Implication and Biconditional in some Three-valued Logics

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Abstract. The interpretation for $\neg$, $\vee$ and $\wedge$ is quite standard in most of the best known three-valued logics. However, this is not the case for the connectives $\rightarrow$ and $\leftrightarrow$, which are the relevant ones in order to study the notions of consequence and equivalence as well as translatability and synonymity. The purpose of this paper is to focus on the behavior of these last connectives in some Three-valued Logics, namely **K3**, **LP**, **Ł3**, $\mathbf{G'_3}$, $\mathbf{CG'_3}$, $\mathbf{L3A_g}$ and $\mathbf{L3B_g}$. Particularly we find equivalence connectives in logics **Ł3**, $\mathbf{G'_3}$, $\mathbf{CG'_3}$, $\mathbf{L3A_g}$ and $\mathbf{L3B_g}$.

Keywords: Implication · equivalence · many-valued logic.

1 Introduction

Aristotle restricted logic to statements, i.e. sentences that can be true or false, as a result, Classical Logic is bivalent and it assumes that all statements are actually true or false. Many logicians think that there are good reasons to avoid such assumption and there is where Many-valued logics allowing for additional truth values or truth value gaps appear. Many-valued logic has become a vast field in logic and it continues growing as an independent discipline. One can find excellent introductions to the topic as those in [11] or [6].

Many-valued logics are a big family of non-classical logics. Though they reject the bivalent principle, they accept the principle of truth-functionality, namely, that the truth of a compound sentence is determined by the truth values of its component sentences. Once one there are extra truth values for sentences or they have run away of them, one must rethink the meaning of the logical connectives as well as the definition of notions such as validity or consequence. In both scenarios a huge amount of options arises; e.g. in case of a third option apart from true and false is included, we have $3^3 = 27$ different unary connectives and $3^9 = 19683$ binary connectives to select a new interpretation for the usual connectives namely $\neg$, $\vee$ $\wedge$ $\rightarrow$ and $\leftrightarrow$.

One can feel overwhelmed by such a bunch of different options. At the same time, one can wonder if it is possible to select different connectives and that they lead us to similar logics. In fact, even if two logics are defined using different languages or approaches they can have the same expressive power, (translatability

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equivalent logics) or even more, they can be the same logic (synonymous logics).This problem is not an easy one, as it is revealed by Pelletier and Urquhart in [9].

The interpretation for $\neg$, $\vee$ and $\wedge$ is quite standard, but it is not unique as we will see in Section 4. However, this is not the case for the connectives $\rightarrow$ and $\leftrightarrow$, which are the relevant ones in order to study the notions of consequence and equivalence as well as translatability and synonymity. The purpose of this paper is to focus on the behavior of these last connectives in some of the most important Three-valued Logics.

2 Basic Concepts

Let us start by introducing the syntax of the language considered in this article as well as some definitions. We suppose that the reader has some familiarity with basic concepts related to mathematical logic such as those given in the first chapter of [7].

We consider a formal language $\mathcal{L}$ built from: an enumerable set of atoms (denoted as $p, q, r, \ldots$), the set of atoms is denoted as $atom(\mathcal{L})$ and the set of connectives $\mathcal{C} = \{\wedge, \vee, \rightarrow, \neg\}$. Formulas are constructed as usual and will be denoted as lowercase Greek letters. The set of all formulas of an language $\mathcal{L}$ is denoted as $Form(\mathcal{L})$. Theories are sets of formulas and will be denoted as uppercase Greek letters. There is no consensus on the definition of logic, but two of the most accepted ones are that a logic can be considered as the set of its theorems or defined instead by the valid inferences it accepts.

Definition 1. *Given a language $\mathcal{L}$, a logic L in the language $\mathcal{L}$ is a subset of $Form(\mathcal{L})$.*

In this case the elements of a logic L are called theorems and the notation $\vdash_L \varphi$ is used to state that the formula φ is a theorem of L.

Definition 2. *Given a language $\mathcal{L}$, a logic L is a relation between sets of formulas and formulas, i.e. a subset of $\mathcal{P}(Form(\mathcal{L})) \times Form(\mathcal{L})$.*

In this case the elements of a logic L are pairs, the valid inferences. The notation $\Gamma \vdash_L \varphi$ is used to state that the formula φ can be inferred from Γ in L. Whenever it is clear by the context which logic you are referring to, the subscript will be dropped.

It is common to impose extra conditions to the previous definitions to define a logic. For example in the context of Definition 1 is common to ask that the set is closed under Modus Ponens (MP) and substitution. Meanwhile regarding Definition 2 it is common to ask that the relation satisfies reflexivity, monotonicity and transitivity. We take no position on this issue and we will consider Definitions 1 and 2 equally valid.

The usefulness of a logic depends on the available connectives in its language, particularly important connectives are: implication, conjunction, disjunction and negation. In the following definition we ask for some conditions on binary connectives so they can be considered as conjunctions, disjunctions or implications according to [1].

Definition 3. *Let L be a logic in the language $\mathcal{L}$ with binary connectives $\wedge$, $\vee$ and $\rightarrow$, then:*

1. *$\wedge$ is a* ***conjunction*** *for L, when: $\Gamma \vdash \varphi \wedge \psi$ iff $\Gamma \vdash_{\varphi}$ and $\Gamma \vdash \psi$.*
2. *$\vee$ is a* ***disjunction*** *for L, when: $\Gamma, \varphi \vee \psi \vdash \sigma$ iff $\Gamma, \varphi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$.*
3. *$\rightarrow$ is an* ***implication*** *for L, when: $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \rightarrow \psi$.*

L is a ***semi-normal logic*** *if it has at least one of this connectives, If the logic has the three of them then it is called* ***normal logic*** *.*

Here some extra important notions on regard to the connectives behavior.

Definition 4. *[5] Let $\vdash_{\mathcal{L}}$ be a logic whose language has some of the following connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$, then:*

1. *$\neg$ is a* ***Classical Negation*** *if $\Gamma, \neg\varphi \vdash \psi$ and $\Gamma, \neg\varphi \vdash \neg\psi$ imply that $\Gamma \vdash \varphi$.*
2. *$\wedge$ is a* ***Classical Conjunction*** *if $\Gamma, \varphi \wedge \psi \vdash \varphi$, $\Gamma, \varphi \wedge \psi \vdash \psi$ and $\Gamma, \varphi, \psi \vdash \varphi \wedge \psi$.*
3. *$\vee$ is a* ***Classical disjunction*** *if $\Gamma, \varphi \vdash \varphi \vee \psi$, $\Gamma, \psi \vdash \varphi \vee \psi$ and if $\Gamma, \varphi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$ then $\Gamma, \varphi \vee \psi \vdash \sigma$.*
4. *$\rightarrow$ is a* ***Classical implication*** *if $\Gamma \vdash \varphi \rightarrow (\psi \rightarrow \varphi)$, if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$ imply that $\Gamma \vdash \psi$, and $\Gamma \vdash \Big(\varphi \rightarrow (\psi \rightarrow \sigma)\Big) \rightarrow \Big((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)\Big)$.*

3 Many-valued logics

The early notions of many-valued logics can be traced back to Boole, Peirce and Vasiliev, however, first papers introducing many-valued logics formally are due to Łukasiewicz and Post around 1920.

The usual manner to define the many-valued logic is by means of a matrix.

Definition 5. *Given a language $\mathcal{L}$, a* matrix *is a structure $M := \langle V, D^{*}, F \rangle$:*

1. *V is a non-empty set of truth values (domain).*
2. *D is a subset of D (set of designated values).*
3. *$F := \{f_c | c \in \mathcal{C}\}$ is a set of truth functions, with a function for each logical connective in $\mathcal{L}$.*

Definition 6. *Given a language $\mathcal{L}$, a* valuation *or an* interpretation *is a function $t : atom(\mathcal{L}) \rightarrow V$ that maps atoms into elements of the domain.*

An interpretation t can be extended to all formulas by $\hat{t} : Form(\mathcal{L}) \rightarrow V$ as usual, i.e. applying recursively the truth functions of logical connectives in F. The interpretations allow us to define the notion of validity.

Definition 7. *Given a matrix M, we say that the formula φ is* valid *under the interpretation t, if $\hat{t}(\varphi) \in D$ and we denote it by $t \models_M \varphi$.*

In this case the validity depends on the interpretation, but if we want to find the "logical truths" of the system then the validity should not depend on the interpretation, in other words we have:

Definition 8. *Given a matrix M, we say that a formula φ is a* tautology *in M (or simply it is valid) if for every possible interpretation, the formula φ is valid and we denote this by $\models_M \varphi$.*

It is also possible to define a consequence relation by means of a matrix.

Definition 9. *Given M, Γ and φ, a matrix, a theory and a formula respectively, we say that φ is a* consequence *of γ in M if for any interpretation t, $t \models_M \varphi$ whenever $t \models_M \Gamma$.*

When one defines a logic L via a matrix M we have two cases. Using Definition 1 the set of theorems of the logic is defined as the set of tautologies that are obtained from the matrix, i.e. $\vdash_L \varphi$ iff $\models_M \varphi$. On the other hand if we use Definition 2 instead, then an inference $\Gamma \vdash_L \varphi$ is valid if φ is a consequence of Γ in M, i.e. $\Gamma \vdash_L \varphi$ iff $\Gamma \models_M \varphi$.

Some extra important notions are those classifying the connectives behavior as:

Definition 10. *Let $M = \langle V, D, F \rangle$ be a matrix in the language $\mathcal{L}$, $A \subseteq V$ and $a_1, \ldots, a_n \in V$. An n-ary connective $\diamond$ in $\mathcal{L}$ is:*

1. *A-closed if $a_1, \ldots, a_n \in A$ implies $\tilde{\diamond}(a_1, \ldots, a_n) \in A$.*
2. *A-limited if $\tilde{\diamond}(a_1, \ldots, a_n) \in A$ implies $a_1, \ldots, a_n \in A$.*

Now we define *neoclassical connectives*, its name can be easily understood if we identify the True value with designated and False with not designated. These conditions are generalizations of those that satisfy and in some way define the nature of the connectives in Classical Logic.

Definition 11. *[5] Let $M = \langle V, D, F \rangle$ be a matrix for the language $\mathcal{L}$ and $\overline{D} = V \setminus D$ then:*

1. *$\neg$ ia a **Neoclassical negation**, when $\neg p \in \overline{D}$ iff $p \in D$.*
2. *$\wedge$ is a **Neoclassical conjunction**, iff it is D-closed and D-limited.*
3. *$\vee$ is a **Neoclassical disjunction**, it is $\overline{D}$-closed and $\overline{D}$-limited.*
4. *$\rightarrow$ is a **Neoclassical implication**, when $a_1 \rightarrow a_2 \in D$ iff $a_1 \in \overline{D}$ or $a_2 \in D$.*

Definition 12. *[2] A multivalued operator $\circledast$ is a **conservative extension** of a bi-valued operator if the restriction of $\circledast$ to the values of the bi-valued operator coincide.*

Definition 13. *Given a bi-valued logic L_1 and a multivalued logic L_2 such that the set of connectives of L_1 is a subset of the connectives of L_2, L_2 is a **conservative extension** of L_1 if all the connectives in common are conservative extensions.*

4 Three-valued logics

Many-valued logics, particularly three-valued logics have been one of the most prolific family of non-classical logics since they formally show up around 1920. Due to its high degree of plasticity, they have served as counterexamples, as models to solve philosophical problems and have found many applications in areas such as computer science. Consequently, it is valuable to continue studying them. In this section we present some of the most important three-valued logics by means of matrices that show the slight but important differences between these logics. In all the cases the domain will be the set $V = \{0, 1, 2\}$ where 0 is identified with False, 2 is identified with True but the interpretation of the third one, namely 1 will vary among systems.

4.1 Logic K3

The original three-valued logic by Kleene has an epistemological motivation. The third logical value was designed to mark indeterminacy of some proposition at a certain stage of scientific investigation. The matrix of **K3** logic is given by: $M = \langle V, D, F\rangle$ where: the domain is $V = \{0, 1, 2\}$ and the set of designated values is $D = \{2\}$ and the set F of truth functions for connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ consists of the functions shown in Table 1. In this case **K3** must be defined as a logic according to Definition 2 due to the absence of tautologies since any valuation which assigns the value 1 to each propositional variable sends any formula into 1.

$f_\neg$	
0	2
1	1
2	0

$f_\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$f_\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$f_\rightarrow$	0	1	2
0	2	2	2
1	1	1	2
2	0	1	2

Table 1. Truth functions of connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ in **K3**.

4.2 Logic LP

A popular paraconsistent logic is the logic **LP** (Logic of Paradox), introduced by Asenjo in 1966. The idea behind LP is to take classical logics two-valued semantics and extend it to a three-valued semantics, with truth values true, false and paradoxical (identified here as 0, 2 and 1 respectively). We think of the third truth value as meaning both true and false. The matrix of **LP** logic is given by: $M = \langle V, D, F\rangle$ where: the domain is $V = \{0, 1, 2\}$ and the set of designated values is $D = \{1, 2\}$ and the set F of truth functions for connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ consists of the same functions shown in Table 1. **LP** validates all classical tautologies; however, it fails to validate all classical inferences.

4.3 Logic Ł3

The matrix of **Ł3** logic is given by: $M = \langle V, D, F\rangle$ where: the domain is $V = \{0, 1, 2\}$ and the set of designated values is $D = \{2\}$ and the set F of truth functions for connectives $\wedge$, $\vee$, $\to$ and $\neg$ consists of the functions shown in Table 2. Truth tables for connectives $\wedge$, $\vee$ and $\neg$ remain the same as for **K3** and **LP**. In this case the law of non-contradiction, $\neg(\varphi \wedge \neg\varphi)$, is not valid in **Ł3**. However, there are some classical tautologies that even can turn out false in **Ł3**: $\neg(\varphi \to \neg\varphi) \vee \neg(\neg\varphi \to \varphi)$.

$f_\neg$	
0	2
1	1
2	0

$f_\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$f_\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$f_\to$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

Table 2. Truth functions of connectives $\wedge$, $\vee$, $\to$ and $\neg$ in **Ł3**.

4.4 Logic $\mathbf{G_3'}$

In [3] Carnielli and Marcos define $\mathbf{G_3'}$ as a paraconsistent logic and use it only as a tool to prove that $(\varphi \vee (\varphi \to \psi))$ is not a theorem of $\mathbf{C}_\omega$. In [8] Osorio and Carballido make a detailed study of $\mathbf{G_3'}$. The matrix of $\mathbf{G_3'}$ logic is given by: $M = \langle V, D, F\rangle$ where: the domain is $V = \{0, 1, 2\}$ and the set of designated values is $D = \{2\}$ and the set F of truth functions for connectives $\wedge$, $\vee$, $\to$ and $\neg$ consists of the functions shown in Table 3.

$f_\neg$	
0	2
1	2
2	0

$f_\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$f_\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$f_\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

Table 3. Truth functions of connectives $\wedge$, $\vee$, $\to$ and $\neg$ in $\mathbf{G_3'}$.

4.5 Logic $\mathbf{CG_3'}$

The logic $\mathbf{CG_3'}$ is a paraconsistent logic that extends $\mathbf{G_3'}$. The logical matrix of $\mathbf{CG_3'}$ is given by $V = \{0, 1, 2\}$, $D = \{1, 2\}$ and the truth functions are those of $\mathbf{G_3'}$ that can be found in the Table 3. In other words we obtain the matrix of $\mathbf{CG_3'}$ by adding 1 as designated value to the matrix of $\mathbf{G_3'}$.

4.6 Logic $\mathbf{L3A_g}$

Genuine Paraconsistent logics **L3A** and **L3B** were defined in 2016 by Béziau et al, including only three logical connectives, namely, negation disjunction and conjunction. Afterwards Hernández-Tello et al, provide implications for both logics and define the logics $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$ in [5]. Logic $\mathbf{L3A_G}$ and it is given by a matrix given by $V = \{0, 1, 2\}$, $D = \{1, 2\}$ and the truth functions can be found in the Table 3. It is worth mentioning that the matrix of $\mathbf{L3A_G}$ is very close to the matrix of $\mathbf{CG'_3}$ since the only differences are on the table of disjunction, specifically $\wedge(1, 2) = 2$ and $\wedge(2, 1) = 2$, while in $\mathbf{CG'_3}$ this table correspond to the minimum as shown in Table 4.

$f_\neg$	
0	2
1	2
2	0

$f_\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$f_\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$f_\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

Table 4. Truth functions of connectives $\wedge$, $\vee$, $\to$ and $\neg$ in $\mathbf{L3A_g}$.

4.7 Logic $\mathbf{L3B_g}$

The matrix of $\mathbf{L3B_G}$ differs from the one for $\mathbf{L3A_g}$ only in the connectives $\neg$ and $\vee$ as shown in Table 5.

$f_\neg$	
0	2
1	1
2	0

$f_\wedge$	0	1	2
0	0	0	0
1	0	2	1
2	0	1	2

$f_\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$f_\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

Table 5. Truth functions of connectives $\wedge$, $\vee$, $\to$ and $\neg$ in $\mathbf{L3A_g}$.

4.8 Some remarks

We have presented seven different logics. Though they share some connectives, their properties are quite different. In the following theorems we summarize some of these properties. Proofs are straightforward from the interpretation of the connectives.

Theorem 1. *Let* $L \in \{\mathbf{K3}, \mathbf{LP}, \mathbf{Ł3}, \mathbf{G'_3}, \mathbf{CG'_3}, \mathbf{L3A_g}, \mathbf{L3B_g}\}$ *then* L *is a conservative extension of Classical Logic.*

Proof. Given a logic L according to Definition 13 it is necessary to verify that each of the connectives $\neg$, $\vee$, $\wedge$ and $\to$ in the corresponding logic are conservative extensions of the classical ones. If we identify 0 as F (False) and 2 as T (True) and we ignore all the entries that involve the third value leaving them as ?, for any of the cited logics we obtain the truth tables in Table 6 which correspond to the classical truth tables in all cases.

$f_\neg$	
F	T
?	?
T	F

$f_\wedge$	F	?	T
F	F	?	F
?	?	?	?
T	F	?	T

$f_\vee$	F	?	T
F	F	?	T
?	?	?	?
T	T	?	T

$f_\to$	F	?	T
F	T	?	T
?	?	?	?
T	F	?	T

Table 6. Truth tables of restrictions of three valued operators

Theorem 2. *The logics* $\mathbf{K3}$*,* $\mathbf{LP}$ *and* $\mathbf{G'_3}$ *are semi-normal logics and* $Ł\mathbf{3}$*,* $\mathbf{CG'_3}$*,* $\mathbf{L3A_g}$ *and* $\mathbf{L3B_g}$ *are normal logics.*

Proof. In all the cases it is enough to check the truth tables for all the connectives. The connective $\to$ is not an implication according to Definition 3 for those logics that said to be semi-normal, for example for logic $\mathbf{LP}$ it is enough to check the case when the antecedent is 1 and the consequent is 0.

On the other hand if we focus on the neoclassicality of the connectives we have the following:

Theorem 3. *Most of the connectives* $\wedge$*,* $\vee$*,* $\to$ *and* $\neg$ *are neoclassical in the logics in* $B = \{\mathbf{K3}, \mathbf{LP}, Ł\mathbf{3}, \mathbf{G'_3}, \mathbf{CG'_3}, \mathbf{L3A_g}, \mathbf{L3B_g}\}$ *as depicted in Table 7 (a check mark in the table establishes the respective connective is neoclassical in the corresponding logic, meanwhile an* $\times$ *mark means the connective is not neoclassical).*

	K3	**LP**	Ł3	$\mathbf{G'_3}$	$\mathbf{CG'_3}$	$\mathbf{L3A_g}$	$\mathbf{L3B_g}$
$\neg$	×	×	×	✓	×	×	×
$\vee$	✓	✓	✓	✓	✓	✓	✓
$\wedge$	✓	✓	✓	✓	✓	✓	✓
$\to$	×	✓	×	×	✓	✓	✓

Table 7. Neoclassicality of connectives $\neg$, $\wedge$, $\vee$ and $\to$.

Proof. In all the cases the truth tables give the evidence or the counterexample of the respective property. For example $\to$ is not a neoclassical implication nor for $\mathbf{K3}$, Ł$\mathbf{3}$ neither for $\mathbf{G'_3}$. In all cases it is enough to check the case when the antecedent is 1 and the consequent is 0.

There is also a close relation between the notions of connective, neoclassicla connective and classical connective as stated by the following propositions.

Proposition 1. *[4] Neoclassical connectives define classical connectives, i. e. if* $\circledast$ *is a connective then if it is a neoclassical connective, it is a classical connective as well.*

Proof. Let see the case of implication, the rest of the proofs are analogous. Let φ and ψ be two formulas and let *rightarrow* be a neoclassical implication. We will verify that each of the conditions required to be a classical implication is fulfiled.

- Suppose that $\varphi \to (\psi \to \varphi)$ is not designated for some valuation v. Since $\to$ is neoclassical, then $v(\varphi) \in D$ and $v(\psi \to \varphi) \notin D$. The last part implies that $v(\varphi) \notin D$, which leads us to a contradiction, then $\varphi \to (\psi \to \varphi)$ must be designated.
- Let suppose that $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then for any valuation that models Γ it also models φ and $\varphi \to \psi$, since $\to$ is neoclassical then that valuation also models ψ as a result $\Gamma \vdash \psi$.
- Suppose that $(\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to (\varphi \to \sigma))$ is not designated for some valuation v. Since $\to$ is neoclassical then $v(\varphi \to (\psi \to \sigma)) \in D$ and $v((\varphi \to \psi) \to (\psi \to \sigma)) \notin D$ and that $v(\varphi), v(\psi) \in D$ and $v(\sigma) \notin D$, which leads us to a contradiction, then $(\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to (\varphi \to \sigma))$ must be designated.

Proposition 2. *[4] A conjunction (disjunction or implication) defines a classical conjunction (disjunction or implication).*

The proofs of previous proposition is based on the properties of consequence relations. This lead us to the following theorem.

Theorem 4. *Most of the connectives* $\wedge$, $\vee$ *and* $\to$ *are classical in the logics in* B *as depicted in Table 8.*

	K3	**LP**	Ł3	$\mathbf{G}'_3$	$\mathbf{CG}'_3$	$\mathbf{L3A_g}$	$\mathbf{L3B_g}$
$\vee$	✓	✓	✓	✓	✓	✓	✓
$\wedge$	✓	✓	✓	✓	✓	✓	✓
$\to$	×	×	✓	✓	✓	✓	✓

Table 8. Classicality of connectives $\neg$, $\wedge$, $\vee$ and $\to$.

As a final remark from Theorems 2, 3 and 4 we can see that the converse of Proposition 1 is not valid and notions of connective, classical connective and neoclassical connective do not agree.

5 Implication and Validity

The existence of a third truth value also forces us to revise the relationship between implication and validity. Some of the principles stated for granted in classical logic are anymore acceptable in three-valued logics.

K3 is a semi-normal logic, in this case the connective $\to$ is not a neoclassical connective nor a classical one. Apart from Modus Ponens any of the usual properties regarding implication is satisfied.

In particular in **LP**, Modus Ponens fails, it can be seen by making $p = 1$ and $q = 0$. In that case both p and $p \to q$ have designated values (both are 1), but the conclusion is 0. But LP is a very simple and intuitive proposal. Various ways around the problem have been proposed, see [10] for more details.

Logic **Ł3** is quite similar to **K3** in fact the unique difference is in its definition of implication in that "1 implies 1" is 2. This small change in the truth table lead us to a completely different logic that has a lot of classical theorems, but not all of them, e.g. law of excluded middle, $\varphi \vee \neg\varphi$, and the law of non-contradiction, $\neg(\varphi \wedge \neg\varphi)$.

Though $\mathbf{G_3'}$, $\mathbf{CG_3'}$, $\mathbf{L3A_g}$ and $\mathbf{L3B_g}$ share the truth table for $\to$, which agrees with implication of the well known three-valued logic of Gödel, the behavior of $\mathbf{G_3'}$ differs from the others since it has only one designated value.

Several principles, rules and theorems made the boundary between implication and validity indistinguishable. Let us focus in some of them such as:

- Identity Principle (**IP** $\vdash \varphi \to \varphi$),
- Modus Ponens (**MP** $\varphi, \varphi \to \psi \vdash \psi$)
- Deduction Theorem (**DT** If $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$)
- Thinning Principle (**TP** $\vdash \varphi \to (\psi \to \varphi)$)

Most of the many-valued logics we study are paraconsistent logics, in the sense that at least one of the following formulations of the principle of non contradiction fails.

- Non Contradiction (**NC** $\Gamma \vdash \neg(\varphi \wedge \neg\varphi)$)
- Explosion by Contradiction (**EC** $\Gamma, \varphi, \neg\varphi \vdash \psi$)

And some of them are genuine paraconsistent logics, i. e. that both **NC** and **EC** fail. In Table 9 it is possible to observe which of the studied properties hold in each logic.

Property	**K3**	**LP**	**Ł3**	$\mathbf{G}'_3$	$\mathbf{CG}'_3$	$\mathbf{L3A_g}$	$\mathbf{L3B_g}$
IP $\vdash \varphi \to \varphi$	×	✓	✓	✓	✓	✓	✓
MP $\varphi, \varphi \to \psi \vdash \psi$	✓	×	×	✓	✓	✓	✓
DT If $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$	×	✓	✓	×	✓	✓	✓
TP $\vdash \varphi \to (\psi \to \varphi)$	×	✓	✓	✓	✓	✓	✓
EC $\varphi, \neg\varphi \vdash \psi$	✓	×	×	✓	×	×	×
NC $\vdash \neg(\varphi \wedge \neg\varphi)$	×	✓	✓	✓	✓	×	×

Table 9. Properties of some three-valued logics paraconsistent logics.

6 Biconditional and Substitution

Up to now we have used matrices whose language includes $\neg$, $\wedge$, $\vee$ and $\to$, but those matrices can be extended by a new connective, $\leftrightarrow$, either as a primitive connective or as an abbreviation in terms of the primitive ones. We are familiar with the classical definition of $\leftrightarrow$ using $\to$ and $\wedge$, $\varphi \leftrightarrow \psi =_{df} (\varphi \to \psi) \wedge (\psi \to \varphi)$. However, as we saw in the previous section the behavior of the implication in our set of logics is varied and conjunction also vary at least in $\mathbf{L3A_g}$ and $\mathbf{L3A_g}$.

Considering the classical definition of $\leftrightarrow$ using $\to$ and $\wedge$, $\varphi \leftrightarrow \psi =_{df} (\varphi \to \psi) \wedge (\psi \to \varphi)$ and the $\to$ connective of each logic we obtain that it is not an equivalence relation:

K3, **LP**

$f_\leftrightarrow$	0	1	2
0	2	1	0
1	1	1	1
2	0	1	2

Ł3

$f_\leftrightarrow$	0	1	2
0	2	1	0
1	1	2	1
2	0	1	2

$\mathbf{G}'_3$, $\mathbf{CG}'_3$, $\mathbf{L3B_g}$

$f_\leftrightarrow$	0	1	2
0	2	0	0
1	0	2	1
2	0	1	2

$\mathbf{L3A_g}$

$f_\leftrightarrow$	0	1	2
0	2	0	0
1	0	2	2
2	0	2	2

Table 10. Truth functions of connective $\leftrightarrow$.

We are particularly interested in logics whose biconditional connective corresponds to equality. As stated in the following propositions, it is possible to define an equivalence connective between formulas by means of the binary connective $\Leftrightarrow$ as follows.

Proposition 3. *In* **Ł3** *there is an equivalence connective, namely:*

$$\varphi \Leftrightarrow \psi =_{df} \neg\big(\neg\neg(\varphi \leftrightarrow \psi) \to \neg(\varphi \leftrightarrow \psi)\big)$$

Proposition 4. *In* $\mathbf{G}'_3$ *and* $\mathbf{CG}'_3$ *there is an equivalence connective, namely:*

$$\varphi \Leftrightarrow \psi =_{df} \neg\neg(\varphi \leftrightarrow \psi)$$

Proposition 5. *In* $\mathbf{L3A_g}$ *there is an equivalence connective, namely:*

$$\varphi \Leftrightarrow \psi =_{df} \neg\big((\varphi \to \psi) \wedge \neg(\psi \to \varphi)\big) \wedge \neg\big((\psi \to \varphi) \wedge \neg(\varphi \to \psi)\big)$$

Proposition 6. *In* $\mathbf{L3B_g}$ *there is an equivalence connective, namely:*

$$\varphi \Leftrightarrow \psi =_{df}$$

$$\neg\big((\varphi \to \psi) \wedge (\varphi \to \psi) \wedge \neg(\psi \to \varphi) \wedge \neg(\psi \to \varphi)\big) \wedge \neg\big((\psi \to \varphi) \wedge (\psi \to \varphi) \wedge \neg(\varphi \to \psi) \wedge \neg(\varphi \to \psi)\big)$$

The propositions follows directly from the truth tables of the connectives involved in the definition and in Table 11 one can find the truth table for $\Leftrightarrow$ according to their definition.

$f_{\leftrightarrow}$	0	1	2
0	2	0	0
1	0	2	0
2	0	0	2

Table 11. Truth functions for $\Leftrightarrow$ in Ł**3**, $\mathbf{G_3'}$, $\mathbf{cG_3'}$ $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$

Definition 14. *[9] Let* $\leftrightarrow$ *be a connective in a matrix* $M = \langle V, D, F\rangle$*, we say that* M *is an algebraic matrix if it satisfies the condition that for any elements* $a, b \in V$*,* $a \leftrightarrow b \in D$ *iff* $a = b$*.*

Let B' be the set of logics containing Ł**3**, $\mathbf{G_3'}$, $\mathbf{CG_3'}$ $\mathbf{L3A_G}$ and $\mathbf{L3B_G}$. As a result of the previous propositions and definition we obtain the following theorem:

Theorem 5. *The matrices for Ł***3***,* $\mathbf{G_3'}$*,* $\mathbf{CG_3'}$ $\mathbf{L3A_G}$ *and* $\mathbf{L3B_G}$ *are algebraic matrices.*

If M is an algebraic matrix, then we can think of the elements of the algebra formed by $\langle V, F\rangle$ as propositions, and of the designated elements in D as the true propositions. Then an algebraic matrix is a matrix in which equivalent propositions are identified. Algebraic matrices are an important tool to define and identify notions such as translatable equivalent logics or canonical models (see [9]).

Another benefit of the equivalence connective $\Leftrightarrow$ is that the substitution of logical equivalents, also known as the replacement theorem, holds.

Definition 15. *Let* $\sigma_1 = \{p/\varphi\}$ *and* $\sigma_2 = \{p/\psi\}$ *be two substitutions in a logic* L*, logic* L *satisfies substitution of logical equivalents if* $\vdash_L \varphi \Leftrightarrow \psi$ *implies that for any formula* ρ *it holds that* $\vdash_L \sigma_1(\rho) \Leftrightarrow \sigma_2(\rho)$*.*

Theorem 6. *The replacement theorem holds in Ł***3***,* $\mathbf{G_3'}$*,* $\mathbf{CG_3'}$ $\mathbf{L3A_G}$ *and* $\mathbf{L3B_G}$*.*

Proof. L For $L \in \{\text{Ł}\mathbf{3}, \mathbf{G_3'}, \mathbf{CG_3'}\mathbf{L3A_G}, \mathbf{L3B_G}\}$ if $\vdash_L \varphi \Leftrightarrow \psi$, then for any interpretation t in L, $\hat{t}(\varphi) = \hat{t}(\psi)$ and due to the recursive definition of $\hat{t}$, we have that $\hat{t}(\sigma_1(\rho)) = \hat{t}(\sigma_2(\rho))$ for any interpretation, which means that $\vdash_L \sigma_1(\rho) \Leftrightarrow \sigma_2(\rho)$.

The theorem states that, if any part of a formula is replaced by an equivalent of that part, the resulting formula and the original are also equivalents in the underlying logic. Such replacements need not be uniform.

7 Conclusions

The interpretation for connectives $\rightarrow$ and $\leftrightarrow$ in many-valued logics is not standard and it is important to study them since they are the relevant connectives in order to define notions of consequence and equivalence as well as many others related to this like translatability and synonymity. We selected seven of the most important Three-valued Logics namely, **K3**, **LP**, **Ł3**, $\mathbf{G'_3}$, $\mathbf{CG'_3}$, $\mathbf{L3A_G}$, $\mathbf{L3B_G}$ and analyzed notions of classicality and neoclassicality of connective $\rightarrow$ as well as its behavior with respect to some principles, rules and theorems that involves this connective. After that we look for an equivalence connective for each of the presented logics obtaining that in five of them this is possible, which has interesting consequences.

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