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On Possible Approaches to Differentiation of Rough Real Functions

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Abstract

In the mid 1990s Z. Pawlak relying on the rough set theory initiated the study of rough calculus in his many papers. He invented the investigation of its different subfields such as rough continuity, rough derivatives–integrals, rough differential equations, etc. Some authors have systematically investigated the rough continuity of rough real functions in Pawlak's sense. The following reasonable step would be to define the derivative of rough functions. However, it does not seem clear how it could be carried through this important step. In the paper, a possible approach will be outlined.

Keywords: Rough functions, discrete calculus, digital calculus, rough calculus

MSC: 03E70, 26A24

1. Introduction

In the mid 1990s Z. Pawlak relying on the rough set theory (RST) [6, 7, 15] initiated the study of rough calculus in his many papers [9, 11, 13, 14]. He invented the investigation of its different subfields such as rough continuity–discontinuity, rough derivatives–integrals, rough differential equations, etc.

The paper [3] systematically investigates the rough continuity–discontinuity of rough real functions in Pawlak's sense. The next reasonable step would be to define the derivative of rough functions.

Pawlak defined the rough derivatives based on discrete calculus. This paper basically, but not completely follows Pawlak's method.

The rest of paper is organized as follows. After the introduction, Section 1, the rough real numbers are defined in Section 2. Section 3 surveys different possible approximations of rough functions. Section 4 defines the rough derivatives and discusses some special features of this approach.

2. Rough real numbers

Let U, V be two classical nonempty sets. A *function* f with domain U and co-domain V is denoted by $f : U \to V$, $u \mapsto f(u)$. V^U denotes the set of all functions with domain U, in notation $\mathsf{Dom} f = U$, and co-domain V, in notation $\mathsf{Im} f = V$.

If $f, g \in V^U$, the operation $f \odot g$, $\odot \in \{+, -, \cdot, /\}$ and the relation $f \boxdot g$, $\boxdot \in \{=, \neq, \leq, <, \geq, >\}$ are understood by pointwise.

For any $S \subseteq U$, $f(S) = \{f(u) \mid u \in S\} \subseteq V$ is the direct image of S. Especially, $f(U) \subseteq V$ is the *range* of f.

If $a, b \in \mathbb{R}$ $(a \leq b)$, $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ and $]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$ denote *closed* and *open* intervals. $[a, a] = \{a\}$ is identified with the real number $a \in \mathbb{R}$. It is easy to interpret the open-closed $]a, b]$ and closed-open $[a, b[$ intervals.

(a, b) means an ordered pair of real numbers a and b.

Let $\mathbb{R}^{\geq 0}$ denote the set of nonnegative real numbers. Let $[n] = \{0, 1, \ldots, n\} \subseteq \mathbb{N}$ be a finite set of natural numbers. Accordingly, $]n] = 1, \ldots, n$, $[n[= 1, \ldots, n-1$, and $]n[= 1, \ldots, n-1$.

Definition of rough real numbers can be found in Pawlak's different papers such as [13, 14, 9, 8, 11, 12]. Here, it is briefly summarized.

Let I denote a closed interval $I = [0, a]$ $(a \in \mathbb{R}^{\geq 0}$, $a > 0)$.

Definition 2.1. A *categorization* of I is a sequence $S_I = \{x_i\}_{i \in [n]} \subseteq \mathbb{R}^{\geq 0}$, where $n \geq 1$ and $0 = x_0 < x_1 < \ldots < x_n = a$. S_I is also called the *discretization* of I.

Let I_S denote an equivalence relation generated by the categorization S_I. Let $x, y \in I$. $x I_S y$ if $x = y = x_i \in S_I$ for some $i \in [n]$, or $x, y \in]x_i, x_{i+1}[$ for some $i \in [n[$. Hence, the partition I/I_S associated with the equivalence relation I_S is:

$$I/I_S = \{\{x_0\},]x_0, x_1[, \{x_1\}, \ldots, \{x_{n-1}\},]x_{n-1}, x_n[, \{x_n\}\}.$$

It should be noted that in classical analysis, the term "partition of I" is used in a slightly different sense: Two compact real intervals *nonoverlapping* if either they are disjoint or their intersection contains at most one point, which necessarily an endpoint of both intervals ([2], p. 4). In the classical analysis context, a *partition* of I is a collection of nonoverlapping closed intervals whose union is I ([1], p. 149).

The block of the partition I/I_S containing $x \in I$ is denoted by $[\![x]\!]_{I_S}$. In particular, if $x \in S_I$, $[\![x]\!]_{I_S} = \{x\}$. If $x \in [\![x]\!]_{I_S} =]x_i, x_{i+1}[$, $\overline{[\![x]\!]}_{I_S} = [x_i, x_{i+1}]$ is the *closure* of $[\![x]\!]_{I_S}$. Of course, when $x \in S_I$, $[\![x]\!]_{I_S} = \overline{[\![x]\!]}_{I_S} = \{x\}$.

In terms of RST terminology, I_S is an indiscernibility relation on I. Hence, the naming of the following notions is consistent with the standard terminology of RST.

The members of I/I_S are called *elementary* or *base* sets. Any union of base sets are referred to as *definable* sets. By definition, $\emptyset$ is definable. Their collection is denoted by $\mathcal{D}_{I/I_S}$.

The principal notions of RST are the lower and upper approximation functions, l_S and u_S, respectively. Most commonly, their domain and co-domain are the power set of I. In the following, however, the closed intervals of the form $[0,x]$ $(x \in I)$ will only be approximated. Therefore,

$$\mathsf{l}_S([0,x]) = \{x' \in I \mid [\![x']\!]_{I_S} \subseteq [0,x]\} = \cup\{[\![x']\!]_{I_S} \in I/I_S \mid [\![x']\!]_{I_S} \subseteq [0,x]\},$$
$$\mathsf{u}_S([0,x]) = \{x' \in I \mid [\![x']\!]_{I_S} \cap [0,x] \neq \emptyset\} = \cup\{[\![x']\!]_{I_S} \in I/I_S \mid [\![x']\!]_{I_S} \cap [0,x] \neq \emptyset\}.$$

$\mathsf{PAS}(I) = (I, I/I_S, \mathcal{D}_{I/I_S}, \mathsf{l}_S, \mathsf{u}_S)$ is called *Pawlak approximation space.*

With a slight abuse of notation, let us define the following numbers:

$$\mathsf{l}_S(x) = \max\{x' \in S_I \mid x' \leq x\} \text{ and } \mathsf{u}_S(x) = \min\{x' \in S_I \mid x' \geq x\}.$$

Of course, $\mathsf{l}(x) \leq x \leq \mathsf{u}_S(x)$, and $\mathsf{l}_S(x) = \mathsf{u}_S(x) = x$ iff $x \in S_I$. Moreover,

- $\mathsf{l}_S([0,x]) = [0, \mathsf{l}_S(x)] = [0,x]$ and $\mathsf{u}_S([0,x]) = [0, \mathsf{u}_S(x)] = [0,x]$ (if $x \in S_I$);
- $\mathsf{l}_S([0,x]) = [0, \mathsf{l}_S(x)] \subsetneqq [0,x]$ and $\mathsf{u}_S([0,x]) = [0, \mathsf{u}_S(x)[\supsetneqq [0,x]$ (if $x \notin S_I$).

It is said that the number $x \in I$ is *exact* with respect to $\mathsf{PAS}(I)$ if $\mathsf{l}_S(x) = \mathsf{u}_S(x)$, otherwise x is *inexact* or *rough* [13]. Of course, $x \in I$ is exact iff $x \in S_I$.

Members of I/I_S are called *rough numbers* with respect to $\mathsf{PAS}(I)$. They can be represented as $[\![x]\!]_{I_S} = [\mathsf{l}_S(x), \mathsf{u}_S(x)] = \{x\}$ if $x \in S_I$, $[\![x]\!]_{I_S} =]\mathsf{l}_S(x), \mathsf{u}_S(x)[$ if $x \notin S_I$.

3. Approximation of rough real functions

Let $I = [0, a_I]$, $J = [0, a_J]$ be two closed intervals with a_I, $a_J \in \mathbb{R}^{\geq 0}$, $a_I, a_J > 0$. Let S_I, P_J be categorizations of I and J, where $S_I = \{x_i\}_{i \in [n]}$, $P_J = \{y_j\}_{j \in [m]} \subseteq \mathbb{R}^{\geq 0}$ with $m, n \geq 1$, $0 = x_0 < x_1 < \cdots < x_n = a_I$ and $0 = y_0 < y_1 < \cdots < y_m = a_J$. The corresponding approximation spaces are $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$.

A function J^I is called a *rough real function* with respect to $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$.

To make the blocks of I/I_S easier to handle technically, they are enumerated as follows.

$$N_I : I/I_S \to [2n],\ [\![x]\!]_{I_S} \mapsto \begin{cases} B_{2i} = 2i, & \text{if } \exists i \in [n]\ ([\![x]\!]_{I_S} = \{x_i\} \subset S_I), \\ B_{2i+1} = 2i+1, & \text{if } \exists i \in [n[\ ([\![x]\!]_{I_S} =]x_i, x_{i+1}[). \end{cases}$$

The inverse of N_I is:

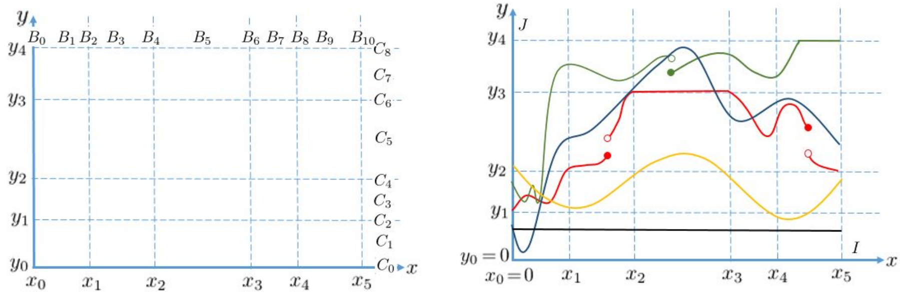
$$N_I^{-1} : [2n] \to I/I_S, \quad B_i \mapsto \begin{cases} \{x_{i/2}\}, & \text{if } i \equiv 0 \pmod 2, \\ ]x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}}[, & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

The equivalence classes of J/J_P can be enumerated in the same way by the help of an enumeration function N_J. Its values are referred to as C_j's $(j \in [2m])$.

Example 3.1. In the running example, let $I = [0, x_5]$ with $S_I = \{x_0, x_1, x_2, x_3, x_4, x_5\}$, and $J = [0, y_4]$ with $P_J = \{y_0, y_1, y_2, y_3, y_4\}$.

Figure 1 (a) shows the rough coordinate system with respect to $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$, and the enumeration of I/I_S and J/J_P. Figure 1 (b) presents some rough real functions in this rough coordinate system.

The purpose of this section is to show how rough real functions can be represented taken into account the features of approximation spaces $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$. The ideas of these representations mainly rely on Papwlak's paper [9, 10, 11, 12, 13].



(a) (b)

Figure 1: Rough coordinate system and rough real functions

3.1. Pointwise approximation of rough real functions

Definition 3.2 ([9, 13])**.** Let $f \in J^I$. The *pointwise* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$\underline{f} : I \to P_J,\ x \mapsto \mathsf{l}_P(f(x)), \quad \overline{f} : I \to P_J,\ x \mapsto \mathsf{u}_P(f(x)).$$

f is *exact* at x, if $\underline{f}(x) = \overline{f}(x)$, otherwise f is *inexact* (*rough*) at x.

f is *pointwise exact on* $I' \subseteq I$, if $\underline{f}(x) = \overline{f}(x)$ for all $x \in I'$, otherwise f is *pointwise inexact* (*rough*) *on* I'.

Remark 3.3. Rough real functions are *ab ovo* treated in (S_I, P_J)–coordinate systems. Hence, it seems reasonable to define the pointwise approximation in this context, too. Nevertheless, a "pointwise" feature much better fits to the whole interval $I = [a, b]$. But with the choice $S = \{x_0 = 0, x_1 = a\}$, this case is also included in the above definition.

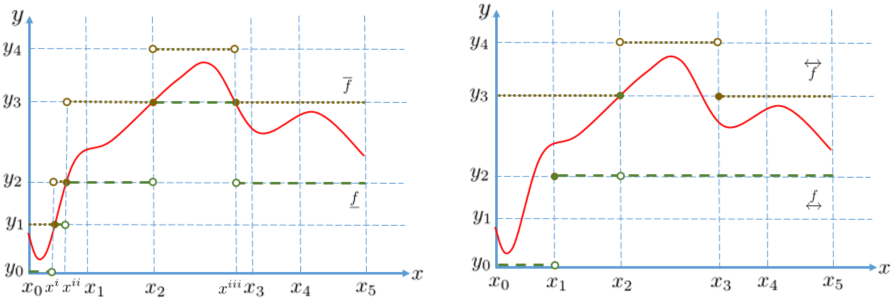
$f \in J^I$ is exact at $x \in I$ iff $f(x) = y_j \in P_J$ for some $j \in [m]$. Geometrically it means that f is exact at a point in I iff in this point f touches or intersects a line segment $y = y_j$, where $y_j \in P_J$.

Example 3.4. Figure 2 (a) shows the *pointwise* (S_I, P_J)–lower and (S_I, P_J)–upper approximations, $\underline{f}$ and $\overline{f}$, of a function $f \in J^I$. f is exact at points $x^i, x^{ii}, x_2, x^{iii}$ and rough at all other points.

3.2. Blockwise approximation of rough real functions

Definition 3.5. Let $f \in J^I$. The *blockwise* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$\underset{\leftrightarrow}{f} : I/I_S \to P_J,\ [\![x]\!]_{I_S} \mapsto \mathsf{l}_P(\inf f([\![x]\!]_{I_S})),\quad \overset{\leftrightarrow}{f} : I/I_S \to P_J,\ [\![x]\!]_{I_S} \mapsto \mathsf{u}_P(\sup f([\![x]\!]_{I_S})).$$



(a) (b)

Figure 2: Pointwise and blockwise approximations of f

f is *blockwise exact on* B_i for some $i \in [2n]$ if $\underset{\leftrightarrow}{f}(B_i) = \overset{\leftrightarrow}{f}(B_i)$, otherwise f is *blockwise inexact* (*rough*) on B_i.

Let $f \in J^I$, $i \in [2n]$. f is blockwise exact on B_i iff $f(B_i) = \{y_j\} \subset P_J$ for some $j \in [m]$. Geometrically it means that f is exact on $B_i \in I/I_S$ iff f touches or intersects a line segment $y = y_j$ for some $y_j \in P_J$ at the point $x_{i/2}$ if $i \equiv 0 \,(\mathrm{mod}\ 2)$, or f coincides with a line segment $y = y_j$ for some $y_j \in P_J$ on $]x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}}[$ when $i \equiv 1 \,(\mathrm{mod}\ 2)$.

Example 3.6. Figure 2 (b) shows the *blockwise* (S_I, P_J)–lower and (S_I, P_J)–upper approximations, $\underset{\leftrightarrow}{f}$ and $\overset{\leftrightarrow}{f}$, of f. f is blockwise exact on $B_4 = \{x_2\}$ only, and blockwise rough on all other blocks.

It should be noted that the pointwise and blockwise approximations defined above substantially differ from those considered in function approximation theory. Essentially because the lower and upper approximations (pointwise or blockwise) delimit a family of functions.

For instance, in the case of blockwise approximation, the domain and co–domain of every function f in this family, in their most general form, are $\mathsf{Dom} f = I$ and $\mathsf{Im} f = \{y \in J \mid \underset{\leftrightarrow}{f} \le y \le \overset{\leftrightarrow}{f}\}$, *without any assignment rules.* Certain functions in this family can also be interpreted as partially specified ones with *unknown values.* With the help of getting finer and finer rough coordinates systems, actual values of these unknown values are becoming more and more recognizable.

3.3. Finite sequence approximations

Definition 3.7. Let $f \in J^I$. The *finite sequence* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$f_\circ : [n] \to P_J,\ i \mapsto \mathsf{l}_P(f(x_i)), \quad f^\circ : [n] \to P_J,\ i \mapsto \mathsf{u}_P(f(x_i)).$$

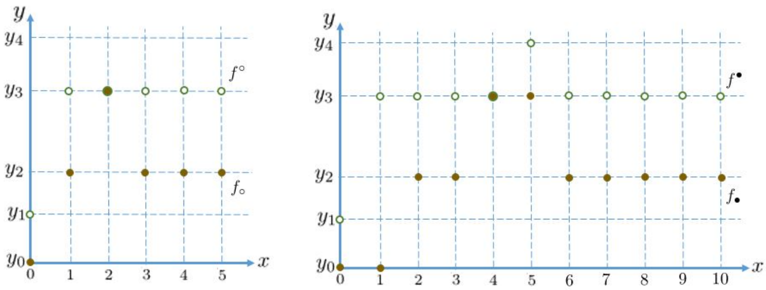
This approximation characterizes the rough functions at the categorization points. However, it does not say anything about how a rough function behaves on the open intervals $]x_i, x_{i+1}[$ $(i \in [n[)$.

Definition 3.8. Let $f \in J^I$. The *extended finite sequence* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$f_\bullet : [2n] \to P_J,\ i \mapsto \mathsf{l}_P(\inf f(B_i)), \quad f^\bullet : [2n] \to P_J,\ i \mapsto \mathsf{u}_P(\sup f(B_i)).$$

Of course, $f_\circ(i) = f_\bullet(2i)$, $f^\circ(i) = f^\bullet(2i)$ $(i \in [n])$, i.e., they are equal at the categorization points.

Example 3.9. Figure 3 (a) and Figure 3 (b) illustrate the finite sequence and extended finite sequence approximations of the function f (see Figure 2).



(a) (b)

Figure 3: Finite sequence approximations of f

3.4. Discrete sequence approximations

The most abstract approximations of rough functions is the discrete sequence one. In the definitions, the following function will be needed.

Let I and J two intervals with categorizations S_I and P_J be given as above.

$$\mathsf{l}_S^{enum} : J \to [m],\ y \mapsto \max\{i \in [m] \mid y_i \leq y\},$$
$$\mathsf{u}_S^{enum} : J \to [m],\ y \mapsto \min\{i \in [m] \mid y_i \geq y\}.$$

Definition 3.10. Let $f \in J^I$. The *discrete sequence* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$f_\star : [n] \to [m],\ i \mapsto \mathsf{l}_S^{enum}(f(x_i)), \quad f^\star : [n] \to [m],\ i \mapsto \mathsf{u}_S^{enum}(f(x_i)).$$

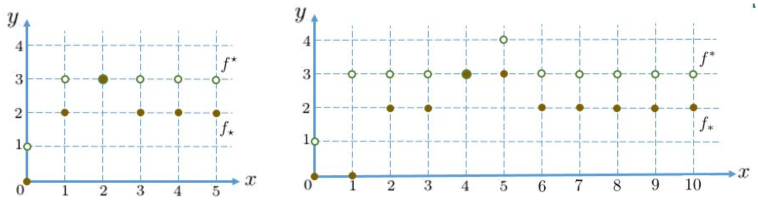
This approximation also characterizes the rough functions at the categorization points only. It will be extended on the whole partition I/I_S of I.

Definition 3.11. Let $f \in J^I$. The *extended discrete sequence* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$f_* : [2n] \to [m],\ i \mapsto \mathsf{l}_S^{enum}(\inf f(B_i)), \quad f^* : [2n] \to [m],\ i \mapsto \mathsf{u}_S^{enum}(\sup f(B_i)).$$

Of course, $f_\star(i) = f_*(2i)$, $f^\star(i) = f^*(2i)$ $(i \in [n])$, i.e., they are equal at the categorization points.

Example 3.12. Figure 4 (a) and Figure 4 (b) illustrate the discrete sequence and extended discrete sequence approximations of f.



(a) (b)

Figure 4: Discrete sequence approximations of f

4. Rough derivatives

Classically, discrete derivatives, or more general, discrete calculus of functions was studied in the theory of "calculus of finite differences", see, e.g., [4], one of the classical treatments of this subject. Traditionally, the subject of the discrete calculus is the functions of the form $f:\mathbb{N}\to\mathbb{R}$, i.e., f's are finite or infinite sequences.

In his papers, however, Pawlak defined the rough derivatives of such functions whose both domain and co–domain are finite set of natural numbers which he called *discrete functions*. In the following, under the discrete functions it is meant a function of the form $f:\mathbb{N}\to\mathbb{Z}$. Obviously, $f:[n]\to[m]$ is discrete function as well. The calculus of such functions is called a rough calculus [12], or digital calculus [5].

4.1. Pawlak's approach

Pawlak's approach is based on the discrete representation of rough functions. This section mainly relies on Pawlak's papers [10, 11, 12], in addition Nakamura, Rosenfeld's paper [5]. Although, Nakamura and Rosenfeld defined derivatives in a slightly more general context, their results can be applied here.

Definition 4.1. Let $f : [n] \to [m]$ be a discrete function. The *rough derivative* f' of f is the function

$$f' : [n[= [n-1] \to \mathbb{Z}, \quad i \mapsto f(i+1) - f(i).$$

The relationship between derivation and function operations slightly differ from the classical rules.

Proposition 4.2 ([5], Theorem 1.4)**.** *Let $f, g : [n] \to [m]$ be two discrete functions. Then, for $f \pm g$, $k \cdot f$, $f \cdot g$, f/g, we have*

- $(f \pm g)'(i) = f'(i) \pm g'(i)$;
- $(k \cdot f)'(i) = k \cdot f'(i)$ $(k \in \mathbb{Z})$;
- $(f \cdot g)'(i) = f'(i) \cdot g(i) + f(i) \cdot g'(i) + f'(i) \cdot g'(i) = f(i) \cdot g'(i) + g(i+1) \cdot f'(i)$;
- $\left(\frac{f}{g}\right)'(i) = \frac{f'(i) \cdot g(i) - f(i) \cdot g'(i)}{g^2(i) + g(i) \cdot g'(i)}$, *provided that* $g^2(i) + g(i) \cdot g'(i) \neq 0$,

where $(f \pm g)'$,$(k \cdot f)'$,$(f \cdot g)'$,$(f/g)' : [n-1] \to \mathbb{Z}$.

Higher order derivatives of f can be defined as usual: the second order derivative of f is the derivative of f', etc. The following notations are used commonly for higher order derivatives: $f = f^{(0)}$, $f' = f^{(1)}$, $f^{(2)} = f^{(1)'}$, $f^{(3)} = f^{(2)'}$, etc.

One can observe that $\mathsf{Dom} f^{(0)} = [n]$, $\mathsf{Dom} f^{(1)} = [n[= [n-1]$, $\mathsf{Dom} f^{(2)} = [n-1[= [n-2]$, and, in general, $\mathsf{Dom} f^{(k)} = [n-(k-1)[= [n-k]$. It also means that the discrete function $f : [n] \to [m]$ has at most derivatives up to n-th order.

4.2. Discussion of rough derivatives

Digital calculus is applied to digital image processing [5, 16]. In this context, direct interpretation of rough derivatives is simple. Let $f : [n] \to [m]$ be a discrete function, and $f' : [n-1] \to \mathbb{Z}$ its derivative. Then, f can be interpreted as a piecewise linear function with slopes $f'(i)$'s $(i \in [n-1])$. Of course, when $f'(i) = 0$, f is constant between $f(i)$ and $f(i+1)$, i.e., $f(i) = f(i+1)$. In this interpretation, the domain of f can be considered as bounded compact real interval $[0, m] \subset \mathbb{R}$.

Example 4.3. Figures 5 (a) and 5 (b) show the discrete derivatives of $f_\star$ and $f^\star$. Figures 5 (c) and 5 (d) depict their piecewise linear function interpretations.

On the other hand, rough calculus was motivated by the setting up a possible calculus of rough functions. Pawlak used the rough calculus, which is also known as digital calculus or discrete calculus, to achieve this goal. However, Pawlak did not establish a connection between the rough derivatives and rough functions. Indeed, it is hard to interpret rough derivatives as rough functions.

Different approximations of rough functions lead to different *sets* of rough functions. More specifically, let $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$ be two approximation spaces as defined above. In addition, let $f \in J^I$ be a fixed rough function. Then, all different approximations of f set up *sets of rough functions*. In regard to Pawlak's approach, here we will focus on the discrete sequence approximation of rough functions only.

The discrete sequence approximation determines the following set of rough functions: $\mathcal{R}_f^{ds} = \{g \in J^I \mid \forall i \in [n](y_{f_\star(i)} \leq g(x_i) \leq y_{f^\star(i)})\}$. That is, $\mathcal{R}_f^{ds}$ consists of such the rough functions g's which are bounded by $y_{f_\star(i)}$ and $y_{f^\star(i)}$ $(i \in [n]$, $y_{f_\star(i)}, y_{f^\star(i)} \in P_J)$ at the categorization points of I, but they are not constrained on the open intervals formed by the categorization points of I.

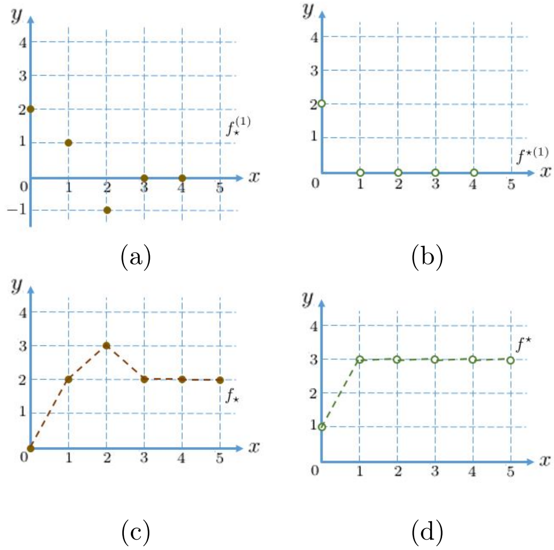


Figure 5: Discrete derivatives of $f_\star$, $f^\star$, and their interpretations

Example 4.4. Figure 6 depicts the discrete approximation of f (Figure 6 (a)), and the set of rough functions determined by it (Figure 6 (b)). One can observe that the discrete approximation constraints the rough functions at the categorizations points, but it does not say anything about how they behave on the open intervals.

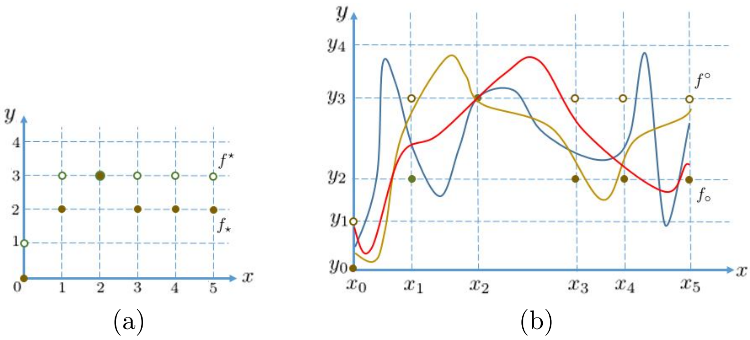


Figure 6: Discrete approximation of f and its interpretation $\mathcal{R}_f^{ds}$

Finding out rough derivatives of a set of rough functions, e.g., $\mathcal{R}_f^{ds}$, may be approached, for instance, by differentiating both lower and upper discrete sequence approximations. Some special difficulties of this approach are the following:

(i) Of course, $f_\star \leq f^\star$ does not imply $f_\star^{(1)} \leq f^{\star(1)}$.

At an $i \in [n[$ may occur $f_\star^{(1)}(i) < f^{\star(1)}(i)$, $f_\star^{(1)}(i) > f^{\star(1)}(i)$, or $f_\star^{(1)}(i) = f^{\star(1)}(i)$. Their interpretations are simple. $f_\star^{(1)}(i) < f^{\star(1)}(i)$ means that the lower approximation changes at i *to a lesser extent* than the upper approximation; whereas $f_\star^{(1)}(i) > f^{\star(1)}(i)$ indicates that the lower approximation changes at i *to a greater extent* than the upper approximation. If $f_\star^{(1)}(i) = f^{\star(1)}(i)$, the lower and upper approximations change at i *to the same extent.*

(*ii*) $f_\star^{(1)}(i) < 0$ and/or $f^{\star(1)}(j) < 0$ for some $i, j \in [n[$.

A possible solution is the following. If $f_\star^{(1)}(i) < 0$ for some $i \in [n[$, then $\mathsf{Im} f_\star^{(1)}$ may/should be extended to the extent necessary. This can be done, of course, analogously for $f^{\star(1)}$, too.

5. Concluding remarks

The paper, first, has presented four different rough approximation methods representing the rough real functions. The chances are that a definition of the rough differentiation should rely on one of these representations. In this paper, basically but not completely following Pawlak's method, the rough differentiation based on discrete sequence approximation has been considered only. Of course, additional differentiation definitions should also be studied based on the other representations which may be the subject of many subsequent papers in the future.

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