On (Maximal, Tractable) Fragments of the Branching Algebra ⋆

Alessandro Bertagnon[1] [0000−0003−2390−0629], Marco Gavanelli[1] [0000−0001−7433−5899], Guido Sciavicco[2] [0000−0002−9221−879X], and Stefano Trevisani[2] [0000−0002−5375−0214]

[1] Department of Engineering, University of Ferrara
alessandro.bertagnon@unife.it, marco.gavanelli@unife.it
[2] Department of Mathematics and Computer Science, University of Ferrara
guido.sciavicco@unife.it, stefan.trevisani@student.unife.it

Abstract. Branching Algebra is the natural branching-time generalization of Allen's Interval Algebra. Its potential applications range from planning with alternatives, to automatic story-telling with alternative timelines, to checking version systems with several branches. As in the linear case, the consistency problem of Branching Algebra is computationally hard, and, in particular, NP-complete. Recently, tractable fragments of it have been studied, but the landscape of tractability of fragments is far from being complete. In this paper, we identify three interesting fragments of the Branching Algebra: the Horn fragment, which was already known, the Pointsable fragment, and the Linear fragment. We study their tractability as well as their tractability via *Path-Consistency*, and we discuss their maximality.

Keywords: Constraint programming · Consistency · Branching time · Tractability of fragments

1 Introduction

When dealing with automated temporal reasoning, one of the most prominent formalisms is certainly Allen's Interval Algebra [1] (IA). Applications of the IA encompass a large number of fields, including scheduling, planning, database theory, natural language processing, among others. Events in the IA are represented as intervals on a *linearly* ordered set, and can be related to each other through one of the thirteen basic *relations* (IA_{basic}). A relation is a constraint that specifies which basic relations may hold between a pair of intervals, so the IA has a total of 2^{13} relations. A *constraint network* is a pair formed by a set of interval variables and a set of constraints between them; the fundamental reasoning task

G. Sciavicco acknowledges partial support by the Italian INDAM GNCS project *Strategic Reasoning and Automated Synthesis of Multi-Agent Systems*

that arises when dealing with constraint networks is the *consistency* problem, that is, the problem of establishing whether it is possible to satisfy all given constraints, asserting the existence of a realization of the network. This problem, which is archetypical for the class of Constraint Satisfaction Problems (CSP), is NP-COMPLETE for generic IA networks, as shown by Vilain and Kautz [17], and therefore, in general, solutions can only be found by exploring the tree of all possible assignments. For this reason, in the nineties and early two thousands the research has focused towards finding the tractable fragments of the IA, that is, subsets of relations for which the consistency problem is tractable. The tractability landscape of the IA is completed by listing all *maximal* tractable sub-algebras of it, that is, all tractable fragments that cannot be further extended without losing the tractability. After [9], such a landscape is fully known, and it encompasses 18 incomparable, maximal, tractable fragments of the IA. Some tractable fragments, even non-maximal, are noteworthy because of the technique used to show their tractability, the naturalness of their relations, and their applications: the *convex* fragment [16] (IA_{convex}, 82 relations), the *pointisable* fragment [17] (IA_{point}, a superset of IA_{convex} with 182 relations), and the *ORD-Horn* fragment [13] (IA_{Horn}, also known as $IA_{preconvex}$ [10], with 868 relations, which extends IA_{point} and is maximal).

The Branching Algebra (BA) [14] is the natural generalization of the IA to tree-shaped orderings. Its potential applications, additional to the classical ones, range from planning with alternatives, to automatic story-telling with alternative timelines, to checking version systems with several branches. The set of basic branching relations (BA_{basic}) contains all of the 13 IA_{basic} relations, plus six new 'branching' basic relations, which take into account the possible incomparability of interval endpoints, for a total of 19 basic pairwise-disjoint and jointly-exhaustive relations. Quite obviously, the consistency problem for the BA is still NP-COMPLETE [14] and therefore, just like in the linear case, we are interested in finding tractable fragments, in particular the maximally tractable ones. Unlike the linear case, however, the landscape of tractability in BA is still far from being complete. Only two tractable fragments are known: the *convex* (BA_{convex}) fragment [8], which resembles its linear counterpart, and the TORD-HORN fragment (BA_{Horn}) [3], inspired by the ORD-HORN fragment of the IA.

In this paper we focus on the tractability of fragments of the BA, and we study both tractable fragments and PC-tractable ones, that is, tractable by *Path-Consistency* (PC-tractability is a desirable property which is not guaranteed by tractability). In particular, we show that BA_{Horn} is maximally tractable (its PC-tractability was already known from [3]), we introduce two new tractable fragments, called BA_{lin} (the *linear* fragment of the BA) and BA_{point} (the *pointisable* fragment of the BA), and we discuss both their PC-tractability as well as their maximality.

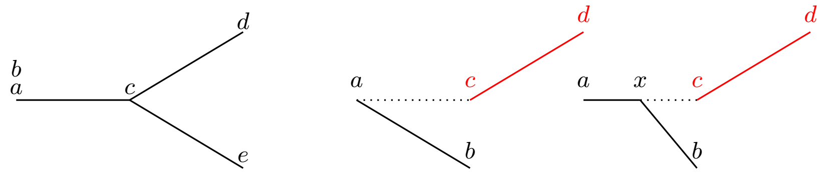

Fig. 1. A pictorial representation of the four basic branching point relations, where $a = b$, $a < c$, $d > c$, and $d||e$ (left-hand side), and an example of two situations that require quantification to be distinguished in the language of endpoints (right-hand side).

2 Preliminaries

Notation. Let $(\mathcal{T}, <)$, often denoted by $\mathcal{T}$, a *right-branching tree-order*, i.e. a partial order where:

$$\forall x, y, z \in \mathcal{T} : (x < z) \wedge (y < z) \Rightarrow (x \gtreqless y).$$

When variables are interpreted as *time points*, $(\mathcal{T}, <)$ is called a *future branching model of time* (or, simply, a *branching model*). Elements of $\mathcal{T}$ are denoted by $a, b, \ldots$, and $a \parallel b$ (resp., $a \gtreqless b$) denotes that a and b are incomparable (resp., comparable) with respect to the ordering relation $<$. We use $x, y, \ldots$ to denote variables in the domain of points, and $x \leq y$ to denote $x < y \vee x = y$. There are four basic relations that may hold between two points on a branching model: *equals* ($=$), *incomparable* ($\parallel$), *less than* ($<$), and *greater than* ($>$); the first two are symmetric, while the last two are the converse of each other. These relations are depicted in Figure 1 (left-hand side), and are called *basic branching point relations*. The set of basic branching point relations is denoted by BPA_{basic}. In the linear setting, the set of basic relations has only three elements, $<, =$ and $>$, and it is called PA_{basic} (*basic point relations*).

An *interval* in $\mathcal{T}$ is a pair $[a, b]$ where $a < b$, and $[a, b] = \{x \in \mathcal{T} : a \leq x \leq b\}$. Intervals are generically denoted by $I, J, \ldots$ For an interval I, we use I^-, I^+ to denote its endpoints. Following [7], one can describe 24 basic branching relations based on the possible relative position of two pairs of ordered points on a branching model, that is, by directly generalizing the universally known set of 13 *basic interval relations* [1] (IA_{basic}). Some of these relations require first-order quantification to be defined: for example, in Figure 1 (right-hand side) we see that, in order to distinguish the two situations, we need to quantify over the existence, or non-existence, of a point between a and c. This problem becomes relevant when we study the behaviour of branching relations in association with the behaviour of branching point relations (that is, by studying the properties of their point-based translations); to overcome it, Ragni and Wölfl [14] introduce a set of coarser relations, characterized by being translatable to point-based relations using only the language of endpoints, without quantification. These 19

b (bi)	I *before* J	$I^+ < J^-$	
m (mi)	I *meets* J	$I^+ = J^-$	
o (oi)	I *overlaps* J	$I^- < J^- < I^+ < J^+$	
d (di)	I *during* J	$J^- < I^- < I^+ < J^+$	
s (si)	I *starts* J	$I^- = J^- < I^+ < J^+$	
f (fi)	I *finishes* J	$J^- < I^- < I^+ = J^+$	
e	I *equals* J	$I^- = J^- < I^+ = J^+$	
ib (ibi)	I *init. before* J	$I^- < J^- \parallel I^+$	
im (imi)	I *init. meets* J	$I^- < J^- < I^+ \parallel J^+$	
ie	I *init. equals* J	$I^- = J^- < I^+ \parallel J^+$	
u	I *unrelated* J	$I^- \parallel J^-$	

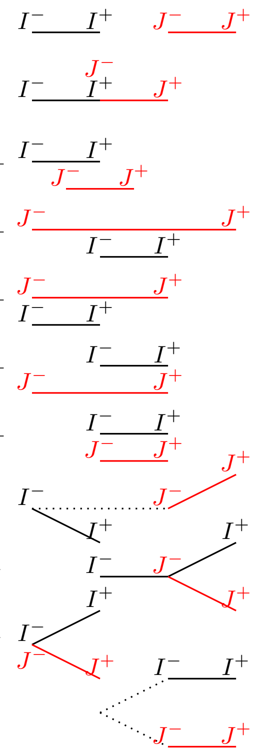

Fig. 2. A pictorial representation of the nineteen basic branching interval relations. In this picture, in which $I = [I^-, I^+]$ and $J = [J^-, J^+]$, we assume $I^- < I^+$ and $J^- < J^+$. Solid lines are actual intervals, dashed lines complete the underlying tree structure. We use $a\ R_1\ b\ R_2\ c$ as a shorthand for $a\ R_1\ b$ and $b\ R_2\ c$.

relations are depicted in Figure 2, and form the set of *basic branching interval relations* (BA_{basic}); for each relation, the symbol in parentheses corresponds to its converse, if the relation is not symmetric. A relation in the set BA_{basic} is either a linear relation, or the relation u (*unrelated*), or it corresponds to the disjunction between a pair of finer relations from the set of 24 [7]. For example, the relation ib is the disjunction of the two relations in Figure 1.

Operations and algebras. Given the basic relations $r_1, \ldots, r_l$, we denote by $R = \{r_1, \ldots, r_l\}$ the disjunctive relation $r_1 \vee \ldots \vee r_l$; thus, a relation is seen as a set, and a basic relation as a singleton. As the set IA_{basic} contains 13 elements, the set IA of all interval relations in the linear setting encompasses 2^{13} elements, including the empty relation; similarly, the set BA_{basic} of 19 basic relations entails 2^{19} interval relations in the branching setting. A *constraint* is an object of the type $x\ R\ y$, where x, y are interval variables and R is a relation.

There are three basic operations with relations: (Boolean) intersection, converse, and weak composition (often called simply composition). The *converse* of a basic relation is defined as:

$$\forall x, y, r : x\ r^{\smile}\ y \Leftrightarrow y\ r\ x,$$

and the converse of a relation is simply the union of the element-wise converse of its basic relations. In our notation, for example, *bi* (*after*) denotes the converse of the basic relation *b* (*before*), and $\{imi, u, o\}$ is the converse of the relation $\{im, u, oi\}$. The *weak composition* of two basic relations is defined as follows:

$$\forall x, y, r, s : x\ (r \circ s)\ y \Leftrightarrow \exists z \mid x\ r\ z \wedge z\ s\ y.$$

Again, the composition of relations is defined by the union of element-wise composition of its basic relation. Intuitively, weak composition is the application of transitivity of relations, and it is usually computed via a *composition table*. For example, given three intervals I, J, K for which $I\ \{s\}\ J$ and $J\ \{o\}\ K$, then we know that $I\ (\{s\} \circ \{o\})\ K = I\ \{b, m, o\}\ K$. The composition table for the IA is shown in [2], and for the BA in [15]. Converse, intersection, and weak composition are classical operations in the literature; however, we can define a new operation that is interesting for us, called *strong composition*, by combining weak composition and intersection. For basic relations we have that:

$$\forall x, y, r, s, t :\ x\ (\diamond(r, s, t))\ y \Leftrightarrow t \in (r \circ s),$$

and for relations, that:

$$\forall x, y, R, S, T :\ x\ (\diamond(R,\ S,\ T))\ y \Leftrightarrow x\ (R \circ S)\ y\ \wedge\ x\ T\ y$$

(recall from the preliminaries that basic relations are represented with lower-case letters, while relations are represented with upper-case letters).
Back to the previous example, suppose we also knew that $I\ \{m, s\}\ K$; the relation $I\ \{b, m, o\}\ K$ could still be refined without using any new variable: indeed, by strong composition, we have that $I\ (\diamond(\{s\}, \{o\}, \{m, s\}))\ K = I\ \{m\}\ K$, which is in fact a *stronger* result. Strong composition associates three relations, and its fix-point application provides Path-Consistency.

Given a relation algebra $\mathcal{A}$, if a set of relations $\mathcal{S} \subseteq \mathcal{A}$ is closed under converse, intersection, and weak composition, we say that it is a *strong subalgebra* of $\mathcal{A}$, while if it is closed only under converse and strong composition, then it is a *weak subalgebra*. Obviously, every strong algebra is also a weak algebra, but the converse is not true in general; a special case occurs when the *unknown* relation $\top$ (i.e. the relation formed by the union of all basic relations, sometimes denoted by "?") is an element of $\mathcal{S}$: then, if $\mathcal{S}$ is a weak algebra it must also be a strong one. Following [13], we can define a *closure operator* $\Gamma_{\pi_1,\pi_2,\ldots}()$ that maps any given set (fragment) $\mathcal{S}$ to its *algebraic closure*, by repeatedly applying the operations $\pi_1, \pi_2, \ldots$ until a fixed point is reached. In particular, $\Gamma_{\smile,\cap,\circ}(\mathcal{S})$ (or simply Γ), computes the *generated (strong) subalgebra* of $\mathcal{S}$.

Decision problems. Given a fragment $\mathcal{S}$ of a relation algebra $\mathcal{A}$, an *instance* Θ of $\mathcal{S}$ is a pair $\langle V, R\rangle$ where V is a set of *variables* (e.g., points or intervals) and R

is a set of relations between the variables. Instances are usually represented by labelled direct digraphs, where the vertices represent the variables and the labels on the edges represent the relations. An *interpretation* $\mathcal{I}$ of Θ is an assignment $R \mapsto \mathcal{A}_{basic}$ such that:

$$\forall r \in R : r \to r' \mid r' \subseteq r \wedge r' \in \mathcal{A}_{basic}.$$

A non-contradictory interpretation, that is an interpretation whose constraint can all be satisfied by at least one set of concrete elements in the underlying order, is called a *model*. Finally, if an instance has at least one model is said to be *consistent*. The problem of determining whether an instance of $\mathcal{S}$ has a model or not is the *consistency* problem $\mathcal{A}$-SAT($\mathcal{S}$), and the problem of determining whether all of the basic relations in each relation of an instance can appear in at least one model is the *minimal labels* problem $\mathcal{A}$-MIN($\mathcal{S}$); since the tractability of $\mathcal{A}$-MIN depends on the tractability of $\mathcal{A}$-SAT (see [17]), we can focus solely on the latter. In particular, we say that a fragment $\mathcal{S}$ is tractable if and only if $\mathcal{A}$-SAT($\mathcal{S}$) is tractable; a tractable fragment $\mathcal{S} \subseteq \mathcal{A}$ which cannot be extended without losing its tractability, is said to be *maximally tractable*, or, simply, a *maximal* fragment.

Assuming that $\mathcal{A}$ is NP-COMPLETE, as it is the case for both the IA ([17]) and the BA ([14]), and that P $\neq$ NP, we are interested in finding all the *maximal tractable subalgebras* of $\mathcal{A}$, because the tractability of any fragment depends on the tractability of its generated subalgebra [13]. Finally, a fragment is said to be PC-*tractable* if its consistency problem is decided by the so called *path-consistency* algorithm (which is a special case of the *local consistency* algorithm [11]). PC-tractability of a fragment is a desirable property, as a PC-tractable fragment $\mathcal{S}$ can be used as an heuristics to speed up a consistency checking brute force algorithm for a non-polynomial algebra $\mathcal{A}$ (see, e.g. [5]). The path-consistency algorithm has also a fixed complexity: $O(n^3)$, where n is the number of distinct variables of an instance, and being a very well-known algorithm, optimized implementations abound. Studying the PC-tractability of tractable fragments is therefore an interesting problem.

3 Some Tractable Fragments of the Branching Algebra

Tractable fragments of the IA. In the linear case there are 18 tractable fragments of the IA [9]. Three of them are noteworthy: IA_{convex}, IA_{point}, and IA_{Horn}. In particular, IA_{point} is characterized by the fact that every instance of IA_{point} can be polynomially translated to an instance of the PA, effectively reducing the interval-based problem to a point-based one. In general, for any intervals-based relation algebra $\mathcal{A}$, there always exists a *pointisable fragment* $\mathcal{A}_{point}$ whose relations can all be exhaustively expressed only by conjunctive constraints between intervals endpoints: this allows us to create a *point mapping operator* ξ which translates any interval algebra instance Θ into an equisatisfiable point algebra instance Θ' in polynomial time. IA_{point} is obtained by applying the operator ξ on IA. IA_{convex} [16] is a fragment of IA_{point} that uses only convex point-based relations; while in both cases, *Path-Consistency* decides the consistency of an instance, in IA_{convex} it also decides its minimal labels problem, unlike IA_{point} [17]. IA_{Horn} [13] includes IA_{point} as a fragment, and is PC-tractable as well. The common characteristics to these three fragments, besides the fact that they all are strong algebras, is that they have been studied via their point-based translation. In the case of IA_{Horn}, unlike IA_{point} and IA_{convex}, an interval-based instance cannot be simply translated to a point-based one; yet, its PC-tractability is a consequence of developing a Horn point-based logical theory (known as the ORD-HORN theory [13]). The remaining 15 other tractable fragments have been studied mostly in [6] and [9], in a rather systematic way, and the problem of their PC-tractability has not been posed; their importance relies in the fact that they complete the tractability landscape of the fragments of the IA, more than the naturalness of their definition or their actual practical implications.

The Convex and the Horn Fragment of the BA. In the branching case, only two tractable fragments are known so far. The BA_{convex} fragment [8] is the natural branching counterpart of IA_{convex}. Unlike the latter, however, it is only a weak algebra; yet it is PC-tractable and, as it is for IA_{convex}, *Path-Consistency* decides the minimal label problems as well. The BA_{Horn} fragment, which extends BA_{convex}, and that has been introduced in [3], has been proven to be PC-tractable by developing a tree-order point Horn theory (TORD-HORN), in a way similar to the linear counterpart. The question we pose now is: which additional fragments of the BA can be discovered by a systematic analysis of their translations to the branching point-based framework?

The Point and the Linear Fragment of the BA. In [4], Broxvall introduces the branching *disjunctive point algebra*, which extends the standard (conjunctive) branching point algebra by allowing disjunctive constraints in its instances; for example, in a disjunctive algebra, a constraint such as:

$$(x < y \vee y \geq z) \vee (x \parallel y \vee z = t),$$

inadmissible in a classical algebra, is allowed. Formally, given a conjunctive point algebra $\mathcal{A}$ and two fragments $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{A}$, we can build the derived disjunctive fragment $\mathcal{S}_1 \overset{\times}{\vee} \mathcal{S}_2$ which contains the relations given by binary disjunctions over $\mathcal{S}_1$ and $\mathcal{S}_2$. Furthermore, we have that $\mathcal{S}^1 = \mathcal{S}$ and $\mathcal{S}^i = \mathcal{S}^{i-1} \overset{\times}{\vee} \mathcal{S}$, so $\mathcal{S}^k$ means that relations are k-disjunctions of the relations in $\mathcal{S}$. Finally, $\mathcal{S}^* = \bigcup_{i=0}^{\infty} \mathcal{S}^i$, indicates that we can use an *arbitrary* amount of disjunctions for our relations. Besides their naturalness, disjunctive fragments in the point-based case are interesting because they allow to define non-pointisable fragments in the interval case. In the linear setting, for example, IA_{Horn} can be translated to a certain disjunctive fragment of PA^* (although the disjunctive point-based fragments have not been explicitly studied in the linear case). An analysis of the disjunctive fragments of the BPA led Broxvall to determine that there are exactly five maximal tractable

Table 1. Broxvall's basic tractable fragments.

	Γ_A	Γ_B	Δ_B	Δ_C	Γ_D	Δ_D	Γ_E	Δ_E
$<$	✓	✓					✓	
$\leq$	✓	✓		✓			✓	
$\lessgtr$	✓	✓	✓				✓	
$\lesseqgtr$	✓	✓	✓	✓			✓	
$\parallel$	✓				✓	✓	✓	
$=\parallel$	✓			✓	✓	✓		
$=$	✓	✓		✓	✓		✓	
$\neq$	✓	✓	✓		✓	✓	✓	✓
$<\parallel$	✓				✓	✓	✓	
$\leq\parallel$	✓			✓	✓	✓		

subalgebras of BPA^*, which are:

$$\mathcal{T}_A = \Gamma_A \qquad \mathcal{T}_B = \Gamma_B \overset{\times}{\vee} \Delta_B^* \qquad \mathcal{T}_C = \Delta_C^*$$

$$\mathcal{T}_D = \Gamma_D \overset{\times}{\vee} \Delta_D^* \qquad \mathcal{T}_E = \Gamma_E \overset{\times}{\vee} \Delta_E^*.$$

The definition of the Γ and Δ *'base fragments'* is provided in Tab. 1. The tractability of these fragments if proven by devising a specific algorithm; it does not imply their PC-tractability, and, as a side note, the complexity of their consistency algorithm is slightly worse than $O(n^3)$.

Given any two fragments $\mathcal{S}_1$ and $\mathcal{S}_2$ of BPA, we can compute $\Gamma(\xi^{-1}(\mathcal{S}_1 \overset{\times}{\vee} \mathcal{S}_2^*))$, where ξ^{-1} is the inverse of the point mapping operator ξ extended to disjunctive point algebras. Note that there is only a finite number of disjunctions which can be allowed in BA; also, from an implementation perspective, instead of systematically trying all possible combinations, $\xi^{-1}(\mathcal{S}_1 \overset{\times}{\vee} \mathcal{S}_2^*)$ can be efficiently computed by a slightly modified version of $\Gamma_\cup(\xi^{-1}(\mathcal{S}_1) \cup \xi^{-1}(\mathcal{S}_2))$, which assures that the relations of $\mathcal{S}_1$ (or their derived) are not combined between themselves (by definition, we allow arbitrary disjunctions only of $\mathcal{S}_2$). By looking at Tab. 1, we see that $\mathcal{T}_C$ and $\mathcal{T}_D$ can be immediately excluded from our analysis since they do not contain the relation $<$, which is needed to state the natural constraint $\forall I : I^- < I^+$ (it is interesting to note that their mappings *degenerate* into the whole BA). Also, it is the case that $BA_{Horn} = \Gamma(\xi^{-1}(\mathcal{T}_E))$: this equivalence, which is also an alternative way to show the tractability of BA_{Horn} (but not to show its PC-tractability), becomes clear when comparing the definitions of $\mathcal{T}_E$ in Tab. 1 and those of TORD clauses given in [3]. By applying the mapping on $\mathcal{T}_A$ which, after closure by converse, is equal to BPA, we obtain the branching equivalent of IA_{point}, that is, BA_{point}, which is a first new tractable fragment of the BA. If, instead, we apply this mapping to $\mathcal{T}_B$, we get another previously unknown fragment, which we called BA_{lin} because all the relations contained in $\mathcal{T}_B$ are linear except one: the relation $\neq$. As the $\neq$ relation is interpreted as

$< \vee > \vee ||$, it is not linear, so BA_{lin} is not a subset of IA, as its name may suggest.

Theorem 1. *The fragments BA_{Horn}, BA_{lin} and BA_{point} are tractable strong subalgebras of the BA.*

The fact that they are strong subalgebras can be proved by a systematic computer-assisted check; we provide the proof of tractability.

Proof. It is easy to see that $BA_{Horn} = \Gamma(\xi^{-1}(\mathcal{T}_E))$, $BA_{point} = \Gamma(\xi^{-1}(\mathcal{T}_A))$, and $BA_{lin} = \Gamma(\xi^{-1}(\mathcal{T}_B))$, therefore it is possible to convert any instance of BA_{Horn}, BA_{lin} and BA_{point} to an instance of (respectively) $\mathcal{T}_E$, $\mathcal{T}_B$ and $\mathcal{T}_A$ via ξ, which operates in polynomial time. □

Tractability and PC-tractability. BA_{Horn}, BA_{point}, and BA_{lin} are tractable; their tractability, however, does not imply their PC-tractability. As a matter of fact, BA_{Horn} is PC-tractable, as shown in [3]. For the other two fragments, unfortunately, we only have partial results; in particular, the following holds:

Theorem 2. *The Path-Consistency algorithm is complete for checking the consistency of instances of BA_{Horn}. On the contrary, it is not complete for checking the consistency of instances of BA_{point}.*

The fact that the consistency of instances of BA_{Horn} can be checked by *Path-Consistency* has been shown in [3].

Proof. The fact that *Path-Consistency* is incomplete for checking the consistency of instances of BA_{point} can be shown by proving the existence of at least one inconsistent, but PC-consistent, instance. One such example is given in Fig. 3. □

Whether BA_{lin} instances can be checked by *Path-Consistency* or not is an open problem; extensive search for counterexamples gave negative results.

The known fragments of BA are reported in Fig. 4. As we can see, IA_{convex} and BA_{convex} are the smallest tractable fragments different from the set of basic relations only; as we have already observed, however, IA_{convex} is a strong algebra, while BA_{convex} is a weak algebra. Its strong closure, $\Gamma(BA_{convex})$, is slightly bigger. The importance of BA_{convex} lies in the fact that, besides being PC-tractable, the *Path-Consistency* is also complete for minimal labels. The PC-tractability of its strong closure, instead, is a mere consequence of the PC-tractability of BA_{Horn}. IA_{Horn} and its strong closure in BA are also, obviously, PC-tractable, as they are both subsets of BA_{Horn}. Finally, observe that BA_{point} is not included in BA_{Horn}, unlike its linear counterpart, and that BA_{lin} does not even have a linear counterpart, although it is a proper superset of IA_{Horn} (and its closure). While it cannot be seen in the figure, it is interesting to point out that BA_{lin} is the only set which does not contain any basic branching relation (like u or ibi), and if we try to extend it by adding any one of them, we always end up with a non tractable fragment of BA.

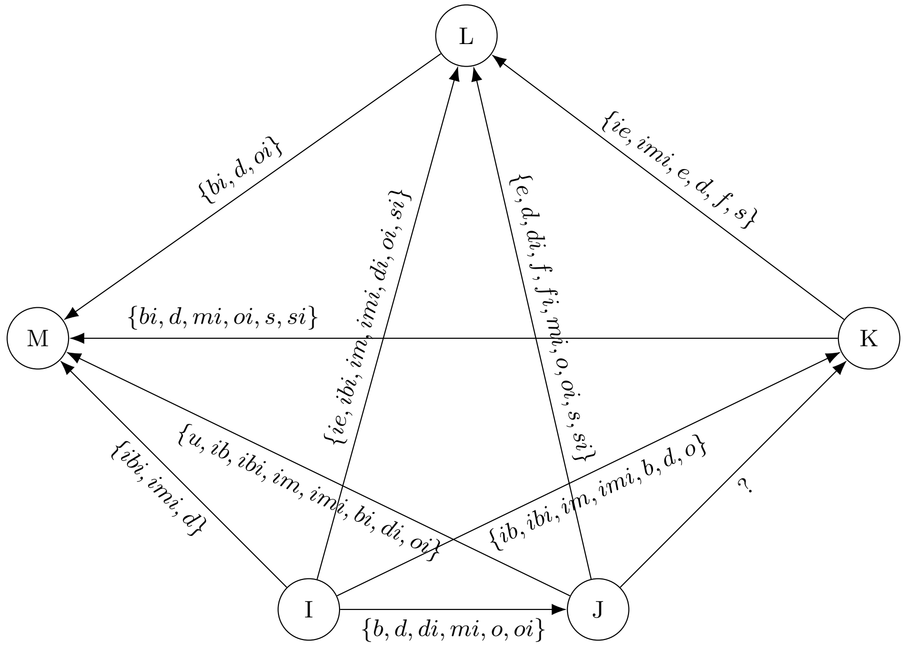

Fig. 3. A path-consistent, but not consistent, instance of BA_{point}.

4 Maximality Results

The landscape of tractable and PC-tractable fragments of the BA is still incomplete. Towards its completion, it is important to establish which tractable (resp., PC-tractable) fragment is also maximally so. In [6], and earlier in [13], the fundamental tool to check the (non) maximality of a certain tractable fragment in the linear case is the introduction of the so-called *corner sets*, i.e. small sets of relations which allow the construction of polynomial reduction of some NP-COMPLETE problem (usually 3-SAT and 3-COLOR). The list of corner sets emerge as a consequence of the systematic analysis of the fragments of the IA. So, in the linear case, it holds that, given a certain tractable fragment $\mathcal{S}$, it the case that $\mathcal{S}$ is maximal w.r.t. tractability if and only if every possible extension, that is, every set of the type $\Gamma(\mathcal{S} \cup \{R\})$ with $R \notin \mathcal{S}$, contains at least one of the corner sets. The complete list of corner sets [6] is:

$$\mathcal{N}_1 = \{\{b, di, fi, m, o\}, \{b, d, m, o, s\}, \{d, di, fi, oi, si\}\}$$
$$\mathcal{N}_2 = \{\{b, di, fi, m, o\}, \{b, d, m, o, s\}, \{di, fi, o, oi, si\}\}$$
$$\mathcal{N}_3 = \{\{b, bi\}, \{o, oi\}\}$$
$$\mathcal{N}_4 = \{\{b, bi\}, \{m, mi, o, oi\}\}$$
$$\mathcal{N}_5 = \{\{m, mi\}, \{b, bi, f, fi, s, si\}\}$$
$$\mathcal{N}_6 = \{\{b, bi, m, mi\}, \{o, oi\}\}$$

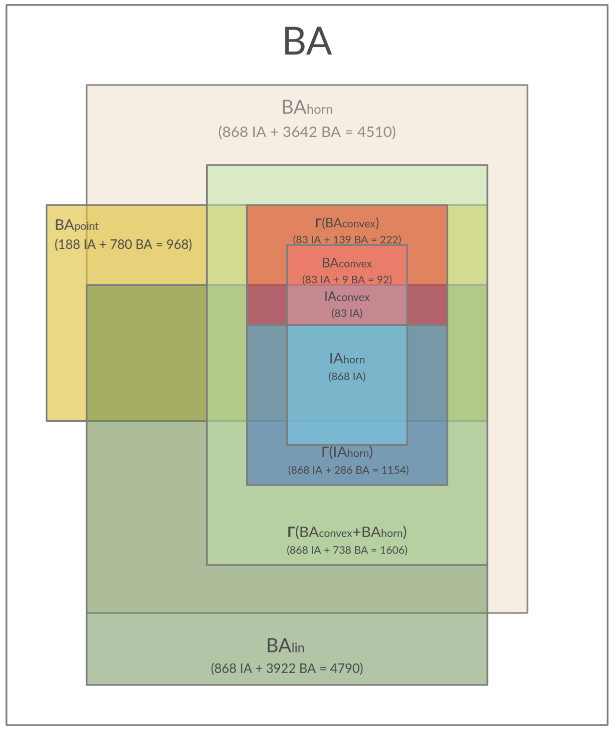

Fig. 4. An Euler-Venn diagram representation of the known tractable fragments in BA. For each set, we also show the number of relations it contains.

To study the maximality of tractable fragments in the branching algebra, the linear corner sets are a possible starting point. Indeed, if a fragment of the BA contains a linear corner set and the relation $\{e, b, bi, d, di, f, fi, m, mi, o, oi, s, si\}$ (which we denote as l), that is, a relation that constrains two intervals to be in some linear relation, then it is certainly not tractable: the reduction that proves NP-completeness is precisely the same as in the linear case, with the addition of the constraint $I\ l\ J$ for each pair I, J of intervals that are not explicitly constrained. Since BA_{Horn}, BA_{point} and BA_{lin} all contain l, this argument can be applied in all three subalgebras. On the other hand, the opposite does not necessarily hold: if a certain fragment of BA does not contain any corner set, then it is not necessarily tractable. Therefore, by applying the same strategy as in the linear case, two outcomes are possible: *(i)* if all the possible extensions of a tractable fragment $\mathcal{S}$ contain a linear corner set, then $\mathcal{S}$ is maximal w.r.t. tractability, and *(ii)* if there exists an extension $\mathcal{S}_R$ $(=\Gamma(\mathcal{S} \cup \{R\}))$, for some relation R that does not contain any corner set, then $\mathcal{S}_R$ is a new candidate for being tractable — in that case, it would also be a candidate to be maximally tractable. In the latter case, the process can be recursively applied, resulting

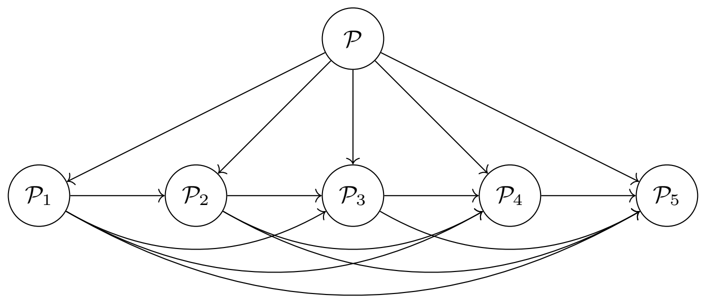

Fig. 5. Extension graph of BA_{point} (denoted, here, by $\mathcal{P}$).

in new, bigger subalgerbas, all potentially interesting, whose tractability is an open issue. We call *extension graph* of a certain fragment $\mathcal{S}$ a directed graph $G = \langle V, E\rangle$ that represents the set of algebras that one obtains by such a systematic search: vertexes represent algebras, and edges indicate that one set can be extended into another, and may be optionally labelled by a relation which brings such extension.

Theorem 3. *The following results hold:*

- BA_{Horn} *is maximally (*PC*-)tractable;*
- BA_{point} *has the extension graph shown in Fig. 5;*
- BA_{lin} *has the extension graph shown in Fig. 6.*

The result was obtained through computer-assisted enumeration. By computer-assisted enumeration, one can see that BA_{Horn} cannot be extended in any fragment that is not a superset of some corner set. Also one can see that the every extension of BA_{point} and BA_{lin} that is not the superset of any corner set is depicted in Fig. 5 and in Fig. 6. There are precisely five supersets of BA_{point} whose tractability is unknown, and nine supersets of BA_{lin}. While in the case of BA_{point} such supersets form a chain w.r.t. set containment, in the case of BA_{lin} the situation is more complex, with six supersets that form three chains, and three supersets formed by combinations of other supersets.

The maximality of both BA_{point} and BA_{lin} is therefore an open issue, as so are the PC-tractability of BA_{lin} and the (non) existence of some other tractable fragment of the BA.

One way in which the tractable fragments are exploited in the literature is to speedup the task of checking the consistency of a network (see, e.g., [12] for IA or [8] for BA). Instead of having a backtracking algorithm that assigns a basic relation to each edge in the network, the search can stop as soon as all the edges are assigned relations (possibly, non basic) that are in a fragment tractable with PC. For this task, BA_{lin} is very promising as it is the largest tractable known fragment; if it was PC-tractable, then it could be exploited to possibly obtain

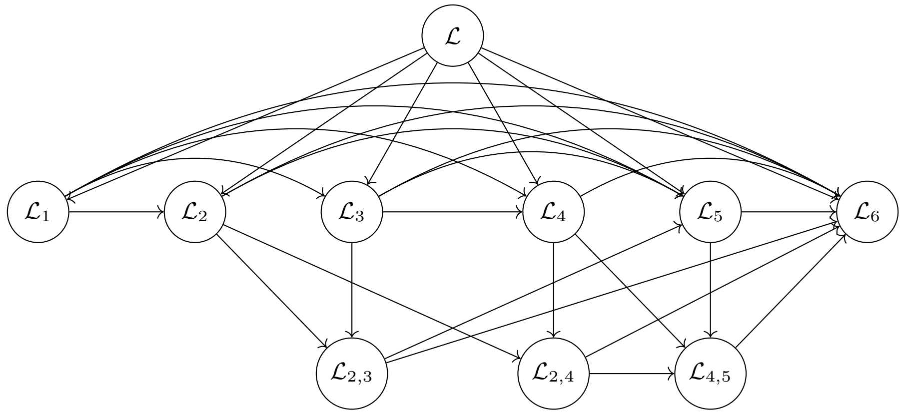

Fig. 6. Extension graph of BA_{lin} (denoted, here, by $\mathcal{L}$).

a higher speedup than that obtained using BA_{Horn} in [3]. However, as already said, the PC-tractability of BA_{lin} is an open issue.

5 Conclusions

The branching interval algebra (BA) is the tree order generalization of Allen's (linear) interval algebra (IA). Its potential applications include planning with alternatives, automatic story-telling with alternative timelines, and checking version systems with several branches. As in the linear case, the consistency problem of Branching Algebra is NP-COMPLETE, and studying its tractable fragments is a interesting problem. In the linear case, we know every tractable fragment of the full algebra, while in the branching case the entire landscape of tractable fragments is still unknown. In this paper we considered some of the results that are known in the linear case and the branching case in the point-based setting; by combining them we were able to add two new tractable fragments (BA_{point} and BA_{lin}) of the branching interval algebra to the one that was already known (BA_{Horn}). Also, we studied their maximality; we were able to prove BA_{Horn} is maximal w.r.t. tractability; as much as the maximality of BA_{point} and BA_{lin} is concerned, however, the problem is still open, although we proved some possibly useful partial results in this sense. Finally, we considered the problem of the tractability via *Path-Consistency* of these fragments, and proved that BA_{point}, while tractable, is not PC-tractable; the PC-tractability of BA_{Horn} was already known, and the PC-tractability of BA_{lin} is an open issue.

This paper is a stepping stone towards the complete classification of the fragments of the BA, which is, obviously, the main open problem at the moment. The techniques, and the algorithms, needed to perform this classification for an algebra with 19 relations (much bigger than the IA, with 13 relations only) can be certainly re-used for similar studies in other algebras, such as the rectangle algebra, and similar formalism for spatial-temporal reasoning.

References

1. Allen, J.: Maintaining knowledge about temporal intervals. Communications of the ACM **26**(11), 832–843 (1983)
2. Allen, J., Hayes, P.J.: Short time periods. In: Proc. of IJCAI 1987: 10th International Joint Conference on Artificial Intelligence. pp. 981–983 (1987)
3. Bertagnon, A., Gavanelli, M., Passantino, A., Sciavicco, G., Trevisani, S.: The Horn Fragment of Branching Algebra. In: Muñoz-Velasco, E., Ozaki, A., Theobald, M. (eds.) 27th International Symposium on Temporal Representation and Reasoning, TIME 2020, September 23-25, 2020, Bozen-Bolzano, Italy. LIPIcs, vol. 178, pp. 5:1–5:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020). https://doi.org/10.4230/LIPIcs.TIME.2020.5
4. Broxvall, M.: The point algebra for branching time revisited. In: Proc. of KI2001: Advances in Artificial Intelligence. Lecture Notes in Artificial Intelligence, vol. 2174, pp. 106–121. Springer (2001)
5. Condotta, J., D'Almeida, D., Lecoutre, C., Saïs, L.: From qualitative to discrete constraint networks. In: Proc. of KI 2006: Workshop on Qualitative Constraint Calculi. pp. 54–64 (2006)
6. Drakengren, T., Jonsson, P.: A complete classification of tractability in Allen's algebra relative to subsets of basic relations. In: Artificial Intelligence. pp. 205–219. Elsevier (1997)
7. Durhan, S., Sciavicco, G.: Allen-like theory of time for tree-like structures. Information and Computation **259**(3), 375–389 (2018)
8. Gavanelli, M., Passantino, A., Sciavicco, G.: Deciding the consistency of branching time interval networks. In: Proc. of TIME 2018: 25th International Symposium on Temporal Representation and Reasoning. LIPIcs, vol. 120, pp. 12:1–12:15 (2018)
9. Krokhin, A., Jeavons, P., Jonsson, P.: Reasoning about temporal relations: The tractable subalgebras of Allen's interval algebra. Journal of the ACM **50**(5), 591–640 (2003)
10. Ligozat, G.: A new proof of tractability for ORD-Horn relations. In: AAAI-96 Proceedings. pp. 395–401 (1996)
11. Mackworth, A.: Consistency in networks of relations. Artificial Intelligence **8**(1), 99–118 (1977)
12. Nebel, B.: Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-Horn class. Constraints **1**(3), 175–190 (1997)
13. Nebel, B., Bürckert, H.: Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra. Journal of the ACM **42**(1), 43–66 (1995)
14. Ragni, M., Wölfl, S.: Branching Allen. In: Proc. of ISCS 2004: 4th International Conference on Spatial Cognition. Lecture Notes in Computer Science, vol. 3343, pp. 323–343. Springer (2004)
15. Reich, A.: Intervals, points, and branching time. In: Proc. of TIME 1994: 9th International Symposium on Temporal Representation and Reasoning. pp. 121–133. IEEE (1994)
16. van Beek, P., Cohen, R.: Exact and approximate reasoning about temporal relations. Computational Intelligence **6**, 132–144 (1990)
17. Vilain, M.B., Kautz, H.: Constraint propagation algorithms for temporal reasoning. In: Proc. of AAAI 1986: 5th National Conference on Artificial Intelligence. pp. 377–382 (1986)