^m z_i \in Z_j^1$$

is a valid 4LQS-formula (in this case we say that  $(\forall z_1) \dots (\forall z_n) \varphi_0$  is *linked to the variables*  $Z_1^1, \dots, Z_m^1$ );

2. for every purely universal formula  $(\forall Z_1^2) \dots (\forall Z_p^2) \varphi_2$  of level 3 in  $\psi$ :

- every purely universal formula of level 1 occurring negatively in  $\varphi_2$  and not occurring in a purely universal formula of level 2 is only allowed to be of the form

$$(\forall z_1) \dots (\forall z_n) \neg \left( \bigwedge_{i=1}^n \bigwedge_{j=1}^n \langle z_i, z_j \rangle = Y_{ij}^2 \right),$$

with  $Y_{ij}^2 \in \mathcal{V}^2$ , for  $i, j = 1, \dots, n$ ;

- purely universal formulae  $(\forall Z_1^1) \dots (\forall Z_m^1) \varphi_1$  of level 2 may occur only positively in  $\varphi_2$ .

Restriction 1 has been introduced for technical reasons concerning the decidability of the satisfiability problem for the fragment, while restriction 2 allows one to define binary relations and several operations on them (for space reasons details are not included here but can be found in [2]).

The semantics of  $4LQS^R$  plainly coincides with that of  $4LQS$ .

## 2.2 The logic $\mathcal{DL}\langle 4LQS^{R,\times} \rangle(\mathbf{D})$

The description logic  $\mathcal{DL}\langle 4LQS^{R,\times} \rangle(\mathbf{D})$  (more simply referred to as  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ ) is the extension of the logic  $\mathcal{DL}\langle 4LQS^R \rangle(\mathbf{D})$  (for short  $\mathcal{DL}_{\mathbf{D}}^4$ ) presented in [7] in which Boolean operations on concrete roles and the product of concepts are admitted. Analogously to  $\mathcal{DL}_{\mathbf{D}}^4$ , the logic  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  supports concept constructs such as full negation, union and intersection of concepts, concept domain and range, existential quantification and min cardinality on the left-hand side of inclusion axioms, role constructs such as role chains on the left hand side of inclusion axioms, union, intersection, and complement of roles, and properties on roles such as transitivity, symmetry, reflexivity, and irreflexivity.

As far as the construction of role inclusion axioms is concerned,  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  is more liberal than  $SR\mathcal{OIQ}(\mathbf{D})$  [12] (the logic underlying the most expressive Ontology Web Language 2 profile, OWL 2 DL [16]), since the roles involved are not required to be subject to any ordering relationship, and the notion of simple role is not needed.  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  treats derived datatypes by admitting datatype terms constructed from data ranges by means of a finite number of applications of the Boolean operators. Basic and derived datatypes can be used inside inclusion axioms involving concrete roles.

Datatypes are defined according to [14] as follows. Let  $\mathbf{D} = (N_D, N_C, N_F, \cdot^{\mathbf{D}})$  be a *datatype map*, where  $N_D$  is a finite set of datatypes,  $N_C$  is a map assigning a set of constants  $N_C(d)$  to each datatype  $d \in N_D$ ,  $N_F$  is a map assigning a set of facets  $N_F(d)$  to each  $d \in N_D$ , and  $\cdot^{\mathbf{D}}$  is a map assigning (i) a datatype interpretation  $d^{\mathbf{D}}$  to each datatype  $d \in N_D$ , (ii) a facet interpretation  $f^{\mathbf{D}} \subseteq d^{\mathbf{D}}$  to each facet  $f \in N_F(d)$ , and (iii) a data value  $e_d^{\mathbf{D}} \in d^{\mathbf{D}}$  to every constant  $e_d \in N_C(d)$ . We shall assume that the interpretations of the datatypes in  $N_D$  are non-empty pairwise disjoint sets.

A *facet expression* for a datatype  $d \in N_D$  is a formula  $\psi_d$  constructed from the elements of  $N_F(d) \cup \{\top_d, \perp_d\}$  by applying a finite number of times the connectives  $\neg$ ,  $\wedge$ , and  $\vee$ . The function  $\cdot^{\mathbf{D}}$  is extended to facet expressions for

$d \in N_D$  by putting  $\top_d^{\mathbf{D}} = d^{\mathbf{D}}$ ,  $\perp_d^{\mathbf{D}} = \emptyset$ ,  $(\neg f)^{\mathbf{D}} = d^{\mathbf{D}} \setminus f^{\mathbf{D}}$ ,  $(f_1 \wedge f_2)^{\mathbf{D}} = f_1^{\mathbf{D}} \cap f_2^{\mathbf{D}}$ , and  $(f_1 \vee f_2)^{\mathbf{D}} = f_1^{\mathbf{D}} \cup f_2^{\mathbf{D}}$ , for  $f, f_1, f_2 \in N_F(d)$ .

A *data range*  $dr$  for  $\mathbf{D}$  is either a datatype  $d \in N_D$ , or a finite enumeration of datatype constants  $\{e_{d_1}, \dots, e_{d_n}\}$ , with  $e_{d_i} \in N_C(d_i)$  and  $d_i \in N_D$ , or a facet expression  $\psi_d$ , for  $d \in N_D$ , or their complementation.

Let  $\mathbf{R}_A, \mathbf{R}_D, \mathbf{C}, \mathbf{Ind}$  be denumerable pairwise disjoint sets of abstract role names, concrete role names, concept names, and individual names, respectively. We assume that the set of abstract role names  $\mathbf{R}_A$  contains a name  $U$  denoting the universal role.

(a)  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -datatype, (b)  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concept, (c)  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role, and (d)  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concrete role terms are constructed according to the following syntax rules:

- (a)  $t_1, t_2 \longrightarrow dr \mid \neg t_1 \mid t_1 \sqcap t_2 \mid t_1 \sqcup t_2 \mid \{e_d\}$ ,
- (b)  $C_1, C_2 \longrightarrow A \mid \top \mid \perp \mid \neg C_1 \mid C_1 \sqcup C_2 \mid C_1 \sqcap C_2 \mid \{a\} \mid \exists R. Self \mid \exists R. \{a\} \mid \exists P. \{e_d\}$ ,
- (c)  $R_1, R_2 \longrightarrow S \mid U \mid R_1^- \mid \neg R_1 \mid R_1 \sqcup R_2 \mid R_1 \sqcap R_2 \mid R_{C_1} \mid R_{\mid C_1} \mid R_{C_1} \mid c_2 \mid id(C) \mid C_1 \times C_2$ ,
- (d)  $P_1, P_2 \longrightarrow T \mid \neg P_1 \mid P_1 \sqcup P_2 \mid P_1 \sqcap P_2 \mid P_{C_1} \mid P_{\mid t_1} \mid P_{C_1 \mid t_1}$ ,

where  $dr$  is a data range for  $\mathbf{D}$ ,  $t_1, t_2$  are data-type terms,  $e_d$  is a constant in  $N_C(d)$ ,  $a$  is an individual name,  $A$  is a concept name,  $C_1, C_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concept terms,  $S$  is an abstract role name,  $R, R_1, R_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role terms,  $T$  is a concrete role name, and  $P, P_1, P_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concrete role terms.

A  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base is a triple  $\mathcal{KB} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$  such that  $\mathcal{R}$  is a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -RBox,  $\mathcal{T}$  is a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -TBox, and  $\mathcal{A}$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -ABox (see next).

A  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -RBox is a collection of statements of the following forms:

$R_1 \equiv R_2$ ,  $R_1 \sqsubseteq R_2$ ,  $R_1 \dots R_n \sqsubseteq R_{n+1}$ ,  $\text{Sym}(R_1)$ ,  $\text{Asym}(R_1)$ ,  $\text{Ref}(R_1)$ ,  $\text{Irref}(R_1)$ ,  $\text{Dis}(R_1, R_2)$ ,  $\text{Tra}(R_1)$ ,  $\text{Fun}(R_1)$ ,  $R_1 \equiv C_1 \times C_2$ ,  $P_1 \equiv P_2$ ,  $P_1 \sqsubseteq P_2$ ,  $\text{Dis}(P_1, P_2)$ ,  $\text{Fun}(P_1)$ , where  $R_1, R_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role terms,  $C_1, C_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract concept terms, and  $P_1, P_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concrete role terms. Any expression of the type  $w \sqsubseteq R$ , where  $w$  is a finite string of  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role terms and  $R$  is an  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role term is called a *role inclusion axiom (RIA)*.

Next, a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -TBox is a set of statements of the types:

$C_1 \equiv C_2$ ,  $C_1 \sqsubseteq C_2$ ,  $C_1 \sqsubseteq \forall R_1. C_2$ ,  $\exists R_1. C_1 \sqsubseteq C_2$ ,  $\geq_n R_1. C_1 \sqsubseteq C_2$ ,  $C_1 \sqsubseteq \leq_n R_1. C_2$ ,  $t_1 \equiv t_2$ ,  $t_1 \sqsubseteq t_2$ ,  $C_1 \sqsubseteq \forall P_1. t_1$ ,  $\exists P_1. t_1 \sqsubseteq C_1$ ,  $\geq_n P_1. t_1 \sqsubseteq C_1$ ,  $C_1 \sqsubseteq \leq_n P_1. t_1$ , where  $C_1, C_2$  are  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concept terms,  $t_1, t_2$  datatype terms,  $R_1$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role term, and  $P_1$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concrete role term. Any statement  $C \sqsubseteq D$ , with  $C$  and  $D$   $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concept terms, is a *general concept inclusion axiom (GCI)*.

Finally, a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -ABox is a set of *individual assertions* of the forms:  $a : C_1$ ,  $(a, b) : R_1$ ,  $a = b$ ,  $e_d : t_1$ ,  $(a, e_d) : P_1$ , with  $a, b$  individual names,  $C_1$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concept term,  $R_1$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role term,  $d$  a datatype,  $e_d$  a constant in  $N_C(d)$ ,  $t_1$  a datatype term, and  $P_1$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concrete role term. As mentioned above,  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  is more liberal than  $\mathcal{SROIQ}(\mathbf{D})$  in the construction of role inclusion axioms. For example, the role hierarchy  $\{RS \sqsubseteq S, RT \sqsubseteq R, VT \sqsubseteq T, VS \sqsubseteq V\}$  presented in [12] is expressible in  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ , but not in  $\mathcal{SROIQ}(\mathbf{D})$ .

The semantics of  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  is based on interpretations  $\mathbf{I} = (\Delta^{\mathbf{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathbf{I}})$ , where  $\Delta^{\mathbf{I}}$  and  $\Delta_{\mathbf{D}}$  are non-empty disjoint domains such that  $d^{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}$ , for every  $d \in N_{\mathbf{D}}$ , and  $\cdot^{\mathbf{I}}$  is an interpretation function. The interpretation of concepts and roles, and of axioms and assertions is illustrated in [3, Table 1].

Let  $\mathcal{KB} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$  be a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base. An interpretation  $\mathbf{I} = (\Delta^{\mathbf{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathbf{I}})$  is a  $\mathbf{D}$ -model of  $\mathcal{R}$  (and write  $\mathbf{I} \models_{\mathbf{D}} \mathcal{R}$ ) if  $\mathbf{I}$  satisfies each axiom in  $\mathcal{R}$  according to the semantic rules in [3, Table 1]. Similar definitions hold for  $\mathcal{T}$  and  $\mathcal{A}$  too. Then  $\mathbf{I}$  satisfies  $\mathcal{KB}$  (and write  $\mathbf{I} \models_{\mathbf{D}} \mathcal{KB}$ ) if it is a  $\mathbf{D}$ -model of  $\mathcal{R}$ ,  $\mathcal{T}$ , and  $\mathcal{A}$ . A knowledge base is *consistent* if it is satisfied by some interpretation.

### 3 Conjunctive Query Answering for $\mathcal{DL}_{\mathbf{D}}^{4,\times}$

Let  $\mathcal{V} = \{v_1, v_2, \dots\}$  be a denumerable and infinite set of variables disjoint from  $\mathbf{Ind}$  and from  $\bigcup\{N_C(d) : d \in N_{\mathbf{D}}\}$ . A  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -atomic formula is an expression of the following types

$$R(w_1, w_2), \quad P(w_1, u_1), \quad C(w_1), \quad w_1 = w_2, \quad u_1 = u_2,$$

where  $w_1, w_2 \in \mathcal{V} \cup \mathbf{Ind}$ ,  $u_1, u_2 \in \mathcal{V} \cup \bigcup\{N_C(d) : d \in N_{\mathbf{D}}\}$ ,  $R$  is a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -abstract role term,  $P$  is a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concrete role term, and  $C$  is a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -concept term. A  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -atomic formula containing no variables is said to be *closed*. A  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -literal is a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -atomic formula or its negation. A  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive query is a conjunction of  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -literals. Let  $v_1, \dots, v_n \in \mathcal{V}$  and  $o_1, \dots, o_n \in \mathbf{Ind} \cup \bigcup\{N_C(d) : d \in N_{\mathbf{D}}\}$ . A substitution  $\sigma := \{v_1/o_1, \dots, v_n/o_n\}$  is a map such that, for every  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -literal  $L$ ,  $L\sigma$  is obtained from  $L$  by replacing the occurrences of  $v_1, \dots, v_n$  in  $L$  with  $o_1, \dots, o_n$ , respectively. Substitutions can be extended to  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive queries in the usual way. Let  $Q := (L_1 \wedge \dots \wedge L_m)$  be a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive query, and  $\mathcal{KB}$  a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base. A substitution  $\sigma$  involving *exactly* the variables occurring in  $Q$  is a *solution for  $Q$  w.r.t.  $\mathcal{KB}$*  if there exists a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -interpretation  $\mathbf{I}$  such that  $\mathbf{I} \models_{\mathbf{D}} \mathcal{KB}$  and  $\mathbf{I} \models_{\mathbf{D}} Q\sigma$ . The collection  $\Sigma$  of the solutions for  $Q$  w.r.t.  $\mathcal{KB}$  is the *answer set of  $Q$  w.r.t.  $\mathcal{KB}$* . Then the *conjunctive query answering* (CQA) problem for  $Q$  w.r.t.  $\mathcal{KB}$  consists in finding the answer set  $\Sigma$  of  $Q$  w.r.t.  $\mathcal{KB}$ .

We shall solve the CQA problem just stated by reducing it to the analogous problem formulated in the context of the fragment  $4\text{LQS}^{\mathbf{R}}$  (and in turn to the decision procedure for  $4\text{LQS}^{\mathbf{R}}$  presented in [2]). The CQA problem for  $4\text{LQS}^{\mathbf{R}}$ -formulae can be stated as follows. Let  $\phi$  be a  $4\text{LQS}^{\mathbf{R}}$ -formula and let  $\psi$  be a conjunction of  $4\text{LQS}^{\mathbf{R}}$ -quantifier-free atomic formulae of level 0 of the types  $x = y$ ,  $x \in X^1$ ,  $\langle x, y \rangle \in X^3$  or their negations, such that  $\text{Var}_0(\psi) \cap \text{Var}_0(\phi) = \emptyset$  and  $\text{Var}_1(\psi) \cup \text{Var}_3(\psi) \subseteq \text{Var}_1(\phi) \cup \text{Var}_3(\phi)$ . The *CQA problem for  $\psi$  w.r.t.  $\phi$*  consists in computing the *answer set of  $\psi$  w.r.t.  $\phi$* , namely the collection  $\Sigma'$  of all the substitutions  $\sigma' := \{x_1/y_1, \dots, x_n/y_n\}$  (where  $x_1, \dots, x_n$  are the distinct variables of level 0 in  $\psi$  and  $\{y_1, \dots, y_n\} \subseteq \text{Var}_0(\phi)$ ) such that  $\mathcal{M} \models \phi \wedge \psi\sigma'$ , for some  $4\text{LQS}^{\mathbf{R}}$ -interpretation  $\mathcal{M}$ . In view of the decidability of the satisfiability problem for  $4\text{LQS}^{\mathbf{R}}$ -formulae, the CQA problem for  $4\text{LQS}^{\mathbf{R}}$ -formulae is decidable as well. Indeed, given two  $4\text{LQS}^{\mathbf{R}}$ -formulae  $\phi$  and  $\psi$  satisfying the

above requirements, to compute the answer set of  $\psi$  w.r.t.  $\phi$ , for each candidate substitution  $\sigma' := \{x_1/y_1, \dots, x_n/y_n\}$  (with  $\{x_1, \dots, x_n\} = \text{Var}_0(\psi)$  and  $\{y_1, \dots, y_n\} \subseteq \text{Var}_0(\phi)$ ) one has just to test for satisfiability the  $4\text{LQS}^{\text{R}}$ -formula  $\phi \wedge \psi\sigma'$ . Since the number of possible candidate substitutions is  $|\text{Var}_0(\phi)|^{|\text{Var}_0(\psi)|}$  and the satisfiability problem for  $4\text{LQS}^{\text{R}}$ -formulae is decidable, it follows that the answer set of  $\psi$  w.r.t.  $\phi$  can be computed effectively. Summarizing,

**Lemma 1.** *The CQA problem for  $4\text{LQS}^{\text{R}}$ -formulae is decidable.  $\square$*

The following theorem states that also the CQA problem for  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  is decidable.

**Theorem 1.** *Given a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base  $\mathcal{KB}$  and a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive query  $Q$ , the CQA problem for  $Q$  w.r.t.  $\mathcal{KB}$  is decidable.*

*Proof (sketch).* For space reasons we just outline the main ideas of the proof. The interested reader, however, can find complete details in the extended version of this paper (see [3]).

As remarked above, the CQA problem for  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  can be solved via an effective reduction to the CQA problem for  $4\text{LQS}^{\text{R}}$ -formulae, and then exploiting Lemma 1. The reduction is accomplished through a function  $\theta$  that maps effectively variables in  $\mathcal{V}$  and individuals in **Ind** into variables of sort 0 (of the  $4\text{LQS}^{\text{R}}$ -language), etc.,  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -TBoxes, -RBoxes, and -ABoxes, and  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive queries into  $4\text{LQS}^{\text{R}}$ -formulae in conjunctive normal form (CNF), which can be used to map effectively CQA problems from the  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -context into the  $4\text{LQS}^{\text{R}}$ -context. More specifically, given a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base  $\mathcal{KB}$  and a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive query  $Q$ , using the function  $\theta$  we can effectively construct the following  $4\text{LQS}^{\text{R}}$ -formulae in CNF:

$$\phi_{\mathcal{KB}} := \bigwedge_{H \in \mathcal{KB}} \theta(H) \wedge \bigwedge_{i=1}^{12} \xi_i, \quad \psi_Q := \theta(Q).^2$$

Then, if we denote by  $\Sigma$  the answer set of  $Q$  w.r.t.  $\mathcal{KB}$  and by  $\Sigma'$  the answer set of  $\psi_Q$  w.r.t.  $\phi_{\mathcal{KB}}$ , we have that  $\Sigma$  consists of all substitutions  $\sigma$  (involving exactly the variables occurring in  $Q$ ) such that  $\theta(\sigma) \in \Sigma'$ . Since, by Lemma 1,  $\Sigma'$  can be computed effectively, then  $\Sigma$  can be computed effectively too.  $\square$

## 4 A tableau-based procedure

In this section, we illustrate a KE-tableau based procedure that, given a  $4\text{LQS}^{\text{R}}$ -formula  $\phi_{\mathcal{KB}}$  corresponding to a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base and a  $4\text{LQS}^{\text{R}}$ -formula

<sup>2</sup> The definition of the function  $\theta$  is inspired to that of the function  $\tau$  introduced in the proof of Theorem 1 in [7]. Specifically,  $\theta$  differs from  $\tau$  as (i) it allows quantification only on variables of level 0, (ii) it treats Boolean operations on concrete roles and the product of concepts, and (iii) it constructs  $4\text{LQS}^{\text{R}}$ -formulae in CNF. In addition, the constraints  $\xi_1$ – $\xi_{12}$  are similar to the constraints  $\psi_1$ – $\psi_{12}$  introduced in the proof of Theorem 1 in [7]; they are introduced to guarantee that each model of  $\phi_{\mathcal{KB}}$  can be transformed into a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -interpretation. Details of the construction of  $\theta$  and of  $\xi_1$ – $\xi_{12}$  can be found in [3].



$\psi_Q$  corresponding to a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive query  $Q$ , yields all the substitutions  $\sigma = \{x_1/y_1, \dots, x_n/y_n\}$ , with  $\{x_1, \dots, x_n\} = \text{Var}_0(\psi_Q)$  and  $\{y_1, \dots, y_n\} \subseteq \text{Var}_0(\phi_{\mathcal{KB}})$ , belonging to the answer set  $\Sigma'$  of  $\psi_Q$  w.r.t.  $\phi_{\mathcal{KB}}$ .

Let  $\bar{\phi}_{\mathcal{KB}}$  be the formula obtained from  $\phi_{\mathcal{KB}}$  by:

- moving universal quantifiers in  $\phi_{\mathcal{KB}}$  as inwards as possible according to the rule  $(\forall z)(A(z) \wedge B(z)) \longleftrightarrow ((\forall z)A(z) \wedge (\forall z)B(z))$ ,
- renaming universally quantified variables so as to make them pairwise distinct.

Let  $F_1, \dots, F_k$  be the conjuncts of  $\bar{\phi}_{\mathcal{KB}}$  that are 4LQS<sup>R</sup>-quantifier-free atomic formulae and  $S_1, \dots, S_m$  the conjuncts of  $\bar{\phi}_{\mathcal{KB}}$  that are 4LQS<sup>R</sup>-purely universal formulae. For every  $S_i = (\forall z_1^i) \dots (\forall z_{n_i}^i) \chi_i$ ,  $i = 1, \dots, m$ , we put

$$\text{Exp}(S_i) := \bigwedge_{\{x_{a_1}, \dots, x_{a_{n_i}}\} \subseteq \text{Var}_0(\bar{\phi}_{\mathcal{KB}})} S_i\{z_1^i/x_{a_1}, \dots, z_{n_i}^i/x_{a_{n_i}}\}.$$

Let  $\bar{\Phi}_{\mathcal{KB}} := \{F_j : j = 1, \dots, k\} \cup \bigcup_{i=1}^m \text{Exp}(S_i)$ .

To prepare for the KE-tableau based procedure to be described next, we introduce some useful notions and notations (see [10] for a detailed overview of KE-tableau, an optimized variant of semantic tableaux).

Let  $\Phi = \{C_1, \dots, C_p\}$  be a collection of disjunctions of 4LQS<sup>R</sup>-quantifier-free atomic formulae of level 0 of the types:  $x = y$ ,  $x \in X^1$ ,  $\langle x, y \rangle \in X^3$ .  $\mathcal{T}$  is a *KE-tableau* for  $\Phi$  if there exists a finite sequence  $\mathcal{T}_1, \dots, \mathcal{T}_t$  such that (i)  $\mathcal{T}_1$  is a one-branch tree consisting of the sequence  $C_1, \dots, C_p$ , (ii)  $\mathcal{T}_t = \mathcal{T}$ , and (iii) for each  $i < t$ ,  $\mathcal{T}_{i+1}$  is obtained from  $\mathcal{T}_i$  by an application of one of the rules in Fig 1. The set of formulae  $\mathcal{S}_i^{\bar{\beta}} = \{\bar{\beta}_1, \dots, \bar{\beta}_n\} \setminus \{\bar{\beta}_i\}$  occurring as premise in the E-rule contains the complements of all the components of the formula  $\beta$  with the exception of the component  $\beta_i$ .

$$\frac{\beta_1 \vee \dots \vee \beta_n \quad \mathcal{S}_i^{\bar{\beta}}}{\beta_i} \text{ E-Rule} \qquad \frac{}{A \mid \bar{A}} \text{ PB-Rule}$$

where  $\mathcal{S}_i^{\bar{\beta}} := \{\bar{\beta}_1, \dots, \bar{\beta}_n\} \setminus \{\bar{\beta}_i\}$ ,  
for  $i = 1, \dots, n$  with A a literal

**Fig. 1.** Expansion rules for the KE-tableau.

Let  $\mathcal{T}$  be a KE-tableau. A branch  $\vartheta$  of  $\mathcal{T}$  is *closed* if it contains both  $A$  and  $\neg A$ , for some formula  $A$ . Otherwise, the branch is *open*. A formula  $\beta_1 \vee \dots \vee \beta_n$  is *fulfilled* in a branch  $\vartheta$ , if  $\beta_i$  is in  $\vartheta$ , for some  $i = 1, \dots, n$ . A branch  $\vartheta$  is *complete* if every formula  $\beta_1 \vee \dots \vee \beta_n$  occurring in  $\vartheta$  is fulfilled. A KE-tableau is *complete* if all its branches are complete.

Next we introduce the procedure Saturate-KB that takes as input the set  $\Phi_{\mathcal{KB}}$  constructed from a 4LQS<sup>R</sup>-formula  $\phi_{\mathcal{KB}}$  representing a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base  $\mathcal{KB}$  as shown above, and yields a complete KE-tableau  $\mathcal{T}_{\mathcal{KB}}$  for  $\Phi_{\mathcal{KB}}$ .

**Procedure 1** *Saturate-KB*( $\Phi_{\mathcal{KB}}$ )

1.  $\mathcal{T}_{\mathcal{KB}} := \Phi_{\mathcal{KB}}$ ;

2. Select an open branch  $\vartheta$  of  $\mathcal{T}_{\mathcal{KB}}$  that is not yet complete.
  - (a) Select a formula  $\beta_1 \vee \dots \vee \beta_n$  on  $\vartheta$  that is not fulfilled.
  - (b) If  $\mathcal{S}_j^{\bar{\beta}}$  is in  $\vartheta$ , for some  $j \in \{1, \dots, n\}$ , apply the E-Rule to  $\beta_1 \vee \dots \vee \beta_n$  and  $\mathcal{S}_j^{\bar{\beta}}$  on  $\vartheta$  and go to step 2.
  - (c) If  $\mathcal{S}_j^{\bar{\beta}}$  is not in  $\vartheta$ , for every  $j = 1, \dots, n$ , let  $B^{\bar{\beta}}$  be the collection of formulae  $\bar{\beta}_1, \dots, \bar{\beta}_n$  present in  $\vartheta$  and let  $\bar{\beta}_n$  be the lowest index formula such that  $\bar{\beta}_n \in \{\bar{\beta}_1, \dots, \bar{\beta}_n\} \setminus B^{\bar{\beta}}$ , then apply the PB-rule to  $\bar{\beta}_n$  on  $\vartheta$ , and go to step 2.
3. Return  $\mathcal{T}_{\mathcal{KB}}$ .

Soundness of Procedure 1 can be easily proved in a standard way and its completeness can be shown much along the lines of Proposition 36 in [10]. Concerning termination of Procedure 1, our proof is based on the following two facts. The rules in Fig. 1 are applied only to non-fulfilled formulae on open branches and tend to reduce the number of non-fulfilled formulae occurring on the considered branch. In particular, when the E-Rule is applied on a branch  $\vartheta$ , the number of non-fulfilled formulae on  $\vartheta$  decreases. In case of application of the PB-Rule on a formula  $\beta = \beta_1 \vee \dots \vee \beta_n$  on a branch, the rule generates two branches. In one of them the number of non-fulfilled formulae decreases (because  $\beta$  becomes fulfilled). In the other one the number of non-fulfilled formulae stays constant but the subset  $B^{\bar{\beta}}$  of  $\{\bar{\beta}_1, \dots, \bar{\beta}_n\}$  occurring on the branch gains a new element. Once  $|B^{\bar{\beta}}|$  gets equal to  $n - 1$ , namely after at most  $n - 1$  applications of the PB-rule, the E-rule is applied and the formula  $\beta = \beta_1 \vee \dots \vee \beta_n$  becomes fulfilled, thus decrementing the number of non-fulfilled formulae on the branch. Since the number of non-fulfilled formulae on each open branch gets equal to zero after a finite number of steps and the rules of Fig. 1 can be applied only to non-fulfilled formulae on open branches, the procedure terminates.

By the completeness of Procedure 1, each open branch  $\vartheta$  of  $\mathcal{T}_{\mathcal{KB}}$  induces a 4LQS<sup>R</sup>-interpretation  $\mathcal{M}_\vartheta$  such that  $\mathcal{M}_\vartheta \models \Phi_{\mathcal{KB}}$ . We define  $\mathcal{M}_\vartheta = (D_\vartheta, M_\vartheta)$  as follows. We put  $D_\vartheta := \{x \in \mathcal{V}_0 : x \text{ occurs in } \vartheta\}$ ;  $M_\vartheta x := x$ , for every  $x \in D_\vartheta$ ;  $M_\vartheta X_C^1 = \{x : x \in X_C^1 \text{ is in } \vartheta\}$ , for every  $X_C^1 \in \mathcal{V}_1$  occurring in  $\vartheta$ ;  $M_\vartheta X_R^3 = \{\langle x, y \rangle : \langle x, y \rangle \in X_R^3 \text{ is in } \vartheta\}$ , for every  $X_R^3 \in \mathcal{V}_3$  occurring in  $\vartheta$ . It is easy to check that  $\mathcal{M}_\vartheta \models \bar{\phi}_{\mathcal{KB}}$  and thus, plainly, that  $\mathcal{M}_\vartheta \models \phi_{\mathcal{KB}}$ .

Next, we provide some complexity results. Let  $r$  be the maximum number of universal quantifiers in  $S_i$ , and  $k := |\text{Var}_0(\bar{\phi}_{\mathcal{KB}})|$ . Then, each  $S_i$  generates  $k^r$  expansions. Since the knowledge base contains  $m$  such formulae, the number of disjunctions in the initial branch of the KE-tableau is  $m \cdot k^r$ . Next, let  $\ell$  be the maximum number of literals in  $S_i$ , for  $i = 1, \dots, m$ . Then, the maximum depth of the KE-tableau, namely the maximum size of the models of  $\Phi_{\mathcal{KB}}$  constructed as illustrated above, is  $\mathcal{O}(\ell m k^r)$  and the number of leaves of the tableau, that is the number of such models of  $\Phi_{\mathcal{KB}}$ , is  $\mathcal{O}(2^{\ell m k^r})$ .

We now describe a procedure that, given a KE-tableau constructed by Procedure 1 and a 4LQS<sup>R</sup>-formula  $\psi_Q$  representing a  $\mathcal{DL}_{\mathbf{D}}^{4, \times}$ -conjunctive query  $Q$ , yields all the substitutions  $\sigma'$  in the answer set  $\Sigma'$  of  $\psi_Q$  w.r.t.  $\phi_{\mathcal{KB}}$ . By the soundness of Procedure 1, we can limit ourselves to consider only the models  $\mathcal{M}_\vartheta$  of  $\phi_{\mathcal{KB}}$  induced by each open branch  $\vartheta$  of  $\mathcal{T}_{\mathcal{KB}}$ . For every open and complete branch  $\vartheta$

of  $\mathcal{T}_{\mathcal{KB}}$ , we construct a decision tree  $\mathcal{D}_\vartheta$  such that every maximal branch of  $\mathcal{D}_\vartheta$  defines a substitution  $\sigma'$  such that  $\mathcal{M}_\vartheta \models \psi_Q \sigma'$ .

Let  $d$  be the number of literals in  $\psi_Q$ .  $\mathcal{D}_\vartheta$  is a finite labelled tree of depth  $d + 1$  whose labelling satisfies the following conditions, for  $i = 0, \dots, d$ : (i) every node of  $\mathcal{D}_\vartheta$  at level  $i$  is labelled with  $(\sigma_i, \psi_Q \sigma_i)$ , and, in particular, the root is labelled with  $(\sigma'_0, \psi_Q \sigma'_0)$ , where  $\sigma'_0$  is the empty substitution; (ii) if a node at level  $i$  is labelled with  $(\sigma'_i, \psi_Q \sigma'_i)$ , then its  $s$ -successors, with  $s > 0$ , are labelled with  $(\sigma'_i \varrho_1^{q_i+1}, \psi_Q(\sigma'_i \varrho_1^{q_i+1})), \dots, (\sigma'_i \varrho_s^{q_i+1}, \psi_Q(\sigma'_i \varrho_s^{q_i+1}))$ , where  $q_{i+1}$  is the  $(i + 1)$ -st conjunct of  $\psi_Q \sigma'_i$  and  $\mathcal{S}_{q_{i+1}} = \{\varrho_1^{q_i+1}, \dots, \varrho_s^{q_i+1}\}$  is the collection of the substitutions  $\varrho = \{x_1/y_1, \dots, x_j/y_j\}$  with  $\{x_1, \dots, x_j\} = \text{Var}_0(q_{i+1})$  such that  $p = q_{i+1} \varrho$ , for some literal  $p$  on  $\vartheta$ . If  $s = 0$ , the node labelled with  $(\sigma'_i, \psi_Q \sigma'_i)$  is a leaf node and, if  $i = d$ ,  $\sigma'_i$  is added to  $\Sigma'$ .

Let  $\delta(\mathcal{T}_{\mathcal{KB}})$  and  $\lambda(\mathcal{T}_{\mathcal{KB}})$  be, respectively, the maximum depth of  $\mathcal{T}_{\mathcal{KB}}$  and the number of leaves of  $\mathcal{T}_{\mathcal{KB}}$  computed above. Plainly,  $\delta(\mathcal{T}_{\mathcal{KB}}) = \mathcal{O}(\ell m k^r)$  and  $\lambda(\mathcal{T}_{\mathcal{KB}}) = \mathcal{O}(2^{\ell m k^r})$ . It is easy to verify that  $s = 2^k$  is the maximum branching of  $\mathcal{D}_\vartheta$ . Since  $\mathcal{D}_\vartheta$  is a  $s$ -ary tree of depth  $d + 1$ , where  $d$  is the number of literals in  $\psi_Q$ , and the  $s$ -successors of a node are computed in  $\mathcal{O}(\delta(\mathcal{T}_{\mathcal{KB}}))$  time, the number of leaves in  $\mathcal{D}_\vartheta$  is  $\mathcal{O}(s^{(d+1)}) = \mathcal{O}(2^{k(d+1)})$  and they are computed in  $\mathcal{O}(2^{k(d+1)} \delta(\mathcal{T}_{\mathcal{KB}}))$  time. Finally, since we have  $\lambda(\mathcal{T}_{\mathcal{KB}})$  of such decision trees, the answer set of  $\psi_Q$  w.r.t.  $\phi_{\mathcal{KB}}$  is computed in  $\mathcal{O}(2^{k(d+1)} \delta(\mathcal{T}_{\mathcal{KB}}) \lambda(\mathcal{T}_{\mathcal{KB}})) = \mathcal{O}(2^{k(d+1)} \cdot \ell m k^r \cdot 2^{\ell m k^r}) = \mathcal{O}(\ell m k^r 2^{k(d+1) + \ell m k^r})$  time. Since the size of  $\phi_{\mathcal{KB}}$  and of  $\psi_Q$  are polynomially related to those of  $\mathcal{KB}$  and of  $Q$ , respectively (see [3] for details), the construction of the answer set of  $Q$  with respect to  $\mathcal{KB}$  can be done in double-exponential time. In case  $\mathcal{KB}$  contains no role chain axioms and qualified cardinality restrictions, the complexity of our CQA problem is in EXPTIME, since the maximum number of universal quantifiers in  $\phi_{\mathcal{KB}}$ , namely  $r$ , is a constant (in particular  $r = 3$ ). We remark that such result is comparable with the complexity of the CQA problem for a large family of description logics such as *SHIQ* [15]. In particular, the CQA problem for the very expressive description logic *SR<sub>0</sub>IQ* turns out to be 2-NEXPTIME-complete.

## 5 Conclusions

We have introduced the description logic  $\mathcal{DL}\langle 4\text{LQS}^{\text{R},\times} \rangle(\mathbf{D})$  ( $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ , for short) that extends the logic  $\mathcal{DL}\langle 4\text{LQS}^{\text{R}} \rangle(\mathbf{D})$  with Boolean operations on concrete roles and with the product of concepts. We addressed the problem of Conjunctive Query Answering for the description logic  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$  by formalizing  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge bases and  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive queries in terms of formulae of  $4\text{LQS}^{\text{R}}$ . Such formalization seems to be promising for implementation purposes.

In our approach, we first constructed a KE-tableau  $\mathcal{T}_{\mathcal{KB}}$  for  $\phi_{\mathcal{KB}}$ , a  $4\text{LQS}^{\text{R}}$ -formalization of a given  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -knowledge base  $\mathcal{KB}$ , whose branches induce the models of  $\phi_{\mathcal{KB}}$ . Then we computed the answer set of a  $4\text{LQS}^{\text{R}}$ -formula  $\psi_Q$ , representing a  $\mathcal{DL}_{\mathbf{D}}^{4,\times}$ -conjunctive query  $Q$ , with respect to  $\phi_{\mathcal{KB}}$  by means of a forest of decision trees based on the branches of  $\mathcal{T}_{\mathcal{KB}}$  and gave some complexity results.

We plan to generalize our procedure with a data-type checker in order to extend reasoning with data-types, and also to extend  $4LQS^R$  with data-type groups. We also intend to improve the efficiency of the knowledge base saturation algorithm and query answering algorithm, and to extend the expressiveness of the queries. Finally, we intend to study a parallel model of the procedure described and to provide an implementation of it.

## References

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