Cover Problems, Dimensions And The Tensor Product Of Complete Lattices

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Abstract. In this article, we analyze different dimensional concepts of complete (ortho)lattices and their tensor products. The determination of these dimensions can be translated to certain set cover problems and the cardinal product of the complementary underlying formal contexts. To treat this cover problems in a unified manner, we take a more universal approach via the general set cover problem and its product. This yields a sufficient condition for the multiplicativity of various lattice dimensions with respect to the tensor product of complete lattices.

Keywords: formal concept analysis, tolerance relation, cardinal product, tensor product, order dimension, rectangle cover, square cover, block cover.

1 Motivation

The *order* 2*-dimension* of a complete lattice $\mathbb{L} := (L, \leq)$, $\dim_2(\mathbb{L})$, is the smallest n such that an order embedding, that is an order preserving and reflecting map, from $\mathbb{L}$ to the powerset of an n-element set $\mathfrak{P}(\underline{n})$ exists. This can be seen as a measure of $\mathbb{L}$'s "complexity" with respect to set representations.

For two complete lattices $\mathbb{L}_1$ and $\mathbb{L}_2$ with $\dim_2(\mathbb{L}_1) = m$ and $\dim_2(\mathbb{L}_2) = n$ the tensor product, $\mathbb{L}_1 \otimes \mathbb{L}_2$, admits an order embedding to $\mathfrak{P}(\underline{mn})$[1]. Hence, $\dim_2(\mathbb{L}_1 \otimes \mathbb{L}_2)$ is less or equal to $\dim_2(\mathbb{L}_1)\dim_2(\mathbb{L}_2)$. Equality would be an analogue to vector spaces where $\dim(\mathbb{V}_1 \otimes \mathbb{V}_2) = \dim(\mathbb{V}_1)\dim(\mathbb{V}_2)$ holds. In the case of complete lattices, this is also a question whether the "complexity" of $\mathbb{L}_1 \otimes \mathbb{L}_2$ always grows in the same way as the one of $\mathfrak{P}(\underline{mn}) \cong \mathfrak{P}(\underline{m}) \otimes \mathfrak{P}(\underline{n})$ does. But, it turns out that the tensor product is generally not multiplicative with respect to the order 2-dimension.

In this paper we will analyze a sufficient condition when multiplicativity holds. This will be achieved by studying a set cover problem which is equivalent to the determination of the order 2-dimension. In Section 4 we will take a more universal approach and present a general result about set cover problems and their product. This will be applied to different notions of dimension in formal concept analysis (Section 2) and dimensional concepts of tolerance spaces (Section 3). The latter one have interpretations in graph theory, which we will elaborate on too.

[1] This fact is a consequence of Theorem 2, as we will show in Section 5.

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2 Basics Of Formal Concept Analysis

In this section, we will provide the facts from formal concept analysis that we will use in the sequel. If not mentioned otherwise, all results can be found in [7].

A *formal context* is a triple $\mathbb{K} = (G, M, I)$, where the *incidence* $I \subseteq G \times M$ is a binary relation between finite sets. For $A \subseteq G$ and $B \subseteq M$, we define two derivation operators:

$$A^I := \{m \in M | \forall a \in A : (a, m) \in I\} = \bigcap_{a \in A} \{a\}^I,$$
$$B_I := \{g \in G | \forall b \in B : (g, b) \in I\} = \bigcap_{b \in B} \{b\}_I.$$

If $A^I = B$ and $B_I = A$, the pair (A, B) is called a *formal concept* and the cartesian product $A \times B$ is a *maximal rectangle* of $\mathbb{K}$. The set of all formal concepts of $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$ and defines the *concept lattice* $\underline{\mathfrak{B}}(\mathbb{K})$, via the order $(A_1, B_1) \leq (A_2, B_2) :\Longleftrightarrow A_1 \subseteq A_2$. The *complementary context* is defined as $\mathbb{K}^c = (G, M, I^c) := (G, M, (G \times M) - I)$. Furthermore, two special formal concepts of importance are the *object concepts* $\gamma(g) := (\{g\}^I_I, \{g\}^I)$ and *attribute concepts* $\mu(m) = (\{m\}_I, \{m\}^I_I)$. It holds that:

$$gIm \Longleftrightarrow \gamma(g) \leq \mu(m). \tag{1}$$

For two contexts $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$, we use notation from [4] and define the *direct product* $\mathbb{K}_1 \check{\times} \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, I_1 \check{\times} I_2)$,

$$((g, h), (m, n)) \in I_1 \check{\times} I_2 \;:\Longleftrightarrow\; (g, m) \in I_1 \text{ or } (h, n) \in I_2$$

and the *cardinal product* $\mathbb{K}_1 \hat{\times} \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, I_1 \hat{\times} I_2)$,

$$((g, h), (m, n)) \in I_1 \hat{\times} I_2 \;:\Longleftrightarrow\; (g, m) \in I_1 \text{ and } (h, n) \in I_2.$$

It follows that the direct and cardinal product fulfill De Morgan laws:

$$(\mathbb{K}_1 \check{\times} \mathbb{K}_2)^c = \mathbb{K}_1^c \hat{\times} \mathbb{K}_2^c \quad \text{and} \quad (\mathbb{K}_1 \hat{\times} \mathbb{K}_2)^c = \mathbb{K}_1^c \check{\times} \mathbb{K}_2^c. \tag{2}$$

For two complete lattices $\mathbb{L}_1$ and $\mathbb{L}_2$ the *tensor product* $\mathbb{L}_1 \otimes \mathbb{L}_2$ is the concept lattice $\underline{\mathfrak{B}}(\mathbb{L}_1 \check{\times} \mathbb{L}_2)$, where $\mathbb{L}_1$ and $\mathbb{L}_2$ are considered as formal contexts with respect to their order relations. The concept lattice of the direct product is isomorphic to the tensor product of the factors concept lattices.

$$\underline{\mathfrak{B}}(\mathbb{K}_1 \check{\times} \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \otimes \underline{\mathfrak{B}}(\mathbb{K}_2). \tag{3}$$

Next, we will treat the dimension theory of formal concepts. A *Ferrers relation* is a relation $F \subseteq G \times M$ such that $(g, m), (h, n) \in F$ implies $(g, n) \in F$ or $(h, m) \in F$. The definition states that F can be brought into a stair-shaped form by permuting the rows and columns. This is equivalent to $\underline{\mathfrak{B}}(G, M, F)$ being a chain. The *length* l of F is defined as $l(F) := \#\mathfrak{B}(G, M, F) - 1$ and F is *k-step* if $k = \#\{\{g\}^F \mid g \in G\}$. Furthermore, it holds that the complement F^c of a k-step Ferrers relation is a Ferrers relation of length k.

Let $\mathrm{Ferr}\!\upharpoonright\!(\mathbb{K})$ denote the set of all Ferrers relations contained in I and $\mathrm{Ferr}\!\downharpoonright\!(\mathbb{K})$ the set of all Ferrers relations containing I. We define the *Ferrers cover number*[2] and the *Ferrers dimension* of $\mathbb{K}$:

$$\mathrm{fc}(\mathbb{K}) := \min\{\#\mathcal{F} \mid \mathcal{F} \subseteq \mathrm{Ferr}\!\upharpoonright\!(\mathbb{K}),\ I = \bigcup_{F\in\mathcal{F}} F\},$$
$$\mathrm{fdim}(\mathbb{K}) := \min\{\#\mathcal{F} \mid \mathcal{F} \subseteq \mathrm{Ferr}\!\downharpoonright\!(\mathbb{K}),\ I = \bigcap_{F\in\mathcal{F}} F\}.$$

Analogously, $\mathrm{Ferr}\!\upharpoonright_k\!(\mathbb{K})$ denotes the set of all at most k-step Ferrers relations contained in I and $\mathrm{Ferr}\!\downharpoonright_k\!(\mathbb{K})$ the set of all Ferrers relations with length less than k containing I. We define the the *Ferrers k-cover number*[2] and the *Ferrers k-dimension*:

$$\mathrm{fc}_k(\mathbb{K}) := \min\{\#\mathcal{F} \mid \mathcal{F} \subseteq \mathrm{Ferr}\!\upharpoonright_k\!(\mathbb{K}),\ I = \bigcup_{F\in\mathcal{F}} F\},$$
$$\mathrm{fdim}_k(\mathbb{K}) := \min\{\#\mathcal{F} \mid \mathcal{F} \subseteq \mathrm{Ferr}\!\downharpoonright_k\!(\mathbb{K}),\ I = \bigcap_{F\in\mathcal{F}} F\}.$$

Especially, we want to highlight fc_1, the *rectangle cover number*[3]:

$$\mathrm{rc}(\mathbb{K}) := \min\{\#\mathcal{F} \mid \mathcal{F} \subseteq \mathfrak{B}(\mathbb{K}),\ I = \bigcup_{(A,B)\in\mathcal{F}} A \times B\}.$$

Let $\mathcal{A}_I$ be the adjacency matrix of the incidence relation I. In [1] it is implicitly shown that $\mathrm{rc}(\mathbb{K}) = \mathrm{r_B}(\mathcal{A}_I)$, where $\mathrm{r_B}$ denotes the *Boolean rank*[4].

Lastly, we will state the dimension theory of complete lattices and relate it to the above defined dimensions of formal contexts. The *order dimension* of a complete lattice $\mathbb{L}$, $\dim(\mathbb{L})$, is the least number of chains, such that $\mathbb{L}$ can be order embedded in their product. If we restrict the cardinality of these chains to be at

[2] This term is introduced by us, although the concept itself is described in [7].

[3] In the context of Formal Concept Analysis, this term is introduced by us. With respect to Boolean matrices it has already been used in [10].

[4] The *Boolean rank*, $\mathrm{r_B}$, of an $n \times m$ Boolean matrix C is the least integer k such that Boolean $m \times k$ and $k \times n$ matrices with $C = A \circ B$ exist. This definition is equivalent to the fact that C is the sum of k rank one matrices (see [13]).

most k, we get the *k-dimension* of $\mathbb{L}$, denoted by $\dim_k(\mathbb{L})$. Of special interest will be the 2-*dimension*. That is because the n-fold direct product of chains of cardinality 2 is isomorphic to the powerset lattice of the n-element set $\underline{n}$. It holds that $\mathrm{fc}(\mathbb{K}) = \mathrm{fdim}(\mathbb{K}^c) = \dim(\underline{\mathfrak{B}}(\mathbb{K}^c))$, $\mathrm{fc}_{k-1}(\mathbb{K}) = \mathrm{fdim}_k(\mathbb{K}^c) = \dim_k(\underline{\mathfrak{B}}(\mathbb{K}^c))$ and $\mathrm{r_B}(\mathcal{A}_I) = \mathrm{rc}(\mathbb{K}) = \mathrm{fdim}_2(\mathbb{K}^c) = \dim_2(\underline{\mathfrak{B}}(\mathbb{K}^c))$.

In the next section, we will need a proposition from [6] and we will also make use of some definitions from its proof.

Proposition 1. *[6, Hilfssatz 32] There exists an order embedding from $\underline{\mathfrak{B}}(\mathbb{K})$ into a complete lattice $\mathbb{L}$ if and only if there exist mappings $\alpha : G \to \mathbb{L}$ and $\beta : M \to \mathbb{L}$ with*

$$gIm \Longleftrightarrow \alpha(g) \leq \beta(m).$$

Proof. *"$\Rightarrow$": The required order embedding is given through $\varphi(A, B) := \bigvee_{g \in A} \alpha(g)$.*
"$\Leftarrow$": Define $\alpha := \varphi \circ \gamma$ and $\beta := \varphi \circ \mu$.

3 Tolerance Spaces, Dimension And Graph Theory

A *tolerance relation* or simply a *tolerance* is a reflexive and symmetric binary relation τ on a non-empty finite set V. The pair $(V, \tau) =: \mathbb{T}$ is called *tolerance space* and is a special case of a formal context. An introduction to tolerance spaces together with applications can be found in [8] and [9].

For a tolerance τ on V, a non-empty subset $S \subseteq V$ is called *τ-preblock* if $S \times S$ is contained in τ. A maximal τ-preblock with respect to set inclusion is called *τ-block*. In other words, this means that a τ-block $S \subseteq V$ defines a non-enlargeable *square* $S \times S \subseteq \tau$. The set of all τ-blocks is denoted by $\mathrm{Bl}(\mathbb{T})$. Analogously, the set of all maximal squares of $\mathbb{T}$ is denoted by $\mathrm{Sq}(\mathbb{T})$. This set determines the tolerance τ, that is $\tau = \bigcup \mathrm{Sq}(\mathbb{T})$. But often not all squares are necessary to cover τ. This motivates the definition of the *square cover number*[5], $\mathrm{sc}(\mathbb{T})$, of a tolerance space $\mathbb{T}$, as the minimal number of maximal squares necessary to cover τ:

$$\mathrm{sc}(\mathbb{T}) := \min\{\#\mathcal{S} \mid \mathcal{S} \subseteq \mathrm{Sq}(\mathbb{T}),\ \tau = \bigcup \mathcal{S}\}.$$

Another covering problem of tolerance spaces is the *block cover number*[6]:

$$\mathrm{bc}(\mathbb{T}) := \min\{\#\mathcal{B} \mid \mathcal{B} \subseteq \mathrm{Bl}(\mathbb{T}),\ V = \bigcup \mathcal{B}\}.$$

Similarly to the Ferrers cover numbers and rectangle cover number of general formal contexts, we can relate these cover numbers of tolerance spaces to an

[5] This term is introduced by us and is a logical consequence of the term "rectangle cover number".
[6] This term is introduced by us.

intersection problem and a dimension of the complements concept lattice.

We start with the square cover number and notice that the complement of a square is a symmetric Ferrers relation of length 1. Hence, we define the *symmetric Ferrers* 2-*dimension* of $\mathbb{T}^c$, denoted by sfdim_2, as the smallest number of symmetric Ferrers relations of length 1 whose intersection is equal to τ^c.

The concept lattice[7] $\underline{\mathfrak{B}}(\mathbb{T}^c)$ was characterized in [12] as a *complete ortholattice*. This is a complete bounded lattice $\mathbb{L} = (L, \leq, c)$ with an involutory antiautomorphism c, such that for all $x \in L$ it holds that $x \wedge x^c = 0$ and $x \vee x^c = 1$. An abstract *orthogonality relation* $\perp$ is defined through $x \perp y :\Longleftrightarrow x \leq y^c$. In the special case of concept lattices the *orthocomplement* is given via $(A, B)^c := (B, A)$.

Definition 1. *An* orthoembedding*, between two complete ortholattices* $\mathbb{L}_1 = (L_1, \leq, c_1)$ *and* $\mathbb{L}_2 = (L_2, \leq, c_2)$*, is an order embedding* $\varphi : \mathbb{L}_1 \to \mathbb{L}_2$ *which additionally preserves orthogonality* ($x \perp y \Longrightarrow \varphi(x) \perp \varphi(y)$)*, such that there exists an order preserving map* $\psi : \mathbb{L}_2 \to \mathbb{L}_1$ *satisfying for all* $x \in L_1$*:*

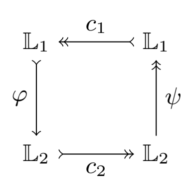
1. $\psi(\varphi(x)) = x$,
2. $\psi(\varphi(x)^{c_2}) = x^{c_1}$.

Remark 1. The map φ is a section from $\mathbb{L}_1$ to $\mathbb{L}_2$, such that φ and the *dual* of φ given by

$$\varphi^d : \mathbb{L}_1 \to \mathbb{L}_2, \;\; x \mapsto (\varphi(x^{c_1}))^{c_2}$$

have ψ as a common retraction.

Fig. 1. The maps φ and $\varphi^d := c_2 \circ \phi \circ c_1$, and their common retraction ψ.



The next proposition provides an equivalent condition for the existence of an orthoembedding from $\underline{\mathfrak{B}}(\mathbb{T}^c)$ to a Boolean algebra (in other words a distributive ortholattice) with the additional property that $x \perp y \Longleftrightarrow x \wedge y = 0$.

[7] The concept lattice was treated under the name *neighborhood ortholattice*.

Proposition 2. *There exists an orthoembedding φ from $\underline{\mathfrak{B}}(\mathbb{T}^c)$ into a Boolean algebra $\mathbb{L} = (L, \leq, c)$, with the special property that for all $x, y \in L$ it holds that $x \perp y \Longleftrightarrow x \wedge y = 0$, if and only if there exists a mapping $\alpha : V \to \mathbb{L}$ with*

$$u\tau^c v \Longleftrightarrow \alpha(u) \perp \alpha(v).$$

Proof. "⇒": We define $\alpha := \varphi \circ \gamma$ and use Equation 1 to conclude that:

$$\begin{aligned} u\,\tau^c v &\Longleftrightarrow \gamma(u) \leq \mu(v) = \gamma(v)^c \\ &\Longleftrightarrow \gamma(u) \perp \gamma(v) \\ &\Longrightarrow \varphi \circ \gamma(u) \perp \varphi \circ \gamma(v). \end{aligned} \qquad \begin{aligned} u\,\tau^c v &\Longleftrightarrow \gamma(u) \leq \gamma(v)^c \\ &\Longleftarrow \psi(\varphi \circ \gamma(u)) \leq \psi((\varphi \circ \gamma(v))^c) \\ &\Longleftarrow \varphi \circ \gamma(u) \leq (\varphi \circ \gamma(v))^c \\ &\Longleftrightarrow \varphi \circ \gamma(u) \perp \varphi \circ \gamma(v). \end{aligned}$$

"⇐": We define $\varphi(A, B) := \bigvee_{a \in A} \alpha(a)$. It follows from Proposition 1 that φ is an order embedding. Let (A, B) and (C, D) be formal concepts of $\underline{\mathfrak{B}}(\mathbb{T}^c)$. We show that $(A, B) \perp (C, D) \Longrightarrow \varphi(A, B) \perp \varphi(C, D)$. From the definition of φ and the complement in $\underline{\mathfrak{B}}(\mathbb{T}^c)$ it follows that:

$$(A, B) \perp (C, D) \Longleftrightarrow (A, B) \leq (D, C) \Longleftrightarrow \bigvee_{a \in A} \alpha(a) \leq \bigvee_{d \in D} \alpha(d), \tag{4}$$

$$\varphi(A, B) \perp \varphi(C, D) \Longleftrightarrow \bigvee_{a \in A} \alpha(a) \wedge \bigvee_{c \in C} \alpha(c) = 0. \tag{5}$$

In the next step, we show that 4 implies 5 by "connecting" the ends.
"(4)⇒(5)": Since, (C, D) is a formal concept it holds that $\alpha(d) \leq \alpha(c)^c$ for all $d \in D$ and all $c \in C$.

$$(4) \Rightarrow \bigvee_{a \in A} \alpha(a) \leq \bigvee_{c \in C} \alpha(c)^c \Rightarrow \bigvee_{a \in A} \alpha(a) = \bigvee_{a \in A} \alpha(a) \wedge \bigvee_{c \in C} \alpha(c)^c.$$

Finally, we take the meet with $\bigvee_{c \in C} \alpha(c)$ and use distributivity.

$$\bigvee_{a \in A} \alpha(a) \wedge \bigvee_{c \in C} \alpha(c) = (\bigvee_{a \in A} \alpha(a) \wedge \bigvee_{c \in C} \alpha(c)^c) \wedge \bigvee_{c \in C} \alpha(c) = 0.$$

Furthermore, we define the desired map $\psi : \mathbb{L} \to \underline{\mathfrak{B}}(\mathbb{T}^c)$ through $\psi(x) := (\{v \in V \mid \alpha(v) \leq x\}, \{v \in V \mid x \leq \alpha(v)^c\})$ and show that $\psi(x)$ is a formal concept of $\underline{\mathfrak{B}}(\mathbb{T}^c)$, as well as that ψ satisfies Property 1 and 2 from Definition 1. The proof is similar to some parts of the proof from the Basic Theorem on Concept Lattices (see [7]). Also note that ψ is order preserving.

The definition of formal concepts states that $A^{\tau^c} = B$ and $B_{\tau^c} = A$ must hold in order for $(A, B) \in \mathfrak{B}(\mathbb{T}^c)$. We only show the second condition as the first one can be shown in the same way.

$$\begin{aligned} u \in \{v \in V \mid \alpha(v) \leq x\} &\iff \alpha(u) \leq x \\ &\iff \alpha(u) \leq \alpha(w)^c \text{ for all } w \in \{v \in V \mid x \leq \alpha(v)^c\} \\ &\iff u\, \tau^c w \text{ for all } w \in \{v \in V \mid x \leq \alpha(v)^c\} \\ &\iff u \in \{v \in V \mid x \leq \alpha(v)^c\}_{\tau^c}. \end{aligned}$$

Next, we show that the second component from $\psi(\varphi(A, B))$ equals B. The first component must then be equal to A, due to the fact shown above.

$$\begin{aligned} \{v \in V \mid \bigvee_{a \in A} \alpha(a) \leq \alpha(v)^c\} &= \{v \in V \mid \alpha(a) \leq \alpha(v)^c \text{ for all } a \in A\} \\ &= \{v \in V \mid a\, \tau^c v \text{ for all } a \in A\} \\ &= A^{\tau^c} = B. \end{aligned}$$

Lastly, we show that the first component from $\psi(\varphi(A, B)^c)$ equals B. The second component must then be equal to A, due to the fact shown above. Hence, we can conclude that $\psi(\varphi(A, B)^c) = \psi(\bigwedge_{a \in A} \alpha(a)^c) = (B, A) = (A, B)^c$.

$$\begin{aligned} \{v \in V \mid \alpha(v) \leq \bigwedge_{a \in A} \alpha(a)^c\} &= \{v \in V \mid \alpha(v) \leq \alpha(a)^c \text{ for all } a \in A\} \\ &= \{v \in V \mid v\, \tau^c a \text{ for all } a \in A\} \\ &= A^{\tau^c} = B. \end{aligned}$$

Definition 2. *The* orthodimension *of a complete ortholattice* $\mathbb{L} = (L, \leq, c)$*, denoted by* $\dim_\perp(\mathbb{L})$*, is the smallest* n *such that there exists an orthoembedding from* $\mathbb{L}$ *to* $\mathfrak{P}(\underline{n})$*.*

Theorem 1. *For a tolerance space* $\mathbb{T}$ *its holds that* $\mathrm{sc}(\mathbb{T}) = \dim_\perp(\underline{\mathfrak{B}}(\mathbb{T}^c))$*.*

Proof. If $\mathrm{sc}(\mathbb{T}) = n$, there exists a minimal square cover $\{S_1 \times S_1, \ldots, S_n \times S_n\}$. It follows from [5] that this is equivalent to the existence of a minimal *set representation* of $\mathbb{T}$, which is a map $\alpha : V \to \mathfrak{P}(\underline{n})$, such that we have $u \tau v \iff \alpha(a) \cap \alpha(v) \neq \emptyset$. From the minimal square cover, this map can be defined via:

$$\alpha : v \mapsto \{i \mid v \in S_i\}.$$

Consequently, α provides a minimal *complementary set representation* for $\mathbb{T}^c$, that is $u\, \tau^c v \iff \alpha(a) \cap \alpha(v) = \emptyset$. Since it holds that $\alpha(a) \cap \alpha(v) = \emptyset \iff \alpha(a) \subseteq \alpha(v)^c \iff \alpha(a) \perp \alpha(v)$, Proposition 2 yields that $\dim_\perp(\underline{\mathfrak{B}}(\mathbb{T}^c)) = n$. As all stated implications are equivalences, it follows that $\mathrm{sc}(\mathbb{T}) = \dim_\perp(\underline{\mathfrak{B}}(\mathbb{T}^c))$.

We have shown that analogue to general formal contexts, in the special case of tolerance spaces, it holds that:

$$\mathrm{sc}(\mathbb{T}) = \mathrm{sfdim}_2(\mathbb{T}^c) = \dim_\perp(\underline{\mathfrak{B}}(\mathbb{T}^c)).$$

Next, we will treat the block cover number. For this, we will use some tools from graph theory which we introduce in the following.

The *underlying graph*, $\mathbb{G}_\mathbb{T}$, of a tolerance space $\mathbb{T} = (V, \tau)$ is defined through the same relation but with all diagonal elements removed. Analogously, to the block and square cover number of $\mathbb{T}$, we can define the *vertex clique cover number*, $\theta_v(\mathbb{G}_\mathbb{T}) = \mathrm{bc}(\mathbb{T})$, and the *edge clique cover number*, $\theta_e(\mathbb{G}_\mathbb{T}) = \mathrm{sc}(\mathbb{T})$ (see [3]). The vertex clique cover number is equal to the *chromatic number* of the complementary graph (see [14]), $\chi(\mathbb{G}^c_\mathbb{T})$[8]. Here the complement is taken in the sense of graph theory, which always yields an irreflexive relation. On the other hand, for tolerance spaces we consider full complements as defined in Section 2. This yields $\mathrm{bc}(\mathbb{T}) = \chi(\mathbb{T}^c)$, since τ^c is irreflexive and symmetric.

We saw that $\underline{\mathfrak{B}}(\mathbb{T}^c)$ is a complete ortholattice. An *orthomap* between complete ortholattices (see [12]) preserves order and orthogonality, and maps only the bottom element of the domain lattice to the bottom element of the codomain lattice.

Definition 3. *We define the* chromatic dimension*,* $\mathrm{cdim}(\mathbb{L})$*, of a complete ortholattice* $\mathbb{L} = (L, \leq, c)$*, to be the minimal* n *such that an orthomap to the powerset lattice* $\mathfrak{P}(\underline{n})$ *exists.*

The nomenclature is justified by the following proposition.

Proposition 3. *For a graph* $\mathbb{G} = (V, E)$ *with an irreflexive and symmetric relation* $E \subseteq V \times V$*, it holds that* $\chi(\mathbb{G}) = \mathrm{cdim}(\underline{\mathfrak{B}}(\mathbb{G}))$*.*

Proof. The chromatic number of $\mathbb{G}$ is n if and only if n is minimal with the property that there exists a graph homomorphism to K_n, the complete graph with n vertices. In [12] it is shown that this is equivalent to the existence of an orthomap from $\underline{\mathfrak{B}}(\mathbb{G})$ to $\mathfrak{P}(\underline{\mathrm{n}}) \cong \underline{\mathfrak{B}}(K_n)$. Consequently, it holds that $\mathrm{cdim}(\underline{\mathfrak{B}}(\mathbb{G})) = n$.

The calculation of the chromatic number is a minimization problem, but not an intersection problem. In order to define an intersection problem, we notice that the complement B^c for a block $B \in \mathrm{Bl}(\mathbb{T})$ is an *independent set* of vertices with respect to τ^c. Hence, we can define an intersection problem with respect to $\mathbb{T}^c$. This yields the *independence dimension* of $\mathbb{T}^c$, denoted by $\mathrm{idim}(\mathbb{T}^c)$, to be the smallest number of independent sets whose intersection is empty.

$$\mathrm{bc}(\mathbb{T}) = \mathrm{idim}(\mathbb{T}^c) = \chi(\mathbb{T}^c) = \mathrm{cdim}(\underline{\mathfrak{B}}(\mathbb{T}^c)).$$

[8] The chromatic number of a graph is the minimal n such that a graph homomorphism, which is an edge preserving vertex map, to the complete graph with n vertices exists.

4 Set Cover Problems And Their Product

A *set cover system* is a triple $\mathbb{S} := (U, X, \mathcal{S})$, with *universe* U, a subset $X \subseteq U$ and $\mathcal{S} \subseteq \mathfrak{P}(X)$. The *cover number* and *isolation number* of $\mathbb{S}$ are defined as:

$$\mathrm{c}(\mathbb{S}) := \min\{\#\tilde{\mathcal{S}} \mid \tilde{\mathcal{S}} \subseteq \mathcal{S},\, X = \bigcup \tilde{\mathcal{S}}\},$$
$$\mathrm{i}(\mathbb{S}) := \max\{\#\tilde{X} \mid \tilde{X} \subseteq X, \forall\, S \in \mathcal{S} : \#(\tilde{X} \cap S) \leq 1\}.$$

The isolation number is the maximal cardinality of an *isolated set* $\tilde{X}$ from $\mathbb{S}$, which means that $\tilde{X}$ is maximal with respect to the property that any pair of its elements is not contained in the same $S \in \mathcal{S}$. Consequently, the isolation number is a lower bound for the cover number.

Remark 2. Both optimization problems can be described as an integer linear program. For this purpose, we define the representation matrix $A \in \{0,1\}^{X \times \mathcal{S}}$ of $\mathbb{S}$ through $A(i, S) = 1 :\Leftrightarrow i \in S$.

$$\begin{array}{llcll} \text{minimize:} & \sum_{S \in \mathcal{S}} x_S & \qquad & \text{maximize:} & \sum_{i \in X} y_i \\ \text{subject to:} & A\boldsymbol{x} \geq \mathbb{1} & & \text{subject to:} & A^T \boldsymbol{y} \leq \mathbb{1} \\ & x_S \in \{0,1\}. & & & y_i \in \{0,1\}. \end{array}$$

Thus, the inequality $\mathrm{i}(\mathbb{S}) \leq \mathrm{c}(\mathbb{S})$ also follows from the weak duality theorem of optimization and the difference $\mathrm{c}(\mathbb{S}) - \mathrm{i}(\mathbb{S})$ is the *duality gap*.

A *set intersection system* is a triple $\mathbb{S} := (U, X, \mathcal{S})$, with *universe* U, a subset $X \subseteq U$ and $\mathcal{S} \subseteq \mathfrak{P}(U)$, such that $X \subseteq S$ for all $S \in \mathcal{S}$. The *intersection number* is defined as:

$$\mathrm{int}(\mathbb{S}) := \min\{\#\tilde{\mathcal{S}} \mid \tilde{\mathcal{S}} \subseteq \mathcal{S},\, X = \bigcap \tilde{\mathcal{S}}\}.$$

It follows that the complement of a set cover system, defined through $\mathbb{S}^c := (U, X^c, \{S^c \mid S \in \mathcal{S}\})$[9], is a set intersection system with $\mathrm{c}(\mathbb{S}) = \mathrm{int}(\mathbb{S}^c)$.

The product of two set cover systems $\mathbb{S}_1 = (U_1, X_1, \mathcal{S}_1)$ and $\mathbb{S}_2 = (U_2, X_2, \mathcal{S}_2)$ is defined as $\mathbb{S}_1 \times \mathbb{S}_2 := (U_1 \times U_2, X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2)$. This definition yields to the next theorem which is a generalization of Theorem 3.2 from [13].

Theorem 2. *For the the product of set cover systems $\mathbb{S}_1$ and $\mathbb{S}_2$, it holds that:*

$$\max(\mathrm{i}(\mathbb{S}_1)\,\mathrm{c}(\mathbb{S}_2), \mathrm{c}(\mathbb{S}_1)\,\mathrm{i}(\mathbb{S}_2)) \leq \mathrm{c}(\mathbb{S}_1 \times \mathbb{S}_2) \leq \mathrm{c}(\mathbb{S}_1)\,\mathrm{c}(\mathbb{S}_2)$$
$$\mathrm{i}(\mathbb{S}_1)\,\mathrm{i}(\mathbb{S}_2) \leq \mathrm{i}(\mathbb{S}_1 \times \mathbb{S}_2) \leq \min(\mathrm{i}(\mathbb{S}_1)\,\mathrm{c}(\mathbb{S}_2), \mathrm{c}(\mathbb{S}_1)\,\mathrm{i}(\mathbb{S}_2)).$$

[9] The complements are defined with respect to U.

Proof. We prove the first inequality and the second one follows from duality. For the upper bound, let $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$ be minimal covers from $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. It is easy to see that $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$ is a cover from $\mathbb{S}_1 \times \mathbb{S}_2$.

For the lower bound, let $\tilde{\mathcal{S}}$ be a minimal cover from $\mathbb{S}_1 \times \mathbb{S}_2$ and $\tilde{X}_1$ a maximal isolated set from $\mathbb{S}_1$. We define for $i \in \tilde{X}_1$ the set $\tilde{\mathcal{S}}_i := \{(S, T) \mid (S, T) \in \tilde{\mathcal{S}},\ i \in S\}$. Since $\tilde{X}_1$ is an isolated set, it follows that for different $i, j \in \tilde{X}_1$, the sets $\tilde{\mathcal{S}}_i$ and $\tilde{\mathcal{S}}_j$ are disjoint. A similar argument implies that every $\tilde{\mathcal{S}}_i$ induces a cover from $\mathbb{S}_2$ and hence $\#\tilde{\mathcal{S}}_i \geq \mathrm{c}(\mathbb{S}_2)$. These facts yield:

$$\#\tilde{\mathcal{S}} \geq \#\Big(\biguplus_{i \in \tilde{X}_1} \tilde{\mathcal{S}}_i\Big) \geq \#\tilde{X}_1 \,\mathrm{c}(\mathbb{S}_2) = \mathrm{i}(\mathbb{S}_1)\,\mathrm{c}(\mathbb{S}_2).$$

The other lower bound's component can be deduced in the same way.

5 Dimension Of The Tensor Product Of Complete Lattices

With Theorem 2, it is easy to provide a sufficient condition when dim, $\dim_k$, $\dim_2$, $\dim_\perp$ and cdim are multiplicative with respect to the tensor product. Since every complete lattice is isomorphic to a concept lattice ([7]), we can shift this problem to the multiplicativity of the associated intersection number with respect to the direct product (Equation 3). Furthermore, due to Equation 2, we transform this problem to the multiplicativity of the corresponding cover number with respect to the cardinal product. All we have to do is to define suitable set cover systems and show that there product expresses the cardinal product of the underlying formal contexts.

In the sense of Section 4, we define the *Ferrers isolation number* fi, the *k-Ferrers isolation number* fi_k, the *rectangle isolation number* ri, the *square isolation number* si and the *block isolation number* bi.

Fig. 2. This table gives an overview about the set cover problems introduced above. Note that $\mathfrak{B}(\mathbb{K})$ actually denotes a set of formal concepts, but here we identify it with the set of all rectangles.

set cover system	cover n.	isolation n.	intersection n.	lattice dim.
$(G \times M, I, \mathrm{Ferr}{\restriction}(\mathbb{K}))$	$\mathrm{fc}(\mathbb{K})$	$\mathrm{fi}(\mathbb{K})$	$\mathrm{fdim}(\mathbb{K}^c)$	$\dim(\underline{\mathfrak{B}}(\mathbb{K}^c))$
$(G \times M, I, \mathrm{Ferr}{\restriction}_{k-1}(\mathbb{K}))$	$\mathrm{fc}_{k-1}(\mathbb{K})$	$\mathrm{fi}_{k-1}(\mathbb{K})$	$\mathrm{fdim}_k(\mathbb{K}^c)$	$\dim_k(\underline{\mathfrak{B}}(\mathbb{K}^c))$
$(G \times M, I, \mathfrak{B}(\mathbb{K}))$	$\mathrm{rc}(\mathbb{K})$	$\mathrm{ri}(\mathbb{K})$	$\mathrm{fdim}_2(\mathbb{K}^c)$	$\dim_2(\underline{\mathfrak{B}}(\mathbb{K}^c))$
$(V \times V, \tau, \mathrm{Sq}(\mathbb{T}))$	$\mathrm{sc}(\mathbb{T})$	$\mathrm{si}(\mathbb{T})$	$\mathrm{sfdim}_2(\mathbb{T}^c)$	$\dim_\perp(\underline{\mathfrak{B}}(\mathbb{T}^c))$
$(V, V, \mathrm{Bl}(\mathbb{T}))$	$\mathrm{bc}(\mathbb{T})$	$\mathrm{bi}(\mathbb{T})$	$\mathrm{idim}(\mathbb{T}^c)$	$\mathrm{cdim}(\underline{\mathfrak{B}}(\mathbb{T}^c))$

For the cardinal product of formal contexts, it holds that $(A, B) \in \mathfrak{B}(\mathbb{K}_1 \hat{\times} \mathbb{K}_2)$ if and only if there exists $(A_1, B_1) \in \mathfrak{B}(\mathbb{K}_1)$ and $(A_2, B_2) \in \mathfrak{B}(\mathbb{K}_2)$, such that $A = A_1 \times A_2$ and $B = B_1 \times B_2$ (see [11]). Only the formal concepts with $A \neq \emptyset$ and $B \neq \emptyset$ are of importance for the covering problems of the cardinal product, since they correspond to maximal rectangles. Hence, these formal concepts (A, B) can be uniquely identified with the pair $((A_1, B_1), (A_2, B_2))$. Comparing this with the definition of the product of set cover systems, we see the desired correspondence to the cardinal product of the respective formal contexts. The same holds for the Ferrers relations. Also note that squares are a special case of rectangles and that blocks are derived from maximal squares. These fact yield the following theorem.

Theorem 3. *We consider the set cover problems of Figure 2 and their products. If for one of the factors, it holds that the isolation number is equal to the cover number, then the respective lattice dimension is multiplicative with respect to the tensor product of the corresponding complete lattices.*

Remark 3. We introduce the *strong product* of two simple graphs $\mathbb{G}_1 = (V_1, E_1)$ and $\mathbb{G}_2 = (V_2, E_2)$, defined as $\mathbb{G}_1 \boxtimes \mathbb{G}_2 := (V_1 \times V_2, \tilde{E})$, with $(u_1, u_2)\tilde{E}(v_1, v_2) :\Longleftrightarrow (u_1 E_1 v_1 \text{ and } u_2 = v_2)$ or $(u_1 = v_1 \text{ and } u_2 E_2 v_2)$ or $(u_1 E_1 v_1 \text{ and } u_2 E_2 v_2)$.

The *reflexive closure* of a graph $\mathbb{G} = (V, E)$ is defined as $\mathbb{G}^{\text{ref}} := (V, E^{\text{ref}})$, where E^{ref} is the reflexive closure of the symmetric relation E. Hence, $\mathbb{G}^{\text{ref}}$ is a tolerance space. It holds that $(\mathbb{G}_1 \boxtimes \mathbb{G}_2)^{\text{ref}} = \mathbb{G}_1^{\text{ref}} \hat{\times} \mathbb{G}_2^{\text{ref}}$. Consequently, we have that $\theta_e(\mathbb{G}_1 \boxtimes \mathbb{G}_2) = \text{sc}(\mathbb{G}_1^{\text{ref}} \hat{\times} \mathbb{G}_2^{\text{ref}})$. In [2] the multiplicativity of the edge clique cover number with respect to the strong product was studied and similar results as with the square cover number of the cardinal product of tolerance spaces have been obtained. That is why this setting would provide another example of a set cover system and its product.

In [2] an example such that $\theta_e(\mathbb{G} \boxtimes \mathbb{G}) < \theta_e(\mathbb{G})\theta_e(\mathbb{G})$ was provided for $\mathbb{G}$ being the join of a 5-cycle with two isolated vertices. We did a computational experiment to find the smallest tolerance space (in terms of the number of vertices) for which the square isolation number is strictly smaller than the square cover number. Thereby, we found out that the smallest tolerance space has 7 vertices and is the reflexive closure of the graph $\mathbb{G}$ described above.

6 Conclusion

We showed that the three fold relationship between cover problem (Ferrers cover number), intersection problem (Ferrers dimension) and lattice dimension (order dimension) also applies to tolerance spaces. That is the equality of the square cover number, the symmetric Ferrers 2-dimension and the orthodimension. Surprisingly, this theme is also present in the case of the block cover number and the chromatic dimension.

Additionally, we highlighted how the cover problems with respect to tolerance spaces have a strong graph theoretic flavor, *i.e.*, interpretations in terms of the chromatic number or the relationship to the strong product of graphs.

Our initial question, about the multiplicativity of the lattice dimension with respect to the tensor product, could, in all investigated examples, be translated to a cover problem of the cardinal product of the related formal contexts complements. This fundamental principle lead to the abstraction to the general set cover problem, which provides a unified setting to treat these various cover problems related to formal contexts. Especially, the question about the multiplicativity of the dimension of the cardinal product could be dealt with in a unified way. This in turn lead to a sufficient condition for the multiplicativity of the lattice dimensions with respect to the tensor product of complete lattices.

The introduced isolation numbers have to our knowledge not been present in the theory of formal concept analysis. It is an open problem to find a purely lattice theoretical interpretation of these isolation numbers.

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