Online Firefighting on Grids*

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Abstract. The Firefighter Problem (FP) is a graph problem originally introduced in 1995 to model the spread of a fire in a graph, which has attracted considerable attention in the literature. The goal is to devise a strategy to employ a given sequence of firefighters on strategic points in the graph in order to contain efficiently the fire (which spreads from each unprotected vertex to all of it neighbours on successive time steps).
Recently, an *online* version of FP— where the number of firefighters available at each turn are revealed in *real-time*— has been introduced in [3, 4] and studied on trees. In this paper, we extend the work in [3, 4] by considering the online containment of fire on square grids. In particular, we provide a set of sufficient conditions that allow to solve the online version of the firefighting problem on infinite square grids, illustrating the corresponding fire containment strategies.

1 Introduction

The Firefighter Problem (FP, from now on) is a combinatorial problem introduced by Bert Hartnell in 1995 [10], providing a deterministic, discrete-time model of the spread of a fire on the vertices of a graph. Suppose that a fire breaks out at time 0 at a vertex v of a graph G. At each subsequent time t, f_t firefighters protect a corresponding number of f_t vertices in G, and then the fire spreads from each burning vertex to all of its undefended neighbours. Once a vertex is burning or defended, it remains so from then onwards. The process terminates when the fire can no longer spread. In the case of finite graphs, the aim is to save as many vertices as possible, while in the infinite case, the aim is that of simply containing the fire.

Since its introduction in [10], FP has been studied intensively in the literature [1, 2, 6, 7, 9]. In particular FP has been shown NP-complete for bipartite graphs [10]. Finbow et al. have strengthened this result [6], proving that FP is NP-complete even if restricted to trees with maximum degree three. In contrast, it is solvable in polynomial time for graphs of maximum degree three, if the fire starts at a vertex of degree two. A polynomial time approximation scheme for FP

* The support of GEO-SAFE, H2020-MSCA-RISE-2015 project # 691161 is fully acknowledged.

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on trees has been recently provided in [1]. Beside trees, FP has been extensively studied on the families of graphs of grids [8, 11–14]. Wang and Moeller [14] proved that one firefighter per turn cannot control a fire sourcing from a vertex $v \in \mathbb{L}_2 = \mathbb{Z} \times \mathbb{Z}$, while two firefighters per turn are sufficient to solve FP on $\mathbb{L}_2$ within 8 turns and 18 burnt vertices. Ng and Raff [13] proved that any periodic function $(f_t)_{t\geq 1}$ whose average exceeds $\frac{3}{2}$ allows the firefighters to control any finite-source fire in $\mathbb{L}_2$. The interested reader can refer to [7–9] for a survey of recent results on the complexity analysis of FP and its variants.

Recently, Coupechoux et al. [3, 4] have considered an *online* version of FP, where the number of firefighters available at each turn are revealed in *real-time*. In [3, 4], the structure of the underlying graph in the online FP is a tree, and suitable competitivity results are provided. In this work, we consider the online version of FP on the infinite Cartesian grid $\mathbb{L}_2 = \mathbb{Z} \times \mathbb{Z}$.

2 Offline vs Online Firefighting on Grids

The classic offline version of the firefighter problem can be understood as a deterministic one-player game, where Player 1 knows in advance the sequence $(f_i)_{i\geq 1}$ of firefighters available overall the game. In contrast, within the online version of the firefighter problem, an adversary called Player 2 reveals to Player 1—turn by turn—how many firefighters are ready to be used in the current turn of the game. Therefore, the online firefighter problem can be understood as a two player game. More precisely, an instance of the online firefighter problem is given by the tuple $(\mathcal{A}, (f_i)_{i\geq 1})$, where both players are aware of the arena of the game, $\mathcal{A} = \langle G, v\rangle$, which is composed by the graph G- the Cartesian grid $\mathbb{L}_2 = \mathbb{Z} \times \mathbb{Z}$ in this paper- and the ignition vertex $v \in \mathbb{L}_2$. Instead, only Player 2 knows the firefighters sequence $(f_i)_{i\geq 1}$. At each turn i of the game, Player 2 reveals f_i to Player 1. Then, Player 1 chooses $m \leq f_i$ vertices (neither protected nor burned), where to place a new firefighter. Finally, the fire spreads on each unprotected neighbour of a vertex on fire, leading to the next turn of the game. Player 2 wins if at each turn a new vertex is burning, otherwise Player 1 wins.

We provide a constraint applying to the sequence of firefighters $(f_i)_{i\geq 1}$ (cf. Condition (1)) that can be shown to be a sufficient condition for Player 1 to win the offline version of the firefighter problem, while there is an instance of the online firefighter problem that fulfills Condition (1) where Player 1 looses.

$$\exists N \geq 1 : \sum_{i=1}^{N} f_i \geq 4N \tag{1}$$

Intuitively, Condition (1) guarantees the existence of a turn of the game for which the *global* number of firefighters deployed (from the beginning of the game up to the current turn) have been 4 times the number of turns played so far. Hence, an offline strategy for Player 1 can rely on the knowledge of $(f_i)_{i\geq 1}$ to surround the fire at the right distance. However, in the online version of the game, Player 1 does not have any information on the turn of the game fulfilling Condition (1). This is formalized by Theorem 1 (cf. [5] for a complete proof).

Theorem 1. *Condition* (1) *is sufficient (resp. not sufficient) for Player* 1 *to win the offline (resp. online) version of the firefighter problem. Moreover,* 5 *turns are enough to make any online strategy for Player* 1 *to fail.*

3 Online Containment of Fires on Grids

In the previous section, we have considered sequences of firefighters $(f_i)_{i\geq 1}$ satisfying Condition (1), showing that such a condition is sufficient for Player 1 to win offline, while there are instances of the online firefighter problem fulfilling Condition(1) where Player 1 loses. The purpose of this section is that of providing sufficient conditions toward the *online containment* of fires on grids. Our first result (cf. Subsection 3.1 below) shows that if $(f_i)_{i\geq 1}$ fulfils Condition (1) and at least one firefighter is eventually always available, then Player 1 has a strategy to win online. The following Subsection 3.2 considers the problem of weakening Condition (1) in order to define further sufficient conditions for the online containment of fires on grids.

3.1 At Least One Firefighter Always Available

Suppose that the sequence of firefighters revealed by Player 2 is such that eventually at least one firefighter will be available on each turn of the game. Then, we show that Condition (1) becomes sufficient for Player 1 to win online. Intuitively, this is because the firefighter(s) available at each turn can be employed *next* to the fire to build incrementally a tight encirclement of it, waiting for later reinforcement. This way, no firefighter is wasted during subsequent turns of the game, while Condition (1) guarantees that eventually, there will be enough firefighters to close the encirclement surrounding the fire.

More precisely, consider the simpler scenario where Player 1 receives *exactly* one firefighter at each turn $1, 2, \ldots, \mu - 1$ and m firefighters at turn μ, where $m \geq 4\mu - (\mu - 1)$. For instance, Figure 1 considers the case where $\mu = 4$, $f_1 = f_2 = f_3 = 1$ and $f_4 = 13$ and illustrates an online winning strategy for Player 1. Such a strategy works as follows: Player 1 uses the available firefighter at each turn to build two diagonal walls (cf. the positioning of the only firefighter received for the first three turns). This way, at the beginning of each turn $1 \leq i \leq \mu$, the fire is always contained within a perimeter of size $4i$, and each firefighter previously employed at some turn $j < i$ has been placed on such a perimeter. Therefore, the encirclement of the fire can be completed as soon as Condition (1) is fulfilled (at turn 4, for the instance illustrated in Figure 1).

The general case, where Player 1 eventually receives *at least* one firefighter on each turn until Condition (1) is accomplished, is slightly more in involved. Roughly, it is solved as follows. As far as Player 1 receives $f_j > 1$ firefighters at each turn j (while Condition (1) still needs to be accomplished) he will place the guaranteed available firefighter on a diagonal wall in front of the advance of the fire, while the extra firefighters (out from the guaranteed one) will be employed to encircle the fire, waiting for later reinforcement. As soon as Condition (1)

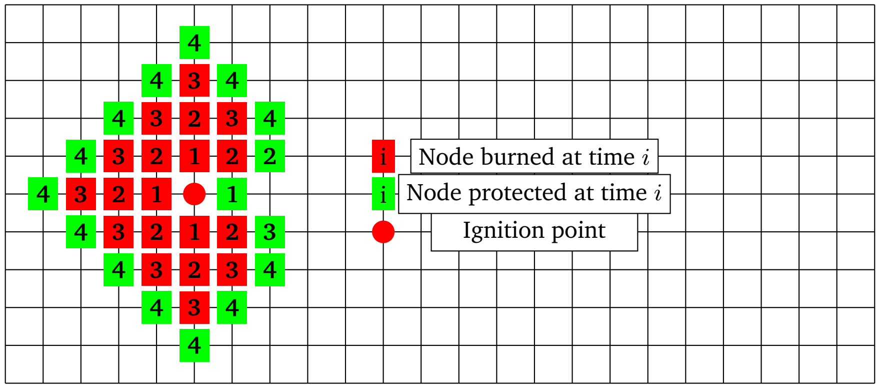

Fig. 1. Online strategy to encircle the fire with (exactly) one firefighter available at each turn of the game $1, \ldots, N-1$, until Condition (1) gets fulfilled at turn N.

is satisfied, the encirclement will be completed. Example 1 below gives more details on the above sketched winning online strategy for Player 2. This leads to the results in Theorem 2, below (cf. [5] for a complete proof).

Theorem 2. *Let the sequence of firefighters* $(f_i)_{i\geq 1}$ *revealed by Player* 2 *be consistent with Condition* (1) *and suppose that there exists an index* M *such that* $f_i \geq 1$ *if* $i \geq M$*, and* $f_i = 0$ *otherwise. Then, Player* 1 *has an online strategy to win the firefighter problem on grids.*

Example 1. Let the sequence of firefighters $(f_i)_{i\geq 1}$ received by Player 1 be such that $f_1 = f_2 = 1$, $f_3 = 4$, $f_4 = f_5 = f_6 = 1$, and $f_7 = 15$. Therefore, Player 1 receives one *or more* firefighters for the first six turns and a number of firefighters leading to the fulfillment of Condition (1) on the seventh turn, i.e. $\sum_{i=1}^{7} f_i \geq 28$.

As illustrated in Figure 2, to win online, Player 1 proceeds as follows. As far as he receives exactly one firefighter per turn, he employs them to build two diagonal walls (cf. the positioning of the first two firefighters) against the front of the fire. In our example, this happens for the first two turns. At the third turn Player 1 receives 4 firefighters, i.e. a number of firefighter that is strictly greater than one but not enough to let Condition (1) being satisfied. Then, Player 1 uses one firefighter on the diagonal walls and the extra three firefighters to start surrounding the fire (cfr the positioning of the firefighters received at turn 3 in Figure 2). At the next turn-i.e. at turn 4 in our example - Player 1 will start building a new diagonal wall from the last enqueued firefighter, so that he always have two diagonal walls under construction, orthogonal to two advancing fronts of fire. Such diagonal walls will be alternatively enlarged (at turn 5 and 6 in our example), until Player 1 will be able to completely surround the fire (at turn 7 in our example), when the available firefighters will allow to have Condition (1) satisfied.

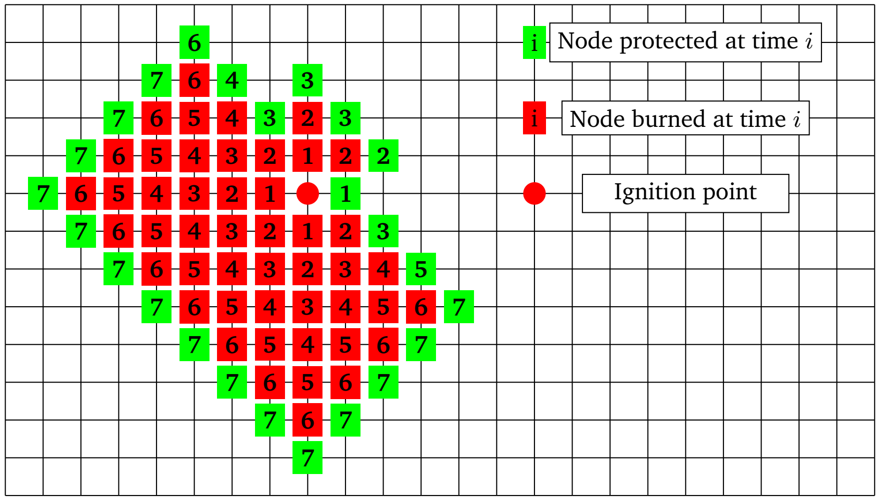

Fig. 2. Winning the fire online with at least one firefighter eventually always available until Condition (1) is satisfied.

3.2 Sufficient Conditions to Win the Fire Online on Grids

Consider the following natural generalisation of Condition (1):

$$\exists N \geq 1 : \sum_{i=1}^{N} f_i \geq \ell \cdot N \tag{2}$$

We study for which $\ell \in \mathbb{N}$ Condition (2) guarantees to contain the fire online. In the offline case, $\ell = 4$ is sufficient to contain the fire as stated in Theorem 1. As already noted, such a strategy does not work online, since Player 1 is not aware of how many firefighters will be available at each turn of the game. Therefore, the attempt of building a diamond-shaped encirclement at distance N could result into a waste of all the firefighters placed on such an encirclement, as soon as the latter gets broken by the spreading fire. However, in this case a maximum number of $4N$ firefighters are lost. Based on this observation, it is possible to come up with an online strategy for Player 1 that allows him to contain the fire under Condition (2) for $\ell = 16$. Such a strategy proceeds as follows: as soon as the encirclement under construction gets broken at turn t of the game, Player 1 starts to build a new diamond-shaped encirclement at distance $2t$ from the ignition point. Let M be the turn of the game at which Condition (2) is satisfied for $\ell = 16$, i.e. assume that a global number of $16M$ firefighters gets available overall the first M turns. We show that at turn M, we are guaranteed to have enough firefighters to complete an encirclement at distance $2M$. In fact, $4 * 2M = 8M$ firefighters are needed for such an encirclement, while the number of firefighters lost in previous turns is bounded by $4M * \sum_{i=0}^{\infty} \frac{1}{2}^i = 8M$. Therefore:

Theorem 3. *Let the sequence of firefighters $(f_i)_{i \geq 1}$ revealed by Player 2 to Player 1 be such that:*

$$\exists N \geq 1 : \sum_{i=1}^{N} f_i \geq 16 \cdot N$$

Then, Player 1 has an online strategy to win the firefighter problem on grids.

References

1. David Adjiashvili, Andrea Baggio, and Rico Zenklusen. Firefighting on trees beyond integrality gaps. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, pages 2364–2383, 2017.
2. Elliot Anshelevich, Deeparnab Chakrabarty, Ameya Hate, and Chaitanya Swamy. Approximability of the firefighter problem. *Algorithmica*, 62(1):520–536, Feb 2012.
3. Pierre Coupechoux, Marc Demange, David Ellison, and Bertrand Jouve. Online firefighting on trees. In *International Symposium on Combinatorial Optimization (ISCO 2018)*, 2018.
4. Pierre Coupechoux, Marc Demange, David Ellison, and Bertrand Jouve. Firefighting on trees. *Theoretical Computer Science*, 2019. In press - available online.
5. Marc Demange, David Ellison, and Raffaella Gentilini. Online firefighting on grids. Technical Report arXiv:1907.07450 [cs.IT], ArXiV, 2019.
6. Stephen Finbow, Andrew King, Gary Macgillivray, and Romeo Rizzi. The firefighter problem for graphs of maximum degree three. *Discrete Mathematics*, 307(16):2094–2105, July 2007.
7. Stephen Finbow and Gary MacGillivray. The firefighter problem: a survey of results, directions and questions. *Australasian Journal of Combinatorics*, 43:57–78, 2009.
8. Fedor V. Fomin, Pinar Heggernes, and Erik Jan van Leeuwen. The firefighter problem on graph classes. *Theoretical Computer Science*, 613(C):38–50, February 2016.
9. Carlos García-Martínez, Christian Blum, Francisco J. Rodriguez, and Manuel Lozano. The firefighter problem. *Computers & Operations Research*, 60(C):55–66, August 2015.
10. Bert L. Hartnell. Firefighter! an application of domination. In *Proc. of 25th Manitoba Conference on Combinatorial Mathematics and Computing*, 1995.
11. Margaret-Ellen Messinger. Firefighting on the triangular grid. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 63:37–45, 2007.
12. Margaret-Ellen Messinger. Average firefighting on infinite grids. *Australasian Journal of Combinatorics*, 41:15–28, 2008.
13. K.L. Ng and Paul Raff. A generalization of the firefighter problem on $\mathbb{Z} \times \mathbb{Z}$. *Discrete Applied Mathematics*, 156(5):730–745, 2008.
14. Ping Wang and Stephanie A. Moeller. Fire control on graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 41:19–34, 2002.