Finite Model Theory of the Triguarded Fragment and Related Logics (Extended Abstract)*

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Ever since first-order logic (FOL) was found to have an undecidable satisfiability problem, researchers have attempted to identify expressive yet decidable fragments of FOL and pinpoint their complexity. In many cases, such fragments embed propositional modal logic as well as many description logics. Two of the most prominent examples in this regard are FO^2 (the *two-variable fragment*) and GF (the *guarded fragment*). For FO^2, decidability is retained through reducing the number of available variables to 2, essentially restricting expressivity to independent pairwise interactions between domain elements. Its satisfiability problem is NExpTime-complete [5]. For GF, which owes its decidability to the restricted "guarded" use of quantifiers, the problem is 2ExpTime-complete [4].

Both FO^2 and GF possess the *finite model property* (FMP), meaning that any satisfiable sentence has a finite model. For satisfiable FO^2 sentences, models of at most exponential size in the sentence exist [5]; for GF, the tight bound on the size of minimal models is doubly exponential [1].

In an attempt to unify FO^2 and GF toward an even more expressive decidable FOL fragment, the *triguarded fragment* (TGF) was introduced [9], extending prior results [6]. TGF brings a new quality, as it allows one to express properties expressible in neither FO^2 nor GF. In particular it embeds Gödel's class, consisting of prenex sentences of the shape $\exists\bar{x}\forall y_1 y_2 \exists\bar{z}\varphi$ (formally we need to replace the variables from $\bar{x}$ by constants and add a dummy guard for $\exists\bar{z}$). Thus, the price to pay for retaining decidability is that equality needs to be disallowed, as Gödel's class with equality is undecidable [2]. Checking satisfiability of TGF is N2ExpTime-complete, dropping to 2ExpTime when disallowing constants – as opposed to FO^2 and GF, where presence or absence of constants does not make a difference, complexity-wise – and to NExpTime if the arity of predicates is bounded.

One central question left wide open in the original work on TGF [9] is if TGF has the FMP. In that paper, it is noted that neither technique used for establishing the FMP for FO^2 and GF seems to directly lend itself for solving the question for TGF, yet it is conjectured that the FMP holds. Indeed, one of our core contributions is to answer this open question in the positive. Let us briefly outline our approach.

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For convenience, we work with the equivalent logic GFU, *the guarded fragment with universal role*. We assume that signatures for GFU always contain the distinguished binary relation symbol U. GFU sentences are then defined precisely like GF sentences, but the set of admissible models is restricted to those which interpret U as the universally true relation. Structures interpreting U in this way will be called U-*biquitous structures*. It is not difficult to see that TGF and GFU have the same expressive power modulo the extra predicate U.

As typical for decidable fragments of first-order logic we introduce a normal form for TGF formulas, similar to those used, e.g., for GF [4] and FO^2 [5].

Given a satisfiable GFU normal form sentence φ, we take its (possibly infinite) U-biquitous model $\mathfrak{A}$ and construct a finite U-biquitous model $\mathfrak{A}'$ of φ as follows (for simplicity, we consider here the case without constants; adding them is routine):

1. Extend φ by conjuncts saying that: exactly the 1-types from $\mathfrak{A}$ are realized; for any two 1-types from $\mathfrak{A}$, there are U-connected representatives; U holds between any pair of elements co-occurring in any relation. The resulting normal form sentence φ^* is still guarded and still satisfies $\mathfrak{A} \models \varphi^*$.

2. Use the FMP for GF to obtain a finite (yet non-ubiquitous) model $\mathfrak{C}$ of φ^*.

3. Obtain $\mathfrak{A}_0 \models \varphi^*$ as $125 \cdot |C|^2$-fold disjoint union of $\mathfrak{C}$ with itself. View $\mathfrak{A}_0$ as a $5|C| \times 5|C|$ table whose each cell contains a copy of the 5-fold disjoint union of $\mathfrak{C}$ with itself. The elements in each cell are numbered from 1 to $5|C|$.

4. U-saturation: Obtain $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ by iteratively picking a pair a, b of yet non-U-connected elements, connecting them, and adjoining them to one of the copies of $\mathfrak{C}$. This is done using an appropriate pair of connected elements as template (hence maintaining φ^*-modelhood). Designing a strategy allowing one to perform this step without conflicts is quite challenging. In our solution, the numbers of a and b in their cells $\mathfrak{B}'$, $\mathfrak{B}''$ determine a cell $\mathfrak{B}$, and the coordinates of $\mathfrak{B}'$ and $\mathfrak{B}''$ in the table are used to choose a particular copy of $\mathfrak{C}$ in $\mathfrak{B}$ to which a and b are adjoined.

5. As the number of elements remains constant, the procedure terminates and yields a U-biquitous $\mathfrak{A}_n = \mathfrak{A}'$.

We also consider a scenario where some distinguished binary symbols have to be interpreted as transitive relations, capturing this way, e.g., some description logics from the family $\mathcal{S}$. One needs to be careful here, since both FO^2 and GF become undecidable under this scenario [3,4]. However, the decidability of GF can be regained if the transitive symbols are allowed to occur only as guards [10] (note that this is sufficient to encode the logic $\mathcal{S}$). The same holds for the corresponding extension of TGF [7].

Results for the finite model case are less extensive: so far, only finite satisfiability of the two-variable variant GF^2+TG of GF with transitive guards was shown to be decidable and 2ExpTime-complete [8]. We note that already this logic does not have the FMP: indeed, a typical infinity axiom saying that, for a transitive relation T, every element has a T-successor but is not related by T to itself is naturally expressible in GF^2+TG. We remark that all the results concerning logics with transitive guards assume the absence of constants. It is conjectured that adding constants to the picture is technically challenging but generally possible without hazarding decidability.

In the current paper we are able to show that (at least in the absence of constants) the *finite* satisfiability problems for GF and TGF with transitive guards are decidable. Our approach incorporates some ideas from the above-described finite model construction for satisfiable TGF sentences and some other concepts, in particular calling as a subprocedure the small-model construction for GF^2+TG from [8].

All our results come with tight complexity bounds and tight bounds on the size of minimal models. Summarising:

Theorem 1. TGF *has the finite model property; every satisfiable TGF sentence has a finite model of size bounded doubly exponentially in its length. Hence, satisfiability and finite satisfiability for* TGF *coincide and are* N2ExpTime-*complete if constants are admitted and 2*ExpTime-*complete otherwise.*

Theorem 2. *Every finitely satisfiable sentence in (constant-free)* GF *or* TGF *with transitive guards has a model of size bounded doubly exponentially in its length. The finite satisfiability problems for (constant-free)* GF *and* TGF *with transitive guards are 2*ExpTime-*complete.*

Decidability and complexity of finite satisfiability of GF and TGF with transitive guards with constant is left open.

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