On the Kleene Algebra of Partial Predicates with Predicate Complement

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Abstract. In the paper we investigate the question of expressibility of partial predicates in the Kleene algebra extended with the composition of predicate complement and give a necessary and sufficient condition of this expressibility in terms of the existence of an optimal solution of an optimization problem. The obtained results may be useful for development of (semi-)automatic deduction tools for an extension of the Floyd-Hoare logic for the case of partial pre- and postconditions.

Keywords: Formal methods, software verification, partial predicate, Floyd-Hoare logic.

1 Introduction

Floyd-Hoare logic [1, 2] is a logic which is useful for proving partial correctness of sequential programs. It is based on properties of triples (assertions) of the form $\{p\}f\{q\}$, where f is a program and p, q are predicates which specify pre- and post-conditions. An assertion of this kind means that if the program's input d satisfies the pre-condition p, and the program terminates on d, the program's output satisfies the post-condition q. In the classical Floyd-Hoare logic the program is allowed to be non-terminating (or have an undefined result of execution), but the pre- and postconditions are assumed to be always defined (have a well defined truth value). In the presence of pre- and postconditions defined by partial predicates (which can be undefined on some data) the inference rules (in particular, the sequence rule) of the classical Floyd-Hoare logic become unsound [13, 14], when a triple $\{p\}f\{q\}$ is understood in the following way: if a precondition p is defined and true on the program's input, and the program f terminates with a result y, and the postcondition q is defined on y, then q is true on y.

In the previous works [15, 3, 4, 10, 12, 11, 8] we investigated an inference system for an extension of Floyd-Hoare logic which remains sound in the case of partial pre- and postconditions, assuming the above mentioned interpretation of Floyd-Hoare triples. The formulations of the rules of this inference system, however, require introduction of a new composition into the logical language used to express pre- and postconditions. Whereas the formulation of the rules of the classical Floyd-Hoare logic depends on the usual boolean compositions ($\neg$, $\wedge$) of pre- and postcondition predicates (which are assumed to be total), the mentioned extension depends on the compositions of negation ($\neg$) and conjunction

($\wedge$) of partial predicates defined in accordance with the tables of Kleene's strong 3-valued logic, and on one additional unary composition of partial predicates which we call the composition of predicate complement and denote as $\sim$. This composition extends the signature of the Kleene algebra of partial predicates [9]. In this paper we investigate the question of expressibility of partial predicates in the Kleene algebra extended with the composition of predicate complement and give a necessary and sufficient condition of this expressibility in terms of the existence of an optimal solution of a special constrained optimization problem. The obtained results may be useful for development of (semi-)automatic deduction tools for the mentioned extension of the Floyd-Hoare logic for the case of partial pre- and postconditions.

2 Notation

We will use the following notation. The notation $f : A \tilde{\to} B$ means that f is a partial function on a set A with values in a set B, and $f : A \to B$ means that f is a total function from A to B. For a function $f : A \tilde{\to} B$:

- $f(x) \downarrow$ means that f is defined on x;
- $f(x) \downarrow = y$ means that f is defined on x and $f(x) = y$;
- $f(x) \uparrow$ means that f is undefined on x;
- $dom(f) = \{x \in A \mid f(x) \downarrow\}$ is the domain of a function.

We will denote as $f_1(x_1) \cong f_2(x_2)$ the *strong equality*, i.e. $f_1(x_1) \downarrow$ if and only if $f_2(x_2) \downarrow$, and if $f_1(x_1) \downarrow$, then $f_1(x_1) = f_2(x_2)$.

The symbols T, F will denote the "true" and "false" values of predicates.

We will denote $Bool = \{T, F\}$. The symbol $\perp$ will denote a nowhere defined partial predicate.

Let $D \neq \emptyset$ be a set, and P_0, P_1, ... P_n be partial predicates on D.

Let $APr_{P_1,...,P_n}(D) = (D \tilde{\to} \{T, F\}; \vee, \wedge, \neg, \sim, P_1, P_2, ..., P_n)$ be an algebra of partial predicates with constants $P_1, ..., P_n$, where

1. $\vee, \wedge, \neg$ are the operations of disjunction, conjunction and negation on partial predicates defined in accordance with Kleene's strong three-valued logic as follows:

$$(P \vee Q)(d) = \begin{cases} T, & \text{if } P(d) \downarrow = T \text{ or } Q(d) \downarrow = T; \\ F, & \text{if } P(d) \downarrow = F \text{ and } Q(d) \downarrow = F; \\ \text{undefined} & \text{in other cases.} \end{cases}$$

$$(P \wedge Q)(d) = \begin{cases} T, & \text{if } P(d) \downarrow = T \text{ and } Q(d) \downarrow = T; \\ F, & \text{if } P(d) \downarrow = F \text{ or } Q(d) \downarrow = F; \\ \text{undefined} & \text{in other cases.} \end{cases}$$

$$(\neg P)(d) = \begin{cases} T, & \text{if } P(d) \downarrow = F; \\ F, & \text{if } P(d) \downarrow = T; \\ \text{undefined} & \text{in other case.} \end{cases}$$

2. $\sim$ is the unary operation of predicate complement:

$$(\sim P)(d) = \begin{cases} T, & \text{if } P(d)\uparrow; \\ \text{undefined}, & \text{if } P(d)\downarrow. \end{cases}$$

We will call $APr_{P_1,...,P_n}(D)$ the Kleene algebra of partial predicates on D with predicate complement and constants $P_1, ..., P_n$.

3 Main Result

Let $F^{(n)}$ be the set of all n-ary functions (operations) $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$. The elements of $F^{(n)}$ will represent functions of 3-valued logic P_3 (where 1 corresponds to the "true" value and -1 corresponds to the "false" value, and 0 is an intermediate truth value).

Let $F = \bigcup_{n\geq 0} F^{(n)}$.

We will denote as $\bar{x} = (x_1, x_2, ..., x_n)$ a tuple of values $x_i \in \{-1, 0, 1\}$.

Let us consider $\{-1, 0, 1\}^n$ as a metric space with Chebyshev distance:

$$\rho_n((x_1, ..., x_n), (y_1, ..., y_n)) = \max_{i=1}^{n} |x_i - y_i|.$$

We will say that a function $f \in F^{(n)}$ is *short*, if it is a short map, i.e. if for all $\bar{x}$, $\bar{y}$ we have

$$|f(\bar{x}) - f(\bar{y})| \leq \rho_n(\bar{x}, \bar{y}).$$

For any predicate $P : D \tilde{\to} \{T, F\}$ denote by $\Phi(P)$ a function $D \to \{-1, 0, 1\}$ such that for all $d \in D$:

$$\Phi(P)(d) = \begin{cases} 1, & \text{if } P(d)\downarrow = T, \\ 0, & \text{if } P(d)\uparrow, \\ -1, & \text{if } P(d)\downarrow = F. \end{cases}$$

Let $D \neq \emptyset$ be a set, $P_1, P_2, ..., P_n : D \tilde{\to} \{T, F\}$ be partial predicates, and

$$APr_{P_1,...,P_n}(D) = (D \tilde{\to} \{T, F\}; \vee, \wedge, \neg, \sim, P_1, P_2, ..., P_n).$$

Let $p_i = \Phi(P_i)$ for $i = 0, 1, 2, ..., n$.

Denote $||f|| = \sum_{\bar{x}\in\{-1,0,1\}^n} |f(\bar{x})|$ for $f \in F^{(n)}$ and consider the following (constrained) optimization problem[1]:

$$||f|| \to \min \quad (1)$$

$$f(p_1(d), p_2(d), ..., p_n(d)) = p_0(d), \quad d \in D \quad (2)$$

Theorem 1. *If $n \geq 1$, a predicate P_0 is expressible in the algebra $APr_{P_1,...,P_n}(D)$ if and only if on the set $F^{(n)}$ the problem (1)-(2) has an optimal solution which is a short function.*

[1] If one interprets partiality in terms as possibility, minimization of $||f||$ may be related to the principle of minimum specificity of D. Dubois et al. from possibility theory, or other similar principles.

4 Proof of the Main Result

Denote for all $x, y \in \{-1, 0, 1\}$:

$$\neg x = -x$$

$$\sim x = 1 - |x|$$

$$x^{[y]} = \begin{cases} x, & \text{if } y = 1 \\ \sim x, & \text{if } y = 0 \\ \neg x, & \text{if } y = -1 \end{cases}$$

Lemma 1. *$\rho_n(\bar{x}, \bar{y}) = 1 - \min_{i=1}^n x_i^{[y_i]}$ for every $n \geq 1$ and $\bar{x}, \bar{y} \in \{-1, 0, 1\}^n$.*

Proof. It is easy to see that for all $x, y \in \{-1, 0, 1\}$:

$$x^{[y]} = 1 - |x - y|$$

Then $\rho_n(\bar{x}, \bar{y}) = \max_{i=1}^n |x_i - y_i| = \max_{i=1}^n (1 - x_i^{[y_i]}) = 1 - \min_{i=1}^n x_i^{[y_i]}$. □

Consider $\{-1, 0, 1\}$ as a lattice with operations:

$$x \vee y = \max(x, y);$$

$$x \wedge y = \min(x, y).$$

Below we will assume that in expressions involving operations on $\{-1, 0, 1\}$ the operation $x^{[y]}$ has the highest priority, and is followed (by priority) by the unary operations $\neg$, $\sim$, which are followed by the binary operations $\wedge$ and $\vee$. As usual, among $\wedge, \vee$, the operation $\wedge$ has higher priority.

Lemma 2. *For each short function $f \in F^{(n)}$ and $\bar{x} \in \{-1, 0, 1\}^n$:*

$$f(\bar{x}) = \hat{f}(\bar{x}) \wedge f_{\neq 0}(\bar{x}) \vee \neg f_{\neq 0}(\bar{x})$$

where

$$\hat{f}(\bar{x}) = \begin{cases} \bigvee_{\bar{y}: f(\bar{y})=1} \bigwedge_{i=1}^n x_i^{[y_i]}, & \text{if } \exists \bar{y}\ f(\bar{y}) = 1 \\ -1, & \text{otherwise} \end{cases}$$

$$f_{\neq 0}(\bar{x}) = \begin{cases} \bigvee_{\bar{y}: f(\bar{y}) \neq 0} \bigwedge_{i=1}^n \sim (x_i^{[y_i]} \wedge \sim x_i^{[y_i]}) \wedge \sim\sim x_i^{[y_i]}, & \text{if } \exists \bar{y}\ f(\bar{y}) \neq 0 \\ 0, & \text{otherwise.} \end{cases}.$$

Proof. It is easy to see that for each $x, y \in \{-1, 0, 1\}$:

$$\sim (x^{[y]} \wedge \sim x^{[y]}) \wedge \sim\sim x^{[y]} = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

Then

$$f_{\neq 0}(\bar{x}) = \begin{cases} 1, & \text{if } f(\bar{x}) \neq 0 \\ 0, & \text{if } f(\bar{x}) = 0. \end{cases}$$

By Lemma 1,

$$\hat{f}(\bar{x}) = \begin{cases} \bigvee_{\bar{y}: f(\bar{y})=1} (1 - \rho_n(\bar{x}, \bar{y})), & \text{if } \exists \bar{y}\ f(\bar{y}) = 1, \\ -1, & \text{otherwise.} \end{cases}$$

If $f(\bar{x}) = 1$, then $\hat{f}(\bar{x}) = 1$ and $f_{\neq 0}(\bar{x}) = 1$, so $\hat{f}(\bar{x}) \wedge f_{\neq 0}(\bar{x}) \vee \neg f_{\neq 0}(\bar{x}) = 1$.
If $f(\bar{x}) = 0$, then $f_{\neq 0}(\bar{x}) = 0$, so
$\hat{f}(\bar{x}) \wedge f_{\neq 0}(\bar{x}) \vee \neg f_{\neq 0}(\bar{x}) = (\hat{f}(\bar{x}) \wedge 0) \vee 0 = 0$.

If $f(\bar{x}) = -1$, then for each $\bar{y}$ such that $f(\bar{y}) = 1$ we have $\rho_n(\bar{x}, \bar{y}) \geq |f(\bar{x}) - f(\bar{y})| = 2$ which implies that $1 - \rho_n(\bar{x}, \bar{y}) = -1$. Then $\hat{f}(\bar{x}) = -1$ and $f_{\neq 0}(\bar{x}) = 1$, so $\hat{f}(\bar{x}) \wedge f_{\neq 0}(\bar{x}) \vee \neg f_{\neq 0}(\bar{x}) = -1$.

Thus

$$f(\bar{x}) = \hat{f}(\bar{x}) \wedge f_{\neq 0}(\bar{x}) \vee \neg f_{\neq 0}(\bar{x}).$$

□

Lemma 3. *The set of all short functions from F is a precomplete class in F and is the functional closure of the set $\{f_0, f_1, f_2, f_3, f_4\}$, where $f_0 \in F^{(0)}$, $f_1, f_2 \in F^{(1)}$, $f_3, f_4 \in F^{(2)}$ and $f_0 = 0$, $f_1(x) = -x$, $f_2(x) = 1 - |x|$, $f_3(x, y) = \max(x, y)$, $f_4(x, y) = \min(x, y)$.*

Proof. Denote by S the set of all short functions from F. In accordance with its definition, a short function from F can be alternatively characterized as a function $\{-1, 0, 1\}^n \to \{-1, 0, 1\}$ $(n \geq 0)$ which does not change sign on each of the sets $\prod_{i=1}^n \{0, a_i\}$, where $a_1, ..., a_n \in \{-1, 1\}^n$. In the terminology of [18], such functions correspond to the precomplete class $T^3_{\mathcal{E}_1,1}$ of functions for which the image of the product of sets, 1-equivalent to $\mathcal{E}_1$ is a subset of a set, 1-equivalent to $\mathcal{E}_1$, where two sets are 1-equivalent, if their symmetric difference has no more than 1 element. Thus S is a precomplete class in F. Obviously, $\{f_0, f_1, f_2, f_3, f_4\} \subseteq S$. On the other hand, since the constant function with value -1 is expressible as $f_1 \circ f_2 \circ f_0$, from Lemma 2 and the definition of $x^{[y]}$ it follows that each $f \in S$ can be expressed as a composition of elements of $\{f_0, f_1, f_2, f_3, f_4\}$ and of projections $\pi^n_k(x_1, ..., x_n) = x_k$ $(n \geq 1,\ k = 1, 2, ..., n)$. Thus S is the functional closure of $\{f_0, f_1, f_2, f_3, f_4\}$. □

Lemma 4. *For each $P, Q : D \tilde{\to} \{T, F\}$ and $d \in D$ we have:*
$\Phi(\bot)(d) = 0$
$\Phi(\neg P)(d) = -(\Phi(P)(d))$
$\Phi(\sim P)(d) = 1 - |\Phi(P)(d)|$
$\Phi(P \vee Q)(d) = \max(\Phi(P)(d), \Phi(Q)(d))$
$\Phi(P \wedge Q)(d) = \min(\Phi(P)(d), \Phi(Q)(d))$

Proof. Follows immediately from the definition Φ and operations $\neg, \sim, \vee, \wedge$ on partial predicates. □

Let $M^{(n)}$ be the set of all short functions from $F^{(n)}$.

Lemma 5. *The problem (1)-(2) has an optimal solution on $F^{(n)}$ if and only if p_0 is continuous in the initial topology on D induced by $p_1, ..., p_n$ (where the codomain of p_i, $\{-1, 0, 1\}$, is considered as a discrete space).*

Proof. "If": assume that p_0 is continuous in the initial topology on D induced by $p_1, ..., p_n$. Then there exists $f \in F^{(n)}$ such that $p_0(d) = f(p_1(d), ..., p_n(d))$ for all $d \in D$. Then since the set $F^{(n)}$ is finite, the problem (1)-(2) has an optimal solution on $F^{(n)}$.

"Only if": assume that the problem (1)-(2) has an optimal solution $f \in F^{(n)}$. Then $p_0(d) = f(p_1(d), ..., p_n(d))$ for all $d \in D$, so p_0 is continuous in the initial topology on D induced by $p_1, ..., p_n$. □

Lemma 6. *If the problem (1)-(2) has an optimal solution on $F^{(n)}$, then this solution is unique.*

Proof. Assume that the problem (1)-(2) has optimal solutions $f, g \in F^{(n)}$. Then $||f|| = ||g||$ and $f(p_1(d), ..., p_n(d)) = p_0(d) = g(p_1(d), ..., p_n(d))$ for all $d \in D$.

Suppose that $f \neq g$. Then there exists $\bar{x}^* = (x_1^*, ..., x_n^*) \in \{-1, 0, 1\}^n$ such that $f(\bar{x}^*) \neq g(\bar{x}^*)$.

Consider the case when $f(\bar{x}^*) \neq 0$. Let us define a function $h \in F^{(n)}$ as follows: $h(\bar{x}) = f(\bar{x})$, if $\bar{x} \neq \bar{x}^*$, and $h(\bar{x}) = 0$, if $\bar{x} = \bar{x}^*$. Then for all $d \in D$, $(p_1(d), ..., p_n(d)) \neq \bar{x}^*$, so $h(p_1(d), ..., p_n(d)) = p_0(d)$. Moreover, $||h|| = ||f|| - |f(\bar{x}^*)| = ||f|| - 1 < ||f||$ which contradicts the assumption that f is an optimal solution of (1)-(2).

Consider the case when $f(\bar{x}^*) = 0$. Then $|g(\bar{x}^*)| = 1$. Let us define a function $h \in F^{(n)}$ as follows: $h(\bar{x}) = g(\bar{x})$, if $\bar{x} \neq \bar{x}^*$, and $h(\bar{x}) = 0$, if $\bar{x} = \bar{x}^*$. Then for all $d \in D$, $(p_1(d), ..., p_n(d)) \neq \bar{x}^*$, so $h(p_1(d), ..., p_n(d)) = p_0(d)$. Moreover, $||h|| = ||g|| - |g(\bar{x}^*)| = ||g|| - 1 < ||g||$ which contradicts the assumption that g is an optimal solution of (1)-(2).

Thus $f = g$. So if the problem (1)-(2) has an optimal solution on $F^{(n)}$, then this solution is unique. □

Lemma 7. *Let $f \in M^{(n)}$, $g \in F^{(n)}$ and $g(\bar{x}) \in \{f(\bar{x}), 0\}$ for each $\bar{x} \in \{-1, 0, 1\}^n$. Then $g \in M^{(n)}$.*

Proof. Let $\bar{x}, \bar{y} \in \{-1, 0, 1\}^n$. Consider the following cases.

1) $g(\bar{x}) = f(\bar{x}), g(\bar{y}) = f(\bar{y})$. Then $|g(\bar{x}) - g(\bar{y})| = |f(\bar{x}) - f(\bar{y})| \leq \rho(\bar{x}, \bar{y})$.

2) $g(\bar{x}) = f(\bar{x}), g(\bar{y}) = 0$. Then $|g(\bar{x}) - g(\bar{y})| = |f(\bar{x})| \leq \rho(\bar{x}, \bar{y})$, if $\bar{x} \neq \bar{y}$, and $|g(\bar{x}) - g(\bar{y})| = 0 \leq \rho(\bar{x}, \bar{y})$, if $\bar{x} = \bar{y}$.

3) $g(\bar{x}) = 0, g(\bar{y}) = f(\bar{y})$. Then $|g(\bar{x}) - g(\bar{y})| = |f(\bar{y})| \leq \rho(\bar{x}, \bar{y})$, if $\bar{x} \neq \bar{y}$, and $|g(\bar{x}) - g(\bar{y})| = 0 \leq \rho(\bar{x}, \bar{y})$, if $\bar{x} = \bar{y}$.

4) $g(\bar{x}) = 0$, $g(\bar{y}) = 0$. Then $|g(\bar{x}) - g(\bar{y})| \leq \rho(\bar{x}, \bar{y})$.

Thus $g \in M^{(n)}$. □

Lemma 8. *The problem (1)-(2) has an optimal solution on $M^{(n)}$ if and only if it has an optimal solution on $F^{(n)}$ which belongs to $M^{(n)}$.*

Proof. "If": assume that the problem (1)-(2) has an optimal solution $f \in F^{(n)}$ which belongs to $M^{(n)}$. Then $f(p_1(d), p_2(d), ..., p_n(d)) = p_0(d)$ for all $d \in D$. Moreover, for each $g \in M^{(n)}$ such that $g(p_1(d), p_2(d), ..., p_n(d)) = p_0(d)$ for all $d \in D$, we have $g \in F^{(n)}$, so $||f|| \leq ||g||$. So f is an optimal solution of (1)-(2)on $M^{(n)}$.

"Only if": assume that the problem (1)-(2) has an optimal solution f on $M^{(n)}$. Then $f(p_1(d), p_2(d), ..., p_n(d)) = p_0(d)$ for all $d \in D$. Then since $F^{(n)}$ is finite, the problem (1)-(2) has an optimal solution on $F^{(n)}$. By Lemma 6, the problem (1)-(2) has a unique optimal solution of $F^{(n)}$. Denote it as g. Then $g(p_1(d), p_2(d), ..., p_n(d)) = p_0(d)$ for all $d \in D$ and $||g|| \leq ||f||$. Let us define a function $h \in F^{(n)}$ as follows: for each $\bar{x} \in \{-1, 0, 1\}^n$, $h(\bar{x}) = f(\bar{x})$, if $g(\bar{x}) \neq 0$, and $h(\bar{x}) = g(\bar{x})$, if $g(\bar{x}) = 0$. Then for all $d \in D$, $h(p_1(d), ..., p_n(d)) = p_0(d)$. Moreover, $h \in M^{(n)}$ by Lemma 7. Then $||h|| = ||f||$, so for each $\bar{x}$ such that $g(\bar{x}) = 0$ we have $f(\bar{x}) = 0$. Then $||f|| \leq ||g||$. Since $||g|| \leq ||f||$ as mentioned above, we have $||f|| = ||g||$. The f is an optimal solution of (1)-(2) on $F^{(n)}$ and f belongs to $M^{(n)}$. □

Now we can give a proof of the main Theorem 1 from the previous section.

Proof (of Theorem 1). "If": assume that the problem (1)-(2) has an optimal solution on the set $F^{(n)}$ which is a short function. Denote by f such a solution. Then we have $p_0(d) = f(p_1(d), p_2(d), ..., p_n(d))$ for all $d \in D$. By Lemma 3, f belongs to the functional closure of $\{f_0, f_1, f_2, f_3, f_4\}$, where f_i are defined as in Lemma 3. From Lemma 4 it follows that $p_0(d) = \Phi(P)(d)$ for all $d \in D$ for some predicate $P : D \tilde{\to} \{T, F\}$ expressible in the algebra $(D \tilde{\to} \{T, F\}; \vee, \wedge, \neg, \sim, \perp, P_1, P_2, ..., P_n)$. Since $n \geq 1$ and the predicate $\perp$ can be expressed as $\sim P_1 \wedge \sim\sim P_1$, we conclude that P is expressible in the algebra $APr_{P_1,...,P_n}(D)$. Then $\Phi(P_0)(d) = \Phi(P)(d)$ for all $d \in D$. Then the definition of Φ implies that $P_0 = P$, so P_0 is expressible in $APr_{P_1,...,P_n}(D)$.

"Only if": assume that a predicate P_0 is expressible in algebra $APr_{P_1,...,P_n}(D)$. Then Lemma 4 implies that $\Phi(P_0)(d) = f(\Phi(P_1)(d), \Phi(P_2)(d), ..., \Phi(P_n)(d))$ for all $d \in D$ for some function $f \in F^{(n)}$ which belongs to the functional closure of $\{f_0, f_1, f_2, f_3, f_4\}$, where f_i are defined as in Lemma 3. Then by Lemma 3, f is a short function and $p_0(d) = f(p_1(d), ..., p_n(d))$ for all $d \in D$. Then since $M^{(n)} \subseteq F^{(n)}$ is a finite set, the problem (1)-(2) has an optimal solution on the set $M^{(n)}$. Then Lemma 8 implies that the problem (1)-(2) has an optimal solution on $F^{(n)}$ which is a short function. □

Note that the problem (1)-(2) has the following addition property.

Lemma 9. *If the problem (1)-(2) has an optimal solution on $M^{(n)}$, then this solution is unique.*

Proof. Assume that f, g are optimal solutions of (1)-(2) on $M^{(n)}$. Then by Lemma 8, (1)-(2) has an optimal solution on $F^{(n)}$ which belongs to $M^{(n)}$. By Lemma 6 this solution is unique. Denote it as h. Then $||h|| \leq ||f||$ and $||h|| \leq ||g||$. Then h is an optimal solution of (1)-(2) on $M^{(n)}$ and $||h|| = ||f|| = ||g||$. Then f, g are optimal solutions of (1)-(2) on $F^{(n)}$. Then by Lemma 6, $f = g$. □

5 Example

In this example of application of the main result of the paper we will use the notation and terminology of the composition-nominative approach to program formalization [16, 17] and [7, 6, 5].

Let v be a fixed name, $V = \{v\}$, $A = \{T, F\}$.

Let $D = {}^V A$ be the set of named sets on V which take values in A. Then

$$D = \{[], [v \mapsto T], [v \mapsto F]\}.$$

Let P_1 be a partial predicate on D such that

$$P_1(d) \cong (v \Rightarrow (d))$$

where $v \Rightarrow$ is the denaming operation [16, 17] (which has undefined value, if $v \notin dom(d)$).

Let P_0 be a partial predicate on D such that

$$P_0(d) = \begin{cases} T, & \text{if } v \Rightarrow (d) \uparrow; \\ F, & \text{if } v \Rightarrow (d) \downarrow. \end{cases}$$

Let us check if P_0 is expressible in the algebra

$$APr_{P_1}(D) = (D \tilde{\to} \{T, F\}; \vee, \wedge, \neg, \sim, P_1).$$

Let $p_i : D \to \{-1, 0, 1\}$, $i = 0, 1$ be functions such that

$$p_i(d) = \begin{cases} 1, & \text{if } P_i(d) \downarrow = T, \\ 0, & \text{if } P_i(d) \uparrow, \\ -1, & \text{if } P_i(d) \downarrow = F. \end{cases}$$

Then

$$p_1(d) = \begin{cases} 1, & \text{if } v \Rightarrow (d) \downarrow = T, \\ 0, & \text{if } v \Rightarrow (d) \uparrow, \\ -1, & \text{if } v \Rightarrow (d) \downarrow = F. \end{cases}$$

$$p_0(d) = \begin{cases} -1, & \text{if } v \Rightarrow (d) \downarrow = T, \\ 1, & \text{if } v \Rightarrow (d) \uparrow, \\ -1, & \text{if } v \Rightarrow (d) \downarrow = F. \end{cases}$$

The initial topology on D induced by p_1 is the power set of D, so p_0 is continuous. We have

$$p_0(\{d \in D \mid p_1(d) = -1\}) = \{-1\}$$
$$p_0(\{d \in D \mid p_1(d) = 0\}) = \{1\}$$
$$p_0(\{d \in D \mid p_1(d) = 1\}) = \{-1\}$$

Then a function with the graph

$$\{(-1, -1), (0, 1), (1, -1)\}$$

is the unique optimal solution of the problem (1)-(2), but it is, obviously, not a short function. Then Theorem 1 implies that P_0 is not expressible in the algebra

$$APr_{P_1}(D) = (D \tilde{\to} \{T, F\}; \vee, \wedge, \neg, \sim, P_1).$$

6 Conclusion

We have investigated the question of expressibility of partial predicates in the Kleene algebra extended with the composition of predicate complement and have given a necessary and sufficient condition of this expressibility in terms of the existence of an optimal solution of a special optimization problem. The obtained results may be useful for development of (semi-)automatic deduction tools for an extension of the Floyd-Hoare logic for the case of partial pre- and postconditions.

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