Strengthening the Rational Closure for Description Logics: An Overview

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Abstract. The paper describes a preferential approach for dealing with exceptions in Description Logics, based on the rational closure. It is well known that the rational closure does not allow an independent handling of the inheritance of different defeasible properties of concepts. Several solutions have been proposed to face this problem and the lexicographic closure is the most notable one. In this work, we provide an overview of the closure constructions that have been proposed to strengthen the rational closure.

1 Introduction

The study of nonmonotonic extensions of DLs is motivated by a problem in standard ontology languages (and, specifically, in OWL Description Logics) where a class inherits the properties of its superclasses, and where the treatment of exceptions is required in many application domains, from those concerning laws and regulations (where new laws override old ones) to medical ontologies.

Many non-monotonic extensions of DLs have been developed incorporating non-monotonic features from most of the non-monotonic formalisms in the literature, from default [3, 4] and autoepistemic logics [23, 41, 49], to circumscription [12, 9] and preferential logics [33, 13, 34, 17, 16, 37, 20, 35, 36], including also the approaches based on Answer Set Programming [25, 24] and, in general, on rule languages [43, 40]. New constructions and semantics have also been developed specifically for dealing with exceptions in DLs as, for instance, the logic of overriding $\mathcal{DL}^N$ in [8, 10] and the context based CKR framework in [11]).

The landscape is very rich and the complexity of the different approaches has been studied, both for low complexity and for high complexity description logics. The case of description logic is an interesting case study for non-monotonic reasoning, which encompasses a limited treatment of first order of non-monotonic logics, namely, the treatment of the decidable fragment including only unary and binary predicates. Some undecidability results have also been found in nonmonotonic extensions, for instance, when roles are fixed in circumscriptive knowledge bases [9].

For dealing with big knowledge bases, tractable constructions are especially important. In this paper we focus on the rational closure for DLs [17, 20, 16, 37, 15] and on its refinements. The rational closure introduced by Lehmann and Magidor [45] is a polynomial construction, and it was first adapted to DLs by Casini and Straccia [17]. On the one hand, the rational closure can be computed by exploiting polynomial reductions to standard DLs [36, 48], and its construction requires a quadratic number of entailments to the underlying DL reasoner. On the other hand, it suffers from different problems, one of them being the well known problem called by Pearl [50] "the blocking of property inheritance problem". There are other problems of the rational closure which are specific to description logics. For logics including some combination of constructs, such as nominals and universal role, the rational closure of a finite knowledge base may be inconsistent. This problem is due to the fact that some DL constructs (or their combination) allow for the specification of very general constraints which are not taken into account in the rational closure construction. For this problem there are some partial solutions in [38, 22] for the low complexity logics of the $\mathcal{EL}^{\perp}$ family, while a new notion of stable rational closure has been proposed by Bonatti [7] to deal with the wider class of expressive description logics. A further problem of rational closure for DLs is that it disregards defeasible information for existential concepts, a problem which has been addressed by Pensel and Turhan [51], who developed a stronger versions of rational and relevant entailment in $\mathcal{EL}$, which considers defeasible information for quantified concepts.

In this paper we focus on the first problem: if a subclass of a class C is exceptional to C for a given aspect, it is exceptional tout court and does not inherit any of the typical properties of C. Refinements of the rational closure construction, avoiding this problem, have been studied in the literature, the first prominent one being the lexicographic closure introduced by Lehmann [46] in the context of propositional logic, later extended to the DL $\mathcal{ALC}$ by Casini and Straccia [19]. In the context of description logics, other approaches have been proposed to deal with this problem. In [20] the same authors have developed an inheritance-based approach for defeasible DLs. In [14] Casini et al. have introduced the notions of basic Relevant Closure and of minimal Relevant Closure as extensions of the rational closure, where relevance is based on the notion of justification. In [39] Gliozzi has defined a semantics for defeasible inclusions in which models are equipped with several preference relations, providing a refinement of the rational closure semantics. It has to be mentioned that other defeasible extensions of DLs still based on preferential logics but not on the rational closure, also suffer from the blocking of property inheritance problem, for instance, the typicality logic $\mathcal{ALC} + \mathbf{T}_{min}$ [35] which, differently from the rational closure, is not based on a non-ranked semantics. A multi-typicality version of this semantics has been studied in [26] to address this problem. Other approaches, such as the logic of overriding $\mathcal{DL}^N$ [8], still build on the rational closure to determine specificity of defaults, but do not suffer from this problem. In a sense, when it builds on the rational closure to determine the ranking of concepts (rather than building on the concept hierarchy), also $\mathcal{DL}^N$ can be regarded as a refinement of the rational closure.

In the following, we give an overview of the lexicographic closure and of other refinements of the rational closure, comparing their outcomes on some examples. In particular, we consider the MP-closure and the Skeptical closure, which are weaker variants of the lexicographic closure, as well as the multi-preference semantics [39], the relevant closure [14], and $\mathcal{DL}^N$. The multi-preference closure was first introduced in [30] as a sound approximation of Gliozzi's multi-preference semantics [39]. As the lexicographic closure, it builds over the rational closure but it defines a preferential, not necessarily ranked, semantics, using a different lexicographic order to compare sets of defaults.

The MP-closure construction generates a superset of the basis generated by the lexicographic closure and, therefore, entailment under the MP-closure (capturing the typicality inclusions which hold from all the MP-closure bases) is weaker than entailment under the lexicographic closure. Unfortunately, both the MP-closure and the lexicographic closure require an exponential number of possible bases to be considered. Having a single base would make reasoning about exceptions in the ontology much faster. The pattern followed by the skeptical closure [27, 28], as by the logic of overriding $\mathcal{DL}^N$ [8], is that of building a single basis which, in essence, is a subset of the intersection of all the basis (soundly approximating the MP-closure), and can be constructed in a polynomial number of steps. Entailment in the skeptical closure is neither weaker nor stronger than entailment in $\mathcal{DL}^N$. On the other hand, the multi-preference semantics, although stronger than the MP-closure (and than the skeptical closure), is incomparable with the lexicographic closure. In the following, we first introduce the rational closure and then we compare the different approaches through some examples.

2 The rational closure for $\mathcal{ALC}$

In this section we recall the extension of $\mathcal{ALC}$ with a typicality operator introduced in [34, 36] under the preferential and ranked semantics. In particular, we recall the logic $\mathcal{ALC} + \mathbf{T}_{\text{R}}$ which is at the basis of a rational closure construction proposed in [36] for $\mathcal{ALC}$. The general idea is that of extending the description logic $\mathcal{ALC}$ with concepts of the form $\mathbf{T}(C)$, whose instances are the *typical* instances of concept C, thus distinguishing between the properties that hold for all instances of concept C (given by strict inclusions $C \sqsubseteq D$), and the properties that only hold for the typical instances of C (given by the defeasible inclusions $\mathbf{T}(C) \sqsubseteq D$). The extended language is defined as follows:

$$C_R := A \mid \top \mid \bot \mid \neg C_R \mid C_R \sqcap C_R \mid C_R \sqcup C_R \mid \forall R.C_R \mid \exists R.C_R$$
$$C_L := C_R \mid \mathbf{T}(C_R),$$

where A is a concept name and R a role name. A knowledge base K is a pair $(\mathcal{T}, \mathcal{A})$, where the TBox $\mathcal{T}$ contains a finite set of concept inclusions $C_L \sqsubseteq C_R$, and the ABox $\mathcal{A}$ contains a finite set of assertions of the form $C_R(a)$ and $R(a, b)$, for a, b individual names, and R role name. Less constrained languages have been considered, in which the typicality operator may also occur on the right hand side of inclusions, for instance, in extensions with typicality of the very expressive logic $\mathcal{SROIQ}$ [36] and of the low complexity logic $\mathcal{SROEL}(\sqcap, \times)$ [38]

For simplicity, however, here we restrict our consideration to inclusions, which are either strict inclusions, or typicality inclusions of the form $\mathbf{T}(C) \sqsubseteq D$ (where C and D are $\mathcal{ALC}$ concepts), which in essence correspond to KLM defeasible inclusions $C \mid\sim D$.

The semantics of $\mathcal{ALC}$ with typicality is defined in terms of preferential models, extending to $\mathcal{ALC}$ the preferential semantics by Kraus, Lehmann and Magidor in [44, 45]: ordinary models of $\mathcal{ALC}$ are equipped with a *preference relation* $<$ on the domain, whose intuitive meaning is to compare the "typicality" of domain elements: $x < y$ means that x is more typical than y. The instances of $\mathbf{T}(C)$ are the instances of concept C that are minimal with respect to $<$. $<$ is further assumed to be *well-founded*[3] (i.e., there is no infinite $<$-descending chain, so that, if $S \neq \emptyset$, also $min_<(S) \neq \emptyset$) and, in ranked models, which characterize $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, $<$ is also assumed to be *modular* (i.e., for all $x, y, z \in \Delta$, if $x < y$ then either $x < z$ or $z < y$). Let us shortly recap the definition of preferential and ranked models of a finite DL knowledge base $K = (\mathcal{T}, \mathcal{A})$.

Definition 1 (Preferential and ranked interpretations of $\mathcal{ALC} + \mathbf{T}$). *A preferential interpretation $\mathcal{M}$ is any structure $\mathcal{M} = \langle \Delta, <, I \rangle$ where: Δ is the domain; $<$ is an irreflexive, transitive and well-founded relation over Δ. I is an interpretation function that maps each concept name $C \in N_C$ to $C^I \subseteq \Delta$, each role name $R \in N_R$ to $R^I \subseteq \Delta^I \times \Delta^I$ and each individual name $a \in N_I$ to $a^I \in \Delta$. For concepts of $\mathcal{ALC}$, C^I is defined in the usual way in $\mathcal{ALC}$ interpetations [2]. In particular:* $\top^I = \Delta$, $\bot^I = \emptyset$, $(\neg C)^I = \Delta \backslash C^I$, $(C \sqcap D)^I = C^I \cap D^I$, $(C \sqcup D)^I = C^I \cup D^I$ *and*

$$(\forall R.C)^I = \{x \in \Delta \mid \text{for all } y \in \Delta,\ (x, y) \in R^I \text{ implies } y \in C^I\}$$
$$(\exists R.C)^I = \{x \in \Delta \mid \text{there is a } y \in \Delta \text{ such that } (x, y) \in R^I \text{ and } y \in C^I\}$$

For the $\mathbf{T}$ operator, we have $(\mathbf{T}(C))^I = min_<(C^I)$.

When the relation $<$ is modular*, I is called a* ranked *interpretation.*

The notion of satisfiability of a KB in an interpretation is defined as usual. Given an $\mathcal{ALC}$ interpretation $\mathcal{M} = \langle \Delta, <, I \rangle$:
- I satisfies an inclusion $C \sqsubseteq D$ if $C^I \subseteq D^I$;
- I satisfies an assertion $C(a)$ if $a^I \in C^I$;
- I satisfies an assertion $R(a, b)$ if $(a^I, b^I) \in R^I$.

Definition 2 (Model of a KB [34]). *A preferential (ranked) model of a knowledge base $K = (\mathcal{T}, \mathcal{A})$ is a preferential (ranked) interpretation $\mathcal{M}$ that satisfies all inclusions in $\mathcal{T}$ and all assertions in $\mathcal{A}$.*

A *query* F (either an assertion $C_L(a)$ or an inclusion relation $C_L \sqsubseteq C_R$) is preferentially (rationally) entailed by a knowledge base K, written $K \models_{\mathcal{ALC}+\mathbf{T}} F$ (resp., $K \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} F$) if F is satisfied in all the models (resp., ranked models) of K.

[3] Since $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ has the finite model property, this is equivalent to having the Smoothness Condition, as shown in [36]. We choose this formulation because it is simpler.

In particular, the definition of the rational closure for $\mathcal{ALC}$ and its semantics in [37, 36] extends the definition introduced by Lehmann and Magidor [45] to the language $\mathcal{ALC} + \mathbf{T}_{\textsf{R}}$ of $\mathcal{ALC}$ plus typicality. Roughly speaking $\mathbf{T}(C) \sqsubseteq D$ holds in the rational closure of K if C is less exceptional than $C \sqcap \neg D$. We shortly recall this construction of the rational closure of a TBox and we refer to [36] for full details.

Definition 3 (Exceptionality of concepts and inclusions). *Let E be a TBox and C a concept. C is* exceptional *for E if and only if $E \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg C$. An inclusion $\mathbf{T}(C) \sqsubseteq D$ is exceptional for E if C is exceptional for E. The set of inclusions which are exceptional for E will be denoted by $\mathcal{E}(E)$.*

Given a TBox $\mathcal{T}$, it is possible to define a sequence of non increasing subsets of the TBox $\mathcal{T}$ ordered according to the exceptionality of the elements $E_0 \supseteq E_1 \supseteq E_2 \dots$ by letting $E_0 = \mathcal{T}$ and, for $i > 0$, $E_i = \mathcal{E}(E_{i-1}) \cup \{C \sqsubseteq D \in \mathcal{T}$ s.t. $\mathbf{T}$ does not occurr in $C\}$. Observe that, being knowledge base finite, there is an $n \geq 0$ such that, for all $m > n, E_m = E_n$ or $E_m = \emptyset$. A concept C has *rank* i (denoted $\mathit{rank}(C) = i$) for TBox, iff i is the least natural number for which C is not exceptional for E_i. If C is exceptional for all E_i then $\mathit{rank}(C) = \infty$ (C has no rank). The rank of a typicality inclusion $\mathbf{T}(C) \sqsubseteq D$ is $\mathit{rank}(C)$. The intuition is that, for $i < j$, E_i contains less specific defeasible properties then E_j. Consider the following example.

Example 1. Let K be the knowledge base with TBox:

1. $\mathbf{T}(\mathit{Student}) \sqsubseteq \neg\exists \mathit{has\_paid}.\mathit{Tax}$
2. $\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Young}$
3. $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \exists \mathit{has\_paid}.\mathit{Tax}$
4. $\mathit{EStudent} \sqsubseteq \mathit{Student}$

stating that typical students have not paid taxes, but typical employed students (which are students) have paid some taxes and that typical students are young. It is possible to see that

$E_0 = \mathcal{T}$

$E_1 = \{\mathbf{T}(\mathit{EStudent}) \sqsubseteq \exists \mathit{has\_paid}.\mathit{Tax},\ \mathit{EStudent} \sqsubseteq \mathit{Student}\}$.

The rank of concept *Student* is 0, as *Student* is non-exceptional for E_0, while concept *EStudent* has rank 1, it is exceptional w.r.t. the property that students typically are not taxpayers. The properties of *EStudent* are more specific than those of Students.

Rational closure builds on this notion of exceptionality:

Definition 4 (Rational closure of TBox). *Let $K = (\mathcal{T}, \mathcal{A})$ be a DL knowledge base. The rational closure of TBox is defined as:*

$$\begin{aligned} RC(\mathcal{T}) = &\{\mathbf{T}(C) \sqsubseteq D \in \mathcal{T} \mid \ \textit{either} \ \ \mathit{rank}(C) < \mathit{rank}(C \sqcap \neg D) \ \textit{or} \\ &\mathit{rank}(C) = \infty\} \ \cup \ \{C \sqsubseteq D \in \mathcal{T} \mid \ \ KB \ \ \models_{\mathcal{ALC}+\mathbf{T}_R} C \sqsubseteq D\} \end{aligned}$$

where C and D are $\mathcal{ALC}$ concepts.

In [36] it is shown that deciding if an inclusion $\mathbf{T}(C) \sqsubseteq D$ belongs to the rational closure of TBox is a problem in ExpTime and that the semantics corresponding to rational closure can be given in terms of *minimal canonical* $\mathcal{ALC} + \mathbf{T}_{\mathrm{R}}$ models. In such models the rank of domain elements is minimized to make each domain element as typical as possible. Furthermore, canonical models are considered in which all possible combinations of concepts are represented. We refer to [36] for a description.

In Example 1, the typicality inclusion $\mathbf{T}(\mathit{Student} \sqcap \mathit{Italian}) \sqsubseteq \neg\exists \mathit{has\_paid}.\mathit{Tax}$ belongs to the rational closure of the TBox, as $\mathit{rank}(\mathit{Student} \sqcap \mathit{Italian}) = 0 < \mathit{rank}(\ \mathit{Student} \sqcap \mathit{Italian} \sqcap \exists \mathit{has\_paid}.\mathit{Tax}) = 1$. Similarly, $\mathbf{T}(\mathit{EStudent} \sqcap \mathit{Italian}) \sqsubseteq \exists \mathit{has\_paid}.\mathit{Tax}$ belongs to the rational closure.

However, the inclusion $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Young}$ does not belong to the rational closure. Indeed, the concept *EStudent* is exceptional for E_0, as it violates the defeasible property of students that, normally, they have not paid taxes ($\mathbf{T}(\mathit{Student}) \sqsubseteq \neg\exists \mathit{has\_paid}.\ \mathit{Tax}$). For this reason, *EStudent* does not inherit "any" of the defeasible properties of *Student*. Indeed, in a language allowing more liberal occurrences of the typicality operator, as the one in [38] one could have actually inferred that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \neg\mathbf{T}(\mathit{Student})$ using preferential (or rational) entailment.

3 Refinements of the rational closure: a comparison

To overcome the weakness of the rational closure, Lehmann introduced the notion of lexicographic closure [46], that has been extended to the description logic $\mathcal{ALC}$ by Casini and Straccia in [19], which strengthens the rational closure by allowing, roughly speaking, a class to inherit as many as possible of the defeasible properties of more general classes, giving preference to the more specific properties. In the example above, the defeasible property of students of being young should be inherited by employed students, as it is consistent with all other (strict and defeasible) properties of employed students and, by "presumption of independence" [46], even if typicality is lost with respect to one consequent ($\exists \mathit{has\_paid}.\mathit{Tax}$) we may still presume typicality of *EStudent* with respect to the property of being *Young*.

In Example 1, the set $D = \{1, 2\}$ of defeasible properties (defaults) from the TBox forms a *basis* for *EStudent*. Using Lehmann terminology, a *basis* D *for* A is a maximally serious set of defaults whose material counterpart $\tilde{D}$ is consistent with A. In our example, following the definition for $\mathcal{ALC}$ in [19], the materialization of D, $\tilde{D} = \{\mathit{Student} \sqsubseteq \mathit{Young},\ \ \mathit{Employee} \sqsubseteq \exists \mathit{has\_paid}.\mathit{Tax}\}$, is consistent with *EStudent* and with the strict part of TBox $\Phi = \{\mathit{Student} \sqsubseteq \mathit{Young}\}$ (i.e., $\Phi \cup \tilde{D} \not\models \neg \mathit{EStudent}$). Since D is the unique set of defaults consistent with *EStudent* and Φ, D is maximally serious, and is the unique basis for *EStudent*. *Young* holds in the basis D, as *Young* follows from $\mathit{EStudent} \cup \Phi \cup \tilde{D}$. Thus, *Young* holds in all the basis for *EStudent*, and (using our notation) the defeasible inclusion $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Young}$ belongs to the lexicographic closure of the KB.

The notion of seriousness comes into play when there are alternative sets D of consistent defaults. Seriousness of D depends on the ranking of defeasible inclusions in the rational closure. In order to compare alternative sets of defaults, in [46] a *seriousness ordering* $\prec$ among sets of defaults is defined by associating with each set of defaults $D \subseteq K$ a tuple of numbers $\langle n_0, n_1, \ldots, n_k \rangle_D$, where n_0 is the number of defaults in D with rank ∞ and, for $1 \leq i \leq k$, n_i is the number of defaults in D with rank $k-i$ (and there is not default with rank k or greater). A modular order $\prec$ among sets of defaults is obtained from the natural lexicographic order over the tuples. This order gives preference to those bases containing more specific defaults (the highest is the rank, the more specific is the default). This is essentially the ordering used in [19] (but for that fact that hidden non-defeasible knowledge is first moved to the strict part of the KB).

Let us consider the following variant of Example 1.

Example 2. Let K' be a knowledge base with TBox:

1. $\mathbf{T}(\mathit{Student}) \sqsubseteq \neg \exists \mathit{has\_paid}.\mathit{Tax}$
2. $\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Young}$
3. $\mathbf{T}(\mathit{Employee}) \sqsubseteq \neg\, \mathit{Young}$
4. $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \exists \mathit{has\_paid}.\mathit{Tax}$
5. $\mathit{EStudent} \sqsubseteq \mathit{Student} \sqcap \mathit{Employee}$

In the rational closure of K', concepts *Student* and *Employee* have rank 0 and concept *EmployedStudent* has rank 1. Default 4 has rank 1 and is more specific than defaults 1, 2 and 3 having rank 0. We want to derive the properties of typical employed students. Clearly, default 4 holds for *EStudent*, as its rank is 1. The property that typical employed students are taxpayers (default 4), overrides the property that students are typically not taxpayers (default 1). Instead, the two defaults 2 and 3 are each one compatible with with the property that employed students have paid taxes (default 4). In the lexicographic closure there are two alternative bases (sets of defaults) for *EStudent*, namely, $\mathcal{D} = \{2, 4\}$ and $\mathcal{E} = \{3, 4\}$, which are not comparable (no one is more serious than the other), while the strict inclusion 5 holds for all bases. In fact, the corresponding tuples $\langle 0, 1, 1 \rangle_D$ ($\mathcal{D}$ contains no default with rank ∞, 1 default with rank 1 and 1 default with rank 0) and, similarly, $\langle 0, 1, 1 \rangle_E$. As the two tuples are incomparable, neither $\mathcal{D} \prec \mathcal{E}$ nor $\mathcal{E} \prec \mathcal{D}$. As *Young* does not hold from basis $\mathcal{E}$, in the lexicographic closure one cannot conclude that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Young}$. Similarly, one cannot conclude that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \neg\, \mathit{Young}$.

Concerning Examples 1 and 2 above, similar results can be obtained when reasoning with the MP-closure, with the Skeptical closure and with the logic $\mathcal{DL}^N$. None of them can conclude that typical employed students are young. In Example 2 the logic of overriding $\mathcal{DL}^N$ finds out that there is a conflict between the defaults 2 and 3, none of which is overridden by more specific properties. In this case the prototype of concept *EStudent* is said to be inconsistent.

In Example 2 the MP-closure and the Skeptical closure, as the lexicographic closure, silently eliminate the conflict and neither infer that typical employed students are young, nor that they are not young. In particular, the MP-closure is a variant of the lexicographic closure which has been studied in [29], and exploits a different lexicographic ordering with respect to the one considered by the lexicographic closure. It compares two sets of defaults $\mathcal{D}$ and $\mathcal{E}$ by considering the tuples of the sets of defaults with different ranks rather than the tuples of their cardinality. The natural lexicographic order to compare such tuples of sets exploits strict subset inclusion, and the resulting seriousness ordering among sets of defaults is a strict partial order, but is not modular in general.

In Example 1, there is a single basis in the MP-closure as in the lexicographic closure, as there is a single set of defaults consistent with *EStudent* (and with the strict inclusions). In Example 2, the two sets $\mathcal{D} = \{2, 4\}$ and $\mathcal{E} = \{3, 4\}$ are incomparable, as neither $\langle \emptyset \{4\}, \{2\} \rangle_{\mathcal{D}}$ is less serious than $\langle \emptyset, \{4\}, \{3\} \rangle_{\mathcal{E}}$, nor vice-versa. Hence $\mathcal{D}$ and $\mathcal{E}$ are both bases of the MP-closure and, as in the lexicographic closure, we can neither conclude that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Young}$ nor that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \neg \mathit{Young}$.

In this same example, the minimal relevant closure [14] would neither conclude that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Young}$ nor that $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \neg \mathit{Young}$. There are two *EStudent*-justifications w.r.t. K', namely $J^1 = \{1, 4\}$ and $J^2 = \{2, 3\}$. Hence, $J^1_{min} = \{1\}$ and $J^2_{min} = \{2, 3\}$. In particular, both defaults 2 and 3 are relevant for determining the subsumption $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Young}$, and both of them are eligible for removal.[4]

Example 3. Consider the case when the TBox in Example 2 is extended with an additional defeasible inclusion

6. $\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Bright}$.

In this case, in both the lexicographic closure and the MP-closure, there are two bases for concept *EStudent*, $\mathcal{D}' = \{2, 4, 6\}$ and $\mathcal{E}' = \{3, 4, 6\}$, and default 6 belongs to both them. Hence, in both the lexicographic closure and the MP-closure we would conclude that typical employed students are bright, i.e. $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Bright}$. This conclusion would not be obtained in $\mathcal{DL}^N$, where the prototype of concept *EStudent* is inconsistent, as before. In the Skeptical closure, the defaults with rank 0 in the two bases, namely $D'_0 = \{2, 6\}$ and $E'_0 = \{3, 6\}$ are conflicting. Hence they are all discarded when constructing the single basis for *EStudent*, and $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Bright}$ cannot be concluded as well.

With the approach in [14], $\mathbf{T}(\mathit{EStudent}) \sqsubseteq \mathit{Bright}$ would belong to the minimal relevant closure of the KB, as the defeasible subsumption 6 is not potentially relevant for resolving the conflict among the other defaults for concept *EStudent*.

To see the difference between the MP-closure and the lexicographic closure, consider the following variant of Example 2:

Example 4. Let the TBox be:

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[4] Using the notation in [14], defeasible subsumptions would be written $C \precsim D$ rather than $\mathbf{T}(C) \sqsubseteq D$, but they rely on the same ranked semantics.

1. $\mathbf{T}(Student) \sqsubseteq \neg\exists has\_paid.Tax$
2. $\mathbf{T}(Student) \sqsubseteq Young$
3. $\mathbf{T}(Employee) \sqsubseteq \neg Young \wedge \exists has\_paid.Tax$
4. $\mathbf{T}(EStudent) \sqsubseteq Graduate\_in\_4Years$
5. $EStudent \sqsubseteq Student \sqcap Employee$

Now, defaults 1, 2 and 3 have rank 0 in the rational closure, while default 4 has rank 1. We have still two alternative bases in the MP-closure, $\{1, 2, 4\}$ and $\{3, 4\}$, but there is a single base $\{1, 2, 4\}$ in the lexicographic closure. The reason is that in the lexicographic order, the set $\mathcal{D} = \{1, 2, 4\}$ (with the associated tuple $\langle 0, 1, 2\rangle_{\mathcal{D}}$) is more serious than $\mathcal{E} = \{3, 4\}$ (with the associated tuple $\langle 0, 1, 1\rangle_{\mathcal{E}}$) as both bases contain 1 default with rank 1, but $\mathcal{D}$ contains two defaults with rank 0, while $\mathcal{E}$ contains just one. From the lexicographic closure one can then conclude that $\mathbf{T}(EStudent) \sqsubseteq Young \wedge \neg\exists has\_paid.\ Tax$.

In the MP-closure, instead, the two bases are not comparable as none of the tuples of sets of defaults $\langle \emptyset, \{4\}, \{1, 2\}\rangle$ and $\langle \emptyset, \{4\}, \{3\}\rangle$ is more serious than the other one, as the two sets of defaults with rank 0, $\{1, 2\}$ and $\{3\}$, are not comparable using subset inclusion.

In this example, the basic relevant closure and the minimal relevant closure would not be able to conclude that $\mathbf{T}(EStudent) \sqsubseteq \neg Young$, as well. In fact, there are two *EStudent*-justifications w.r.t. the KB above namely $J^1 = \{1, 3\}$ and $J^2 = \{2, 3\}$, both of them containing only defaults with rank 0. Hence, $J^1_{min} = J^1$ and $J^2_{min} = J^2$. In particular, defaults 1, 2 and 3 are all relevant for determining the subsumption $\mathbf{T}(EStudent) \sqsubseteq \neg Young$, and all eligible for removal.

In this last example the lexicographic closure appears to be too bold, as the reason to accept that typical employed students are not young and have paid taxes may be disputable. Notice also that if we replace default 3 with two defaults

$\mathbf{T}(Employee) \sqsubseteq \neg Young$
$\mathbf{T}(Employee) \sqsubseteq \exists has\_paid.Tax$

there would be two bases in the lexicographic closure, and one would not conclude any more that typical employed students are young and have not paid taxes. The MP-closure and the relevant closure are more cautious (and less syntax dependent in this example). In particular, both formulations of Example 4 there are two bases in the MP-closure and nothing can be concluded about typical employed students being young or paying taxes. Similarly, in the relevant closure.

In the Skeptical closure, the set of defaults with rank 0 which are not overridden by more specific defaults is $\{1, 2, 3\}$, but they are altogether incompatible with *EStudent* and with the strict inclusion 5. Hence, nothing can be concluded about typical employed students except that they graduate in 4 years. In $\mathcal{DL}^N$ the prototype of concept *EStudent* is found to be inconsistent, as there are conflicting non overridden defaults 1, 2 and 3.

4 Conclusions and related work

In this paper we have compared the behavior of some refinements of the rational closure, namely the Lexicographic closure [46, 19], the Relevant closure [14], the MP-closure [29] and the Skeptical [28] closure, through some examples. We have also considered for comparison the logic $\mathcal{DL}^N$, proposed by Bonatti et al. in [8], which captures a form of "inheritance with overriding": a defeasible inclusion is inherited by a more specific class if it is not overridden by more specific (conflicting) properties. The logic $\mathcal{DL}^N$ is not necessarily applied starting from the ranking given by the rational closure but, when it does, it provides another approach to deal with the problem of inheritance blocking in the rational closure.

The multi-preference closure (or MP-closure) was first introduced in [30] as a sound approximation of Gliozzi's multi-preference semantics [39]. As the lexicographic closure, it builds over the rational closure but it defines a preferential, not necessarily ranked, semantics, using a different lexicographic order to compare sets of defaults. A semantic characterization of the MP-closure for the description logic $\mathcal{ALC}$ was developed in [29] using bi-preferential (BP) interpretations, preferential interpretations developed along the lines of the preferential semantics introduced by Kraus, Lehmann and Magidor [44, 45], but containing two preference relations, the first one $<_1$ playing the role of the ranked preference relations in the models of the RC, and the second one $<_2$ representing a preferential refinement of $<_1$. The skeptical closure [28] was shown to be a weaker variant of the MP-closure in [29].

The relevant closure [14] is based on the idea of relevance of subsumptions to a query, to overcome the limitation of the weakness of rational closure. Relevance is determined based on justifications and, in minimal relevant closure, the idea is that subsumptions with lower ranks are removed first. It was shown by Casini et al. [14] that the relevant closure is a weaker closure than the lexicographic closure. In [31] it was proved that, in the propositional case, the basic and the minimal relevant closure are weaker than the MP-closure, which, in turn, is weaker than the lexicographic closure.

Another refinement of the rational closure, which also deals with this limitation of the rational closure, is the inheritance-based rational closure in [18, 20], a closure construction which is defined by combining the rational closure with defeasible inheritance networks.

Fernandez Gil in [26] developed a multi-typicality version of the typicality logic $\mathcal{ALC} + \mathbf{T}_{min}$ [35], another defeasible description logic based on preferential extension of $\mathcal{ALC}$ with typicality, which, differently from the rational closure, is not based on the ranked models semantics but, nevertheless, suffers from the blocking of property inheritance problem. [26] provides a solution to this problem for $\mathcal{ALC} + \mathbf{T}_{min}$, considering a language with multiple typicality operators.

Other approaches in the literature deal with the problem of inheritance with exceptions. For instance, Bozzato et al. in [11] present an extension of the CKR framework in which defeasible axioms are allowed in the global context and can be overridden by knowledge in a local context. Exceptions have to be justified in terms of semantic consequence. A translation of extended CHRs (with knowledge bases in $\mathcal{SROIQ}$-RL) into Datalog programs under the answer set semantics is also defined.

Concerning the multipreference semantics introduced in [39] to provide a semantic strengthening of the rational closure, we have proved in [30] that the MP-closure provides a sound approximation of such a semantics. As a consequence of the fact that the skeptical closure is weaker than the MP-closure, the skeptical closure is still a sound (a weaker) approximation for the multipreference semantics.

The relationships among the above variants of rational closure for DLs and the notions of rational closure for DLs developed in the contexts of fuzzy logic [21] and probabilistic logics [47] are also worth being investigated. In the propositional logic case, it has been shown in [5] that KLM preferential logics and the rational closure [44, 45], the probabilistic approach [1], the system Z [50], the possibilistic approach [6, 5] are all related with each other, and another related approach is that of c-representations [42]. Similar relations might be expected to hold among the non-monotonic extensions of description logics as well.

Although the skeptical closure has been defined based on the preferential extension of $\mathcal{ALC}$, the same construction could be adopted for more expressive description logics, provided that the rational closure can be consistently defined for the knowledge base under consideration [32]. Indeed, for a KB in an expressive DL, the consistency of rational closure is not guaranteed and, from the semantic point of view, the knowledge base might have no canonical models. Partial solutions to this problem have been considered in [38, 22] for the low complexity logics in the $\mathcal{EL}$ family, and a notion of *stable rational closure* is proposed in [7] to extend the rational closure to a wider class of expressive description logics, which do not satisfy the disjoint model union property.

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