Craig Interpolation on the Logic of Knowledge

Everardo Bárcenas[1], José-de-Jesús Lavalle-Martínez[2], Guillermo Molero-Castillo[3,4], and Alejandro Velázquez-Mena[1]

[1] Universidad Nacional Autónoma de México
[barcenas,mena]@fi-b.unam.mx
[2] Benemérita Universidad Autónoma de Puebla
jlavalle@cs.buap.mx
[3] Consejo Nacional de Ciencia y Tecnología
[4] Universidad Veracruzana
ggmoleroca@conacyt.mx

Abstract. The Craig interpolation property is described as follows: if a formula ϕ implies another formula ψ, then there is a formula β in the common language of ϕ and ψ, such that ϕ also implies β, as well as β implies ψ. In this paper, we provide a constructive proof of the Craig interpolation property on the modal logic of knowledge K_m. The proof is based on the application of the Maehara technique on a tree-hypersequent calculus. We also show, as a consequence of the interpolation property, Beth definability and Robinson joint consistency .

1 Introduction

Philosophy and Artificial Intelligence have been the traditional study framework to reason about knowledge [12]. More recently, reasoning about knowledge has become of much importance in many areas of computer science, such as distributed systems, cryptography, natural language processing and databases [8]. In this paper, we consider as a formal framework to study knowledge, the basic modal logic of knowledge, also known as the multi-modal logic K_m. This logic has been shown to be mathematically well-founded [11]. As consequence of being a bisimulation invariant fragment of first order logic, K_m posses a nice balance of expressiveness and reasoning efficiency [3].

The interpolation property was first proved for classical first order logic by Craig [6]. Considering a formula ϕ implies another formula ψ, the interpolation property consists in the existence of a formula β, called the interpolant, in the common language of ϕ and β, which is assumed to be non-empty, such that ϕ implies β, and β implies ψ. Some logical consequences of interpolation are Beth definability [1] and Robinson joint consistency [22]. Applications of interpolation in computer science have been recently studied for formal verification [18], computational complexity [5], and knowledge representation [13, 4], among others.

In this paper, we give a constructive proof the Craig interpolation theorem for the modal logic of knowledge K_m. The proof implies a straightforward algorithm to compute interpolants.

1.1 Related work

Early studies about the interpolation property in modal logics are reported in [10, 16]. In [10], Gabbay proved interpolation for several mono-modal logics including K and $S4$. Maksimova in [16] indentifies a close conection of amalgamability of modal logics containing $S4$, and proved that only a finite number modal logics conaining $S4$ enjoys interpolation. Maksimova later proved in [15] interpolation of all normal modal logics via amalgamation. This result was extended for multi-modal logics in [14]. Marx proved interpolation in [17] for several modal logics with bisimulation. This work includes interpolation proofs for K, fibered modal logics and the multi-modal logics of knowledge and belief. In all the above works, interpolation is proved by semantics methods. Although these methods are quite general and can be applied to several logics, they not provide an explicit construction of interpolants. In the current paper, we provide a syntactic proof of interpolation for the multi-modal logic K_m. This proof includes an explicit construction of interpolants.

Syntactic interpolation proofs for modal logics KB, KDB, $K5$ and $KD5$ are described in [20]. In this work, interpolation is proved by means of a cut-free complete sequent-like tableau deduction system. Constructive interpolation for modal logics K and T is given in [2]. More precisely, a stronger form of interpolation, called uniform interpolation, is proved in this work. In uniform interpolation, interpolants are composed by the common language language of formulas in the implication, but restricted by a choice of propositional variables. The closest work to our paper is [9]. In this work, constructive interpolation is proved for the entire modal cube, composed by the logics resulting from any combination of K, D, T, B, 5 and 4. The proof technique used in this work is based on nested sequents. In our paper, we obtain a constructive interpolation proof for the multi-modal logic K_m, using the Maehara technique on a cut-free complete tree-hypersequent calculus.

In [7], D'Agostino reports an extensive survey on interpolation for non-classical logics, including modal logics.

1.2 Outline

We first introduce the multi-modal logic of knowledge K_m in Section 2. In Section 3, we describe a complete cut-free tree-hypersequent calculus for K_m. Then, in Section 4, by means of Maehara technique, we extract interpolants from tree-hypersequent proofs of K_m implications. In Section 5, as a consequence of interpolation, we also prove Beth definability and Robinson joint consistency. Finally, in Section 6, we give a summary of the article and briefly argue further research perspectives.

2 Logic of Knowledge

We assume a basic modal language: a non-empty set of propositions PROP; and a non-empty finite set of modalities MOD.

The set of formulas is inductively defined by the following grammar.

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box_m \phi$$

where p is a proposition and m is a modality.

Notation:

$$\top := p \vee \neg p \qquad \bot := \neg\top$$
$$\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi) \qquad \phi \to \psi := \neg\phi \vee \psi$$
$$\Diamond_m\phi := \neg\Box_m\neg\phi$$

A Kripke structure is a tuple $\mathcal{M} = (W, R, V)$ where:

- W is a non-empty set called *domain*;
- R is a finite set of binary relations $R^m : W \times W$, for every modality m; and
- V : PROP $\mapsto 2^W$ is valuation function mapping propositions to domain subsets.

Given a Kripke structure $\mathcal{M} = (W, R, V)$, formulas are interpreted as follows:

$$\begin{aligned}
[\![p]\!]^{\mathcal{M}} &= \{w \in V(p)\} \\
[\![\neg\phi]\!]^{\mathcal{M}} &= W \setminus [\![\phi]\!]^{\mathcal{M}} \\
[\![\phi \wedge \psi]\!]^{\mathcal{M}} &= [\![\phi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}} \\
[\![\Box_m\phi]\!]^{\mathcal{M}} &= \Big\{w \mid \forall w' \in W : \text{if } (w, w') \in R^m, \\
&\qquad \text{then } w' \in [\![\phi]\!]^{\mathcal{M}}\Big\}
\end{aligned}$$

We may also write $\mathcal{M}, w \models \phi$ instead of $w \in [\![\phi]\!]^{\mathcal{M}}$, $\mathcal{M} \models \phi$ when for every w in M, we have that $\mathcal{M}, w \models \phi$, in which case we say $\mathcal{M}$ is a model of ϕ. If any Kripke structure is a model of ϕ, we write $\models \phi$.

Definition 1 (Hilbert derivation system). *We define the derivation system H by the following schemas and rules, for each $m \in$ MOD:*

A_1 $\phi \to (\psi \to \phi)$
A_2 $(\phi \to (\psi \to \beta)) \to ((\phi \to \psi) \to (\phi \to \beta))$
A_3 $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$
A_4 $\Box_m(\phi \to \psi) \to (\Box_m\phi \to \Box_m\psi)$
R_1 $\dfrac{\phi \to \psi \qquad \phi}{\psi}$
R_2 $\dfrac{\phi}{\Box_m\phi}$

We say a formula ϕ_n is derivable from H, written $\vdash_H \phi_n$, if there is a sequence $\phi_1, \phi_2, \ldots, \phi_n$, such that for each $i \in \{1, \ldots, n\}$:

- ϕ_i is either an instance, up to substitution, of a schema in H, or
- there is (are) $j < i$ (and $k < i$) such that ϕ_i and ϕ_j (and ϕ_k) are instances of the conclusion and premises, resp, of a rule in H.

Consider for instance the following derivation of $\Box_m(\phi \wedge \psi) \to \Box_m\phi$:

1. $(\phi \wedge \psi) \to \phi$, which by notation is an instance of A_1, $\neg\phi \vee \psi \vee \phi$;
2. $\Box_m((\phi \wedge \psi) \to \phi)$, from 1 by R_2;
3. $\Box_m((\phi \wedge \psi) \to \phi) \to (\Box_m(\phi \wedge \psi) \to \Box_m\phi)$, from A_4; and
4. $\Box_m(\phi \wedge \psi) \to \Box_m\phi$, from 2 and 3 by R_1.

Theorem 1 (Correctness [11]). *For any formula ϕ, $\vdash_H \phi$, if and only if, $\models \phi$.*

3 Tree-hypersequents

A sequent is an expression $\Gamma \vdash \Delta$, where Γ and Δ are formula multisets, non-empty and finite. Intuitively, a sequent $\Gamma \vdash \Delta$ is interpreted, in terms of logical symbols, as an implication, where the antecedent is composed by the disjunction of formulas in Γ, and the consequent is the conjunction of formulas in Δ. We then the define the following interpretation function:

$$(\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_k)^I := \bigwedge_{i=1}^{n} \phi_i \to \bigvee_{j=1}^{k} \psi_j$$

where n and k are some positive integers.

In sequents, we often write ϕ, Γ or Γ, ϕ instead of $\{\phi\} \cup \Gamma$, also $\vdash \Delta$ instead of $\top \vdash \Delta$, and $\Gamma \vdash$ in place of $\Gamma \vdash \bot$.

Tree hypersequents expressions are inductively defined by the following grammar:

$$\begin{aligned} T :=& S\,[ST] \\ ST :=& \emptyset \mid m : T, ST \end{aligned}$$

where S is a sequent and m is a modality. We extend the interpretation function of sequents for tree hypersequents as follows:

$$\begin{aligned} (S\,[ST])^I :=& S^I \vee (ST)^I \\ (\emptyset)^I :=& \bot \\ (m : T, ST)^I :=& \Box_m T^I \vee (ST)^I \end{aligned}$$

When clear from context, we often write tree instead of tree hypersequent. It is usually written S instead of $S\,[\emptyset]$, also if S is $\Gamma \vdash \Delta$, we write ϕ, S and S, ϕ instead of $\phi, \Gamma \vdash \Delta$ and $\Gamma \vdash \phi, \Delta$, respectively.

We write $T\,\langle S \rangle$ when a sequent S occurs in a tree T, more precisely:

- $S\,[ST]\,\langle S \rangle$;
- $S'\,[ST]\,\langle S \rangle$, provided that S' is different than S and $ST\,\langle S \rangle$; and
- $(m : T', ST)\,\langle S \rangle$, when either $T'\,\langle S \rangle$ or $ST\,\langle S \rangle$.

We extend the occurring relation $T\,\langle T' \rangle$ between trees as expected:

- T and T' are the same;
- $(S\,[ST])\,\langle T' \rangle$, when $ST\,\langle T' \rangle$;
- $(m : T', ST')\,\langle T' \rangle$; and
- $(m : T'', ST')\,\langle T' \rangle$, provided that T'' is different than T' and $ST'\,\langle T' \rangle$.

We also distinguish when $m : T'$ occurs in a tree T, written $T\,\langle m : T' \rangle$:

- $(S\,[ST])\,\langle m : T' \rangle$, when $ST\,\langle m : T' \rangle$;
- $(m : T', ST')\,\langle m : T' \rangle$; and
- $(m' : T'', ST')\,\langle T' \rangle$, provided that either m is different than m' or T'' is different than T', and $ST'\,\langle m : T' \rangle$.

We say a sequent S occurs, under a modality m, in a finite sequence of tree hypersequents $m_1 : T_1, m_2 : T_2, \ldots, m_k : T_k$, when there is an i such that m_i is m and T_i has the form $S\,[ST]$. Moreover, we often write $S\,[m : S']$ instead of $S\,[ST]$, provided S' occurs under m in ST.

Definition 2 (Tree-hypersequents derivation system). *The inference system for tree hypersequents G is defined as follows.*

- *Initial tree hypersequents:*

$$T\,\langle p, S, p\rangle$$

- *Propositional rules:*

$$\frac{T\,\langle S, \phi\rangle}{T\,\langle \neg\phi, S\rangle}\,\neg L \qquad \frac{T\,\langle \phi, S\rangle}{T\,\langle S, \neg\phi\rangle}\,\neg R$$

$$\frac{T\,\langle \phi, \psi, S\rangle}{T\,\langle \phi \wedge \psi, S\rangle}\,\wedge L \qquad \frac{T\,\langle S, \phi\rangle \qquad T\,\langle S, \psi\rangle}{T\,\langle S, \phi \wedge \psi\rangle}\,\wedge R$$

- *Modal rules:*

$$\frac{T\,\langle \Box_m\phi, S\,[m : \phi, S']\rangle}{T\,\langle \Box_m\phi, S\,[m : S']\rangle}\,\Box_m L$$

$$\frac{T\,\langle S\,[m : \vdash \phi, ST]\rangle}{T\,\langle S, \Box_m\phi\,[ST]\rangle}\,\Box_m R$$

We now define the concept of derivation tree:

- any rule (up to subtitution) of G is a derivation tree;
- $\frac{T'}{T}$ and $\frac{T' \quad T''}{T}$ are derivation trees, provided that T' and T'' are derivation trees, and
- $\frac{T'_0}{T}$ and $\frac{T'_0 \quad T''_0}{T}$ are rules in G and T'_0 and T''_0 are the lowest tree hypersequents occurring in T' and T''.

If all the branches of a derivation tree, where T is the lowest tree hypersequent, are finite and ends with an initial tree hypersequent, then we say the derivation tree is a proof tree, or simply a proof, of T, or that T is derivable in G, and we write $\vdash_G T$.

Consider now for instance the following proof of A_4:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T\,\langle \phi \vdash \psi, \phi\rangle \qquad \dfrac{T\,\langle \phi, \psi \vdash \psi\rangle}{T\,\langle \phi \vdash \psi, \neg\psi\rangle}\,\neg R}{T\,\langle \phi \vdash \psi, \phi \wedge \neg\psi\rangle}\,\wedge R}{\Box_m\neg(\phi \wedge \neg\psi), \Box_m\phi \vdash [m : \neg(\phi \wedge \neg\psi), \phi \vdash \psi]}\,\neg L}{\Box_m\neg(\phi \wedge \neg\psi), \Box_m\phi \vdash [m : \phi \vdash \psi]}\,\Box_m L}{\Box_m\neg(\phi \wedge \neg\psi), \Box_m\phi \vdash [m : \vdash \psi]}\,\Box_m L}{\Box_m\neg(\phi \wedge \neg\psi), \Box_m\phi \vdash \Box_m\psi}\,\Box_m R}{\Box_m\neg(\phi \wedge \neg\psi), \Box_m\phi, \neg\Box_m\psi \vdash}\,\neg L}{\Box_m\neg(\phi \wedge \neg\psi) \wedge \Box_m\phi \wedge \neg\Box_m\psi \vdash}\,\wedge L}{\vdash \neg(\Box_m\neg(\phi \wedge \neg\psi) \wedge \Box_m\phi \wedge \neg\Box_m\psi)}\,\neg R$$

Theorem 2 ([21, 19]). *For any sequent S, $\vdash_G S$, if and only if, $\vdash_H S$.*

Corollary 1. *For any sequent S, $\vdash_G S$, if and only if, $\models S^I$.*

4 Interpolation

We define the set of non-logical symbols $Sym(\phi)$ of a formula ϕ as follows:

- $Sym(p) = \{p\}$;
- $Sym(\neg\phi) = Sym(\phi)$;
- $Sym(\phi \wedge \psi) = Sym(\phi) \cup Sym(\psi)$; and
- $Sym(\Box_m \phi) = \{m\} \cup Sym(\phi)$.

The set of non-logical symbols of a (multi-)set of formulas is defined as expected.

For technical convenience, we consider an equivalent extension G' of the derivation system G, where formulas $\top$ are considered *per se* (not as notation). All rules in G are also in G'. Additionally, the initial sequent $T\langle S, \top\rangle$ is also included in G'.

Lemma 1 (Maehara's Lemma). *Let $T\langle \Gamma \vdash \Delta\rangle$ be derivable in G, and let Γ_1, Γ_2 and Δ_1, Δ_2 be partitions of Γ and Δ, respectively. Then there is a formula β, called the interpolant, such that $T\langle \Gamma_1 \vdash \Delta_1, \beta\rangle$ and $T\langle \beta, \Gamma_2 \vdash \Delta_2\rangle$ are derivable in G', and $Sym(\beta) \subseteq (Sym(\Gamma_1) \cup Sym(\Delta_1)) \cap (Sym(\Gamma_2) \cup Sym(\Delta_2))$.*

Proof. By induction on the height of the proof tree.

The base case is $T\langle p, \Gamma \vdash \Delta, p\rangle$. The interpolant β is then defined according to the occurrence of propositions p in partitions:

$$
\begin{array}{ll}
T\langle p, \Gamma_1 \vdash \Delta_1, p, \neg\top\rangle & T\langle \neg\top, \Gamma_2 \vdash \Delta_2\rangle \\
T\langle \Gamma_1 \vdash \Delta_1, \top\rangle & T\langle \top, p, \Gamma_2 \vdash p, \Delta_2\rangle \\
T\langle p, \Gamma_1 \vdash \Delta_1, p\rangle & T\langle p, \Gamma_2 \vdash \Delta_2, p\rangle \\
T\langle \Gamma_1 \vdash \Delta_1, p, \neg p\rangle & T\langle \neg p, p, \Gamma_2 \vdash \Delta_2\rangle
\end{array}
$$

Induction step. Assume the last inference is the following:

$$\frac{T\langle \Gamma \vdash \Delta, \phi\rangle \qquad T\langle \Gamma \vdash \Delta, \psi\rangle}{T\langle \Gamma \vdash \Delta, \phi \wedge \psi\rangle}$$

By induction hypothesis, there are interpolants β_1 and β_2 for the upper tree hypersequents. There are two possible cases according the occurrence of ϕ and ψ in the respective partitions.

$$
\begin{array}{ll}
T\langle \Gamma_1 \vdash \Delta_1, \phi, \beta_1\rangle & T\langle \beta_1, \Gamma_2 \vdash \Delta_2\rangle \\
T\langle \Gamma_1 \vdash \Delta_1, \psi, \beta_2\rangle & T\langle \beta_2, \Gamma_2 \vdash \Delta_2\rangle \\
T\langle \Gamma_1 \vdash \Delta_1, \beta_1\rangle & T\langle \beta_1, \Gamma_2 \vdash \Delta_2, \phi\rangle \\
T\langle \Gamma_1 \vdash \Delta_1, \beta_2\rangle & T\langle \beta_2, \Gamma_2 \vdash \Delta_2, \psi\rangle
\end{array}
$$

Depending on the occurrence of $\phi \wedge \psi$ in the partitions, we then construct the interpolant ϕ as follows:

$$
\begin{array}{l}
T\langle \Gamma_1 \vdash \Delta_1, \phi \wedge \psi, \beta_1 \vee \beta_2\rangle \\
T\langle \beta_1 \vee \beta_2, \Gamma_2 \vdash \Delta_2\rangle \\
T\langle \Gamma_1 \vdash \Delta_1, \beta_1 \wedge \beta_2\rangle \\
T\langle \beta_1 \wedge \beta_2, \Gamma_2 \vdash \Delta_2, \phi \wedge \psi\rangle
\end{array}
$$

In the induction step, now consider the last inference is

$$\frac{T\,\langle \Box_m\phi, \Gamma \vdash \Delta\,[m : \phi, \Gamma' \vdash \Delta']\rangle}{T\,\langle \Box_m\phi, \Gamma \vdash \Delta\,[m : \Gamma' \vdash \Delta']\rangle}$$

By induction, there is an interpolant β for the upper tree hypersequent. By the occurrence of $\Box_m\phi$ in partitions, we distinguish two cases:

$$T\,\langle \Box_m\phi, \Gamma_1 \vdash \Delta_1, \beta\,[m : \phi, \Gamma' \vdash \Delta']\rangle$$
$$T\,\langle \beta, \Gamma_2 \vdash \Delta_2\,[m : \phi, \Gamma' \vdash \Delta']\rangle$$
$$T\,\langle \Gamma_1 \vdash \Delta_1, \beta\,[m : \phi, \Gamma' \vdash \Delta']\rangle$$
$$T\,\langle \beta, \Box_m\phi, \Gamma_2 \vdash \Delta_2\,[m : \phi, \Gamma' \vdash \Delta']\rangle$$

We then construct the following interpolants:

$$T\,\langle \Box_m\phi, \Gamma_1 \vdash \Delta_1, \Box_m\phi \wedge \beta\,[m : \Gamma' \vdash \Delta']\rangle$$
$$T\,\langle \Box_m\phi \wedge \beta, \Gamma_2 \vdash \Delta_2\,[m : \Gamma' \vdash \Delta']\rangle$$
$$T\,\langle \Gamma_1 \vdash \Delta_1, \neg\Box_m\phi \vee \beta\,[m : \Gamma' \vdash \Delta']\rangle$$
$$T\,\langle \neg\Box_m\phi \vee \beta, \Box_m\phi, \Gamma_2 \vdash \Delta_2\,[m : \Gamma' \vdash \Delta']\rangle$$

Consider now the last inference is the following:

$$\frac{T\,\langle \Gamma \vdash \Delta\,[m : \vdash \phi, ST]\rangle}{T\,\langle \Gamma \vdash \Delta, \Box_m\phi\,[ST]\rangle}$$

We obtain the following interpolant β by induction:

$$T\,\langle \Gamma_1 \vdash \Delta_1, \beta\,[m : \vdash \phi, ST]\rangle$$
$$T\,\langle \beta, \Gamma_2 \vdash \Delta_2\,[m : \vdash \phi, ST]\rangle$$

There are then two cases depending on the occurrence of $\Box_m\phi$ in partitions:

$$T\,\langle \Gamma_1 \vdash \Delta_1, \Box_m\phi, \neg\Box_m\phi \wedge \beta\,[ST]\rangle$$
$$T\,\langle \neg\Box_m\phi \wedge \beta, \Gamma_2 \vdash \Delta_2\,[ST]\rangle$$
$$T\,\langle \Gamma_1 \vdash \Delta_1, \Box_m\phi \vee \beta\,[ST]\rangle$$
$$T\,\langle \Box_m\phi \vee \beta, \Gamma_2 \vdash \Delta_2, \Box_m\phi\,[ST]\rangle$$

Theorem 3 (Craig Interpolation). *For any two formulas ϕ and ψ, if $\models \phi \to \psi$, then there is a formula β, such that $\models \phi \to \beta$, $\models \beta \to \psi$ and $Sym(\beta) \subseteq Sym(\phi) \cap Sym(\beta)$, provided that there is a proposition p such that $p \in Sym(\phi) \cap Sym(\beta)$.*

Proof. Assume $\models \phi \to \psi$, then by Corollary 1, $\phi \vdash \psi$ is derivable in G. By Lemma 1, there is a formula β, such that $\phi \vdash \beta$ and $\beta \vdash \phi$ are derivable in G'. Let $p \in Sym(\phi) \cap Sym(\beta)$. Now, let β' be obtained from β by replacing $\top$ by $\neg(p \wedge \neg p)$. It is straightforward that $\phi \vdash \beta$ and $\beta \vdash \phi$ are derivable in G, and hence (by Corollary 1) $\models \phi \to \beta$ and $\models \beta \to \psi$.

5 Definability and Consistency

Definition 3 (Implicit definability). *Let $\phi(p, p_1, \ldots, p_k)$ be a formula, where $p, p_1, \ldots, p_k$ are propositions occurring in it. We say $\phi(p, p_1, \ldots, p_k)$ defines p implicitly if*

$$\models (\phi(p, p_1, \ldots, p_k) \wedge \phi(p', p_1, \ldots, p_k)) \rightarrow (p \leftrightarrow p')$$

where $p \neq p'$.

Definition 4 (Explicit definability). *Let $\phi(p, p_1, \ldots, p_k)$ be a formula, where $p, p_1, \ldots, p_k$ are propositions occurring in it. We say $\phi(p, p_1, \ldots, p_k)$ defines p explicitly, when*

$$\models \phi(p, p_1, \ldots, p_k) \rightarrow (p \leftrightarrow \psi)$$

where $Sym(\psi) \subseteq Sym(\phi(p, p_1, \ldots, p_k)) \setminus \{p\}$.

Theorem 4 (Beth Definability). *Let $\phi(p, p_1, \ldots, p_k)$ be a formula, where $p, p_1, \ldots, p_k$ are propositions occurring in it. If $\phi(p, p_1, \ldots, p_k)$ defines p implicitly, then $\phi(p, p_1, \ldots, p_k)$ defines p explicitly.*

Proof. From the implicit definability assumption, it is easy to see that

$$\models (\phi(p, p_1, \ldots, p_k) \wedge p) \rightarrow (\phi(p', p_1, \ldots, p_k) \rightarrow p')$$

By the Craig Interpolation Theorem 3, we then obtain

$$\models (\phi(p, p_1, \ldots, p_k) \wedge p) \rightarrow \psi$$
$$\models \psi \rightarrow (\phi(p', p_1, \ldots, p_k) \rightarrow p')$$

where $Sym(\psi) \subseteq Sym(\phi(p, p_1, \ldots, p_k)) \setminus \{p\}$.

Before definining the notion of consistency, we need a precise description of some concepts. An axiom system is a finite set of formulas. An axiom sequence is a (possibly empty) subset of an axiom system. We say a sequent S is derivable (provable) in G from an axiom system A, if there is an axiom sequence A' of A, such that $\vdash_G A', S$.

Definition 5 (Consistency). *An axiom system is inconsistent if the empty sequent is derivable from it. We say an axiom system is consistent if it is not inconsistent.*

Theorem 5 (Robinson Joint Consistency). *Consider two consistent axiom systems A_1 and A_2, if for any formula ϕ, such that $Sym(\phi) \subseteq Sym(A_1) \cap Sym(A_2)$, it is not the case that both ϕ and $\neg\phi$ are derivable from A_1 and A_2 (or A_2 and A_1), respectively, then $A_1 \cup A_2$ is consistent.*

Proof. We prove the contrapositive. If $A_1 \cup A_2$ is not consistent, then there are two axiom sequences A_1' and A_2' of A_1 and A_2, resp., such that $A_1, A_2 \vdash$ are derivable in G. Recall each A_1 and A_2 is consistent, then not empty. By Lemma 1, there is an interpolant ϕ, where $Sym(\phi) \subseteq Sym(A_1) \cap Sym(A_2)$, such that $A_1 \vdash \phi$ and $\phi, A_2 \vdash$ (hence $A_2 \vdash \neg\phi$) are both derivable in G'. As in the proof of Theorem 3, it is straight forward that both $A_1 \vdash \phi$ and $A_2 \vdash \neg\phi$ are also derivable in G by replaceing all the occurrences of $\top$ in ϕ by $\neg(p \wedge \neg p)$ for a $p \in Sym(A_1) \cup Sym(A_2)$.

6 Conclusions

In this paper, we describe a constructive proof of the Craig interpolation property. The proof is based on the Maehara technique on a complete cut-free tree-hypersequent calculus. An interpolant algorithm can easily be inferred from the proof. A complexity analysis of this algorithm is prospected as further research. We are also interested in constructive interpolation proofs for other more expressive modal logics, such as K_m with converse, CTL and the μ-calculus.

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