FCA Went (Multi-)Relational, But Does It Make Any Difference?

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Abstract. Relational Concept Analysis (RCA) was designed as an extension of Formal Concept Analysis (FCA) to multi-relational datasets, such as the ones drawn from Linked Open Data (LOD) by the type-wise grouping of the resource into data tables. RCA has been successfully applied to practical problems of AI such as knowledge elicitation, knowledge discovery from data and knowledge structuring. A crucial question, yet to be answered in a rigorous manner, is to what extent RCA is a true extension of FCA, i.e. reveals concepts that are beyond the reach of core FCA even using a suitable encoding of the original data. We show here that the extension is effective: RCA retrieves all concepts found by FCA as well as many further ones.

Keywords: Multi-relational Data · Formal Concept Analysis · RDF · Propositionalization.

1 Introduction

FCA provides the mathematical framework for several Knowledge Discovery in Databases (KDD) tasks whenever the data is purely, or at least predominantly, of categorical nature. Indeed, FCA-based association discovery and conceptual clustering have been applied to knowledge base structuring, ontology learning, anomaly detection, observation classification, etc. Most real datasets, though, stray from being purely categorical. FCA thus provides a set of scaling operators to deal with numerical and otherwise ordered scales. In AI, the majority of interesting data, such as those compatible with the LOD format, have relational structure. They can be represented either as graphs (for instance, named graphs in RDF) or as sets of relational tables. Approaches have been designed for the former, emphasizing the intra-data object links, e.g. *logical FCA* [7] and *pattern structures* [8], for graph datasets. For the latter, the focus is on inter-object links, e.g. in datasets structured as a unique RDF graph. In this second trend, more akin to *power-context families* [15] and *Graph-FCA* [6], we focus on the particular approach of relational concept analysis (RCA). It has already been successfully applied to a wide range of practical problems such as hydroecology [4], industrial decision making [12] or biology [1,13]. Rather than in a global graph, RCA shapes the data as a set of ×-tables, complying to the *Entity-Relationship* framework [2].

Part of the tables have the classical objects × properties format (entity types, FCA contexts) while the remainder represent objects × objects relations.

A natural question is whether RCA does extend the reach of FCA, knowing that for single datasets, whatever the level of complexity of the object descriptions (sequences, trees, graphs), the results of an FCA-based processing on those descriptions can be brought down to FCA on a context made of suitably-chosen derived **attributes**. The question is all the more important as prior studies seem to imply it does not [3] (though, for a reduced version of RCA). We make the case here for RCA as a true extension of FCA, in the sense that due to its multi-relational input and fixed point computation, it detects concepts that are out of reach for FCA while, in turn, retrieving all concepts that FCA is able to reveal. To that end, we chose some plausible re-encodings of a simple relational context family (RCF), the hypothesis being that with more complex datasets, the phenomenon only amplifies.

The remainder of the paper is as follows: Section 2 provides background on RCA while Section 3 presents our FCA-vs-RCA comparison. Next, Section 4 discusses the comparison outcome and Section 5 concludes.

2 Background

Formal concept analysis [14] is a mathematical method for eliciting the conceptual structure of "object × attribute" datasets. Data are gathered within a *(formal) context*, a triple *K = (O, A, I)* where O is a set of objects, A is a set of attributes and $I \subseteq O \times A$ is the context incidence relation, where $(o, a) \in I$, also written oIa, means that the object o bears the attribute a. A context induces two derivation operators: one mapping objects to attributes, and the reciprocal. The object derivation $'$ maps a subset X of objects to the set of attributes shared by all members of X, $' : \wp(O) \longrightarrow \wp(A)$ with $' : X \mapsto \{a \in A \mid oIa\ \forall o \in X\}$. The dual attribute derivation, also denoted by $'$, works the other way around, $' : \wp(A) \longrightarrow \wp(O)$ with $' : Y \mapsto \{o \in O \mid oIa\ \forall a \in Y\}$. Inside a context K, a *(formal) concept* is a pair $(X, Y) \subseteq O \times A$ such that $X' = Y$ and $Y' = X$. The sets X and Y are called *extent* and *intent* of the concept (X, Y), respectively.

FCA extracts conceptual abstractions on objects by factoring out shared attributes. *Relational concept analysis* [10] extends it by factoring in relational information, as available in multi-relational datasets [5]. RCA admits multiple sorts of objects in its input format, each organized as a separate context, plus a set of binary relations between contexts. The input data structure, called *relational context family* (RCF), is thus a pair $(\mathbf{K}, \mathbf{R})$ where $\mathbf{K} = \{K_i\}_{i=1,\ldots,n}$ is a set of distinct contexts $K_i = (O_i, A_i, I_i)$ and $\mathbf{R} = \{r_k\}_{k=1,\ldots,m}$ a set of binary relations $r_k \subseteq O_i \times O_j$ where the contexts K_i and K_j are the domain and range contexts of r_k, respectively. Relational tables are also processed in their own way, as explained below. A cross in the table of relation r for $(domain\_object_i, range\_object_j)$, can be understood as the first order logic term $r(domain\_object_i, range\_object_j)$ being true.

RCA distills the shared relational information (i.e., inter-object links) using *propositionalization* [9]: It integrates new attributes into an extended version of the initial context, say $K_d = (O_d, A_d, I_d)$, to further refine the conceptual structure of the underlying object set. To increase shareability, rather than the individual objects from the target (range) context, say K_t, the new attributes refer to abstractions on them. In its most basic version, RCA exploits the natural conceptual structure provided by the concepts of each context. Indeed, two links of relation $r : d \longrightarrow t$ departing from o_1 and o_2 from O_d and referring to two distinct objects $\bar{o}_1$ and $\bar{o}_2$ from O_t, respectively, are distinct information. However, replacing $\bar{o}_1$ and $\bar{o}_2$ with a common abstraction, say $\{\bar{o}_1, \bar{o}_2\}'$, makes the new information shareable. Relational scaling follows a well-known schema from description logics: Given a relation r, for each concept c_t from the range context of r, it produces, for A_d, an attribute $q\,r : c_r$ where q is an operator chosen before-hand from a set Q. RCA admits, among others, standard description logics restrictions ($Q = \{\exists, \forall, \forall\exists, \ldots\}$), which behold their respective semantics (see [10] for details and example 1 for illustration).

Example 1. Assume a RCF made of contexts on *people* and *cars*, and an ownership relation, or *pos(sesses)*, which are given in Tables 1, 3 and 2, respectively. The cars lattice is shown in Figure 1. Now, an $\exists$-scaling of the relation *pos* using that lattice will add, for each car concept c, a new attribute $\exists pos : c$ to the person context, e.g. $\exists pos : cars\_4$ and $\exists pos : cars\_3$ that can be rewritten as $\exists pos : (cp)$ and $\exists pos : (el, pw)$, respectively, using intents as IDs.

K_P	Senior	Adult	Male	Female	I.T.	Sport
Fa	×		×			
La		×		×	×	
Sh		×		×		
Tr		×	×		×	×

Table 1: Person Context

pos	tw	$t3$	zo	$f5$
Fa	×			
La		×		×
Sh	×			×
Tr				

Table 2: Relation pos

K_C	el	pw	cp	ch
tw			×	×
$t3$	×	×		
zo	×		×	
$f5$		×	×	

Table 3: Car Context

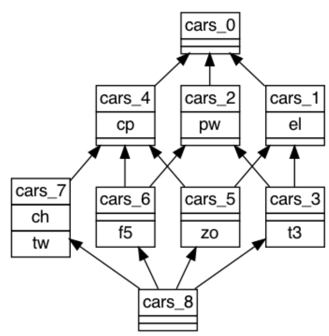

Fig. 1: Car Lattice

A scaling step results in the related contexts being extended, which, in turn, may lead to the emergence of new concepts. Thus, as the set of available abstractions increases, a scaling step with the differential set of concepts would produce further relational attributes and the whole process would go on cycling. The resulting iterative context refinement necessarily ends at a fixed point [10], i.e. a set of lattices whose concepts refer to each other via relational attributes.

3 What can RCA do for AI (that FCA can't) ?

Below, we examine two encoding strategies that bring a multi-relational dataset to a mono-relational one, i.e. aggregate several contexts into a single data table, so that they can be fed to classical FCA.

3.1 Encoding multiple contexts into a single one

Assume a simple RCF made of two contexts and a relation (see Figure 2). We use this simple case for our reasoning, knowing that in more complex cases, i.e. three or more contexts and several relations, it can be extended appropriately. Moreover, while there could be a wide range of concrete encoding disciplines [11], the principle behind them admits only two basic cases, i.e. entity-centric and relation-centric. In our FCA/RCA perspective this boils down to which sort of RCF element, i.e. context or relation, is put center-stage.

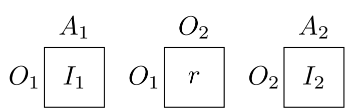

Fig. 2: Fictitious RCF

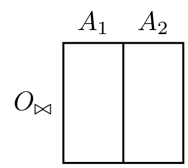

Fig. 3: Semi-join

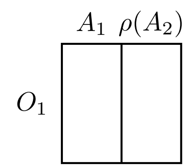

Fig. 4: Aggregation

The first encoding principle we examine below emphasizes the object-to-object relation as a primary construct and pivotal element of the encoding. Its member pairs become first-class objects which carry the attributes of both contexts incident to the relation. Technically speaking, the method is akin to the *(semi-)join* operation of relational algebra. The overall encoding schema is illustrated in Figure 3 whereas Section 3.2 proposes a formal definition thereof. It also provides a detailed comparison of the results from applying RCA to the RCF from Figure 2 with those of FCA on the semi-join of the two initial contexts.

A bit closer to the RCA propositionalization spirit, a second encoding principle emphasizes the context as a main construct and driver of the encoding: The domain context of the relation is extended with some additional attributes that translate the relation while following a technique akin to relational scaling. The main difference here is that the context is the one-shot context extension. The procedure whose details are discussed in Section 3.3, is schematically illustrated in Figure 4. Moreover, the asymmetric encoding of the relation and the one-shot extension amount to processing the range context as if it were aggregated into the domain one. Therefore, we termed the overall encoding principle the *aggregation* and the resulting context the aggregated one.

Finally, please notice that in the detailed investigation of each case (see below), our reasoning follows three steps: 1) We pick an arbitrary formal concept from the FCA output, 2) we show the RCA output comprises a concept with the same objects, and 3) we establish the link between the intents of both concepts.

3.2 Semi-join in single relation RCF

We consider here the concurrent case where the FCA is applied on a context encoding the semi-join of this RCF, as presented in Figure 3. This encoding consists in creating the objects of $O_{\bowtie}$ as the object pairs (o_1, o_2) where $o_1 \in O_1 \cup \{\bot\}$, $o_2 \in O_2 \cup \{\bot\}$, according to the RCF modeling of O_1, O_2 Figure 2. The $\bot$ object is a fictitious empty object with no attributes used to complete the semi-join. There are three cases to define the elements of $O_{\bowtie}$:

- If $o_1 \in O_1$, $o_2 \in O_2$, then $(o_1, o_2) \in O_{\bowtie}$ if and only if $(o_1, o_2) \in r$
- If $o_1 = \bot$, $o_2 \in O_2$, then $(o_1, o_2) \in O_{\bowtie}$ if and only if $r^{-1}(o_2) = \emptyset$, i.e. if there is no $x \in O_1$ such that $(x, o_2) \in r$
- If $o_1 \in O_1$, $o_2 = \bot$, then $(o_1, o_2) \in O_{\bowtie}$ if and only if $r(o_1) = \emptyset$, i.e. if there is no $x \in O_2$ such that $(o_1, x) \in r$

Example 2. As an illustration of the above modeling, assume an RCF made of contexts for people (Table 1) and for cars (Table 3) plus an ownership relation (*pos*session, Table 2). In the first context, *Farley, Lane, Shana,* and *Trudy* are described by being *senior* or *adult*, *male* or *female*, working in *IT*, and practicing a lot of *sports*. Cars –Twingo, Tesla 3, Zoe, and Fiat 500– can be electrical, powerful, compact or (not exclusive) cheap. The corresponding semi-join context is presented in Table 4.

$K_{\bowtie}$	Senior	Adult	Male	Female	I.T.	Sport	*el*	*pw*	*cp*	*ch*
(Fa,tw)	×		×						×	×
(La,$t3$)		×		×	×		×	×		
(La,$f5$)		×		×	×			×	×	
(Sh,tw)		×		×					×	×
(Sh,$f5$)		×		×				×	×	
(Tr,$\bot$)		×	×		×	×				
($\bot$,zo)							×		×	

Table 4: Semi-join context of Example 2 RCF

To avoid ambiguity, we consider the derivations in the K_1 and K_2 contexts always denoted x', while the derivation in the join context is denoted x^{∇} (and the double derivation $x^{\nabla\nabla}$).

We are first interested in describing a formal concept of the joined context. Let $X \subseteq O_{\bowtie}$ be a set of objects. So, for all $(o_1, o_2) \in X$ we have $o_1 \in O_1 \cup \{\bot\}$ and $o_2 \in O_2 \cup \{\bot\}$. Thus, by definition, $C = (X^{\nabla\nabla}, X^{\nabla})$ is a formal concept. Let us now take the projections on the first and second elements of the pairs of $X^{\nabla\nabla}$, i.e. $\pi_1 = \{o_1 \mid \exists o_2, (o_1, o_2) \in X^{\nabla\nabla}\}$ and $\pi_2 = \{o_2 \mid \exists o_1, (o_1, o_2) \in X^{\nabla\nabla}\}$. We start by defining $X^{\nabla\nabla}$ in terms of these projections.

Lemma 1 *We have* $X^{\nabla\nabla} = (\pi_1 \times \pi_2) \cap O_{\bowtie}$

Proof. Let $(u, v) \in X^{\nabla\nabla}$. By definition $X^{\nabla\nabla} \subseteq O_{\bowtie}$, so $(u, v) \in O_{\bowtie}$. Moreover, by construction $u \in \pi_1$ and $v \in \pi_2$ so $(u, v) \in \pi_1 \times \pi_2$. Thus, $X^{\nabla\nabla} \subseteq (\pi_1 \times \pi_2) \cap O_{\bowtie}$.

Let $(u, v) \in (\pi_1 \times \pi_2) \cap O_{\bowtie}$. Since $(u, v) \in (\pi_1 \times \pi_2)$, it exists $\tilde{u}$ and $\tilde{v}$ s.t. $(u, \tilde{u}) \in X^{\nabla\nabla}$ and $(\tilde{v}, v) \in X^{\nabla\nabla}$. But, by construction $\{(u, \tilde{u}), (\tilde{v}, v)\}^{\nabla} \subseteq u' \cup v'$. And, since $(u, v) \in O_{\bowtie}$ we can write $(u, v)^{\nabla} = u' \cup v'$. Thus, by derivation property we have $X^{\nabla\nabla\nabla} \subseteq \{(u, \tilde{u}), (\tilde{v}, v)\}^{\nabla}$, by transitivity $X^{\nabla\nabla\nabla} \subseteq (u, v)^{\nabla}$. Thus, by deriving this expression we obtain $(u, v)^{\nabla\nabla} \subseteq X^{\nabla\nabla\nabla\nabla}$. Finally, as $(u, v) \in (u, v)^{\nabla\nabla}$ and $X^{\nabla\nabla\nabla\nabla} = X^{\nabla\nabla}$ we have $(u, v) \in X^{\nabla\nabla}$. □

We first study the particular cases containing the object $\perp$ by starting with the case where this element appears in both projections.

Proposition 1 *If* $\perp \in \pi_1$ *and* $\perp \in \pi_2$ *then* $X^{\nabla} = \emptyset$ *and* $X^{\nabla\nabla} = O_{\bowtie}$

Proof. Suppose $\perp \in \pi_1$ and $\perp \in \pi_2$ then by definition of $\perp$ we have $X^{\nabla} \cap A_1 = \emptyset$ and $X^{\nabla} \cap A_2 = \emptyset$ and therefore $X^{\nabla} = \emptyset$. By definition of the derivation we have $\emptyset^{\nabla} = O_{\bowtie}$ therefore $X^{\nabla\nabla} = O_{\bowtie}$.The second assertion holds by symmetry. □

In the case described by the lemma 1, it is immediate to show that we can construct $(X^{\nabla\nabla}, X^{\nabla})$. We show that the same is true when only one of the components π_1 or π_2 contains $\perp$ by first describing X^{∇} then $X^{\nabla\nabla}$ in the lemmas 2 and 3.

Lemma 2 *If* $\perp \in \pi_1$ *and* $\perp \notin \pi_2$*,* $X^{\nabla} = \pi'_2$*. If* $\perp \in \pi_2$ *and* $\perp \notin \pi_1$*,* $X^{\nabla} = \pi'_1$*.*

Proof. Let us suppose $\perp \in \pi_1$ and $\perp \notin \pi_2$. We have $a \in X^{\nabla}$ iff $X^{\nabla\nabla} \subseteq a^{\nabla}$. Yet, since $\perp \in \pi_1$ we have construction $X^{\nabla} \cap A_1 = \emptyset$. thus, we have $a \in A_2$.therefore, we have $a \in X^{\nabla}$ iff for all $(o_1, o_2) \in X^{\nabla\nabla}$ o_2 carries the attribute a, i.e. $a \in \pi'_2$. Since we have $a \in X^{\nabla}$ iff $a \in \pi'_2$, we have $X^{\nabla} = \pi'_2$. We show the second assertion symmetrically. □

Lemma 3 *If* $\perp \in \pi_1$ *and* $\perp \notin \pi_2$*,* $X^{\nabla\nabla} = (O_1 \cup \{\perp\} \times \pi_2) \cap O_{\bowtie}$*. If* $\perp \in \pi_2$ *and* $\perp \notin \pi_1$*,* $X^{\nabla\nabla} = (\pi_1 \times O_2 \cup \{\perp\}) \cap O_{\bowtie}$*.*

Proof. Let us suppose $\perp \in \pi_1$ and $\perp \notin \pi_2$. Let $v \in \pi_2$. Any pair $(u, v) \in O_{\bowtie}$ verifies $\pi'_2 \subseteq (u, v)^{\nabla}$. Since, by the lemma 2, we have $X^{\nabla} = \pi'_2$, we can write $X^{\nabla} \subseteq (u, v)^{\nabla}$ and thus, by derivation $(u, v)^{\nabla\nabla} \subseteq X^{\nabla\nabla}$. Finally, for any $u \in O_1 \cup \{\perp\}$, we have $(u, v) \in X^{\nabla\nabla}$ donc $X^{\nabla\nabla} = (O_1 \cup \{\perp\} \times \pi_2) \cap O_{\bowtie}$. We show the second assertion symmetrically. □

The lemmas 2 and 3 allow us to determine that in cases where only one of the projections contains $\perp$ we can write a formal concept of $K_{\bowtie}$ only with the other projection. Let us now study a formal concept based on this projection determined by RCA.

Lemma 4 *If $\bot \in \pi_1$ and $\bot \notin \pi_2$, there exists a concept $C_2 = (\pi_2, \pi_2')$ on K_2. If $\bot \in \pi_2$ and $\bot \notin \pi_1$, there exists a concept $C_1 = (\pi_1, \pi_1')$ on K_1.*

Proof. Let us suppose $\bot \in \pi_1$ and $\bot \notin \pi_2$. Since $\pi_2 \subseteq O_2$, (π_2'', π_2') is a concept son K_2. It is therefore sufficient to show that $\pi_2'' = \pi_2$, or more simply $\pi_2'' \subseteq \pi_2$. Let $o \in \pi_2''$. By construction at least one couple $(\bar{o}, o) \in O_{\bowtie}$ and $o' \subseteq (\bar{o}, o)^{\nabla}$. Now, we have $o \in \pi_2''$ so by derivation, $\pi_2' \subseteq o'$. Moreover, by the lemma 2, we have $X^{\nabla} = \pi_2'$. Thus, $X^{\nabla} \subseteq (\bar{o}, o)^{\nabla}$ so by derivation, $(\bar{o}, o) \in X^{\nabla\nabla}$. Finally, by definition of the projections $o \in \pi_2$. The second assertion holds by symmetry. □

The following proposition gathers the previous lemmas. It emphasizes that, in the case where only one of the two projections contains $\bot$, any concept of $K_{\bowtie}$ can be expressed with the other projection. Moreover, there exists a concept generated by RCA, of the same intent and whose extent corresponds to a projection of the extent of the concept generated by FCA.

Proposition 2 *Let $C = (X^{\nabla\nabla}, X^{\nabla})$. If $\bot \in \pi_1$ and $\bot \notin \pi_2$, $C = ((O_1 \cup \{\bot\} \times \pi_2) \cap O_{\bowtie}, \pi_2')$ and there exists a corresponding concept $C_2 = (\pi_2, \pi_2')$ on K_2. If $\bot \in \pi_2$ and $\bot \notin \pi_1$, $C = ((\pi_1 \times O_2 \cup \{\bot\}) \cap O_{\bowtie}, \pi_1')$ and there exists a corresponding concept $C_1 = (\pi_1, \pi_1')$ on K_1.*

Proof. Follows from the lemmas 2, 3 and 4. □

There remains a specific case, described by the lemma 5, to complete the exhaustive description of a formal concept on the join table.

Lemma 5 *For any $X \subseteq O_1 \times O_2$ we have $X^{\nabla} = \pi_1' \cup \pi_2'$ and $X^{\nabla\nabla} = \{(o_1, o_2) \mid \pi_1' \subseteq o_1' \land \pi_2' \subseteq o_2'\}$*

Proof. Let us show $X^{\nabla} = \pi_1' \cup \pi_2'$ by double inclusion.

(i) $X^{\nabla} \subseteq \pi_1' \cup \pi_2'$.

The RCF modeling assures us that $A_1 \cap A_2 = \emptyset$. Thus, an attribute $a \in X^{\nabla}$ is either in A_1 or in A_2. If $a \in X^{\nabla} \cap A_1$, it must be shared by all the elements of π_1; and so a is in π_1'. Similarly, if a is in $X^{\nabla} \cap A_2$, $a \in \pi_2'$. We deduce that $X^{\nabla} \subseteq \pi_1' \cup \pi_2'$.

(ii) $\pi_1' \cup \pi_2' \subseteq X^{\nabla}$.

On the other hand, if an attribute a is in π_1', then any pair of $X^{\nabla\nabla}$ has a first component that carries the attribute a. Since this property is true for any pair of $X^{\nabla\nabla}$ and $X \subseteq X^{\nabla\nabla}$, then any pair of X carries the attribute a. Therefore, we have $a \in X^{\nabla}$. In the same way, we show that if $a \in \pi_2'$, then $a \in X^{\nabla}$. thus, we have $\pi_1' \subseteq X^{\nabla}$ and $\pi_2' \subseteq X^{\nabla}$. We can therefore affirm that $\pi_1' \cup \pi_2' \subseteq X^{\nabla}$.

Finally, by (i) and (ii) we have $X^{\nabla} = \pi_1' \cup \pi_2'$. As $X^{\nabla\nabla}$ describes exactly the set of couples (o_1, o_2) having the attributes of $\pi_1' \cup \pi_2'$, by construction of the join table we have $\pi_1' \subseteq o_1'$ and $\pi_2' \subseteq o_2'$. □

The cases described by the lemmas 1 and 2 allow for the immediate selection of concepts from the RCA process corresponding in terms of extent to a concept in the join table. The proposition 3 relies on the lemma 5 to state the main result of this subsection, dealing with non-degenerate cases (without $\perp$ element).

Proposition 3 *Let $X \subseteq O_1 \times O_2$. There exists by RCA on K_1 a concept (X_1, Y_1) such that $X_1 = \pi_1$ and $\pi_1' \subseteq Y_1$ and there exists on K_2 a concept (X_2, Y_2) such that $X_2 = \pi_2$ and $\pi_2' \subseteq Y_2$.*

Proof. As $\pi_1' \subseteq A_1$ and $\pi_2' \subseteq A_2$, $C_1 = (\pi_1'', \pi_1')$ and $C_2 = (\pi_2'', \pi_2')$ are formal concepts on their respective contexts computed at step 0 of RCA.

Let us consider the contexts K_1 and K_2 after graduation by the operator $\exists$ on the relations r and r^{-1}. We then have the attributes $\exists r : C_2$ in K_1 and $\exists r^{-1} : C_1$ in K_2. We define the sets of attributes $Y_1 = \pi_1' \cup \{\exists r : C_2\}$ and $Y_2 = \pi_2' \cup \{\exists r^{-1} : C_1\}$ as well as the concepts $C_3 = (Y_1', Y_1'')$ and $C_4 = (Y_2', Y_2'')$ (it is possible that $C_1 = C_3$ or $C_2 = C_4$). We have $Y_1' = \pi_1'' \cap \{\exists r : C_2\}'$, let us show that $Y_1' = \pi_1$ by double inclusion.

Let $o \in \pi_1$, we have $o \in \pi_1''$. Moreover, by construction, any pair of $(o, \bar{o}) \in X^{\nabla\nabla}$ verifies $(o, \bar{o}) \in r$ with $\bar{o} \in \pi_2$ and, by hypothesis, $\bar{o} \neq \perp$. Thus, since $\pi_2 \subseteq \pi_2''$, o carries the attribute $\exists r : C_2$. Thus, we have $\pi_1 \subseteq Y_1'$.

Let $o \in Y_1'$. We have $\pi_1' \cup \{\exists r : C_2\} \subseteq o'$. Since $\{\exists r : C_2\} \subseteq o'$, there exists $\bar{o} \in \pi_2''$ such that $(o, \bar{o}) \in r$ and thus $(o, \bar{o}) \in O_{\bowtie}$. Moreover, since $\bar{o} \in \pi_2''$, we have $\pi_2' \subseteq \bar{o}'$. Since $\pi_2' \subseteq \bar{o}'$, $\pi_1' \subseteq o'$ and that by the lemma 5 we have $X^{\nabla} = \pi_1' \cup \pi_2'$, we can affirm $X^{\nabla} \subseteq (o, \bar{o})^{\nabla}$. Finally $(o, \bar{o})^{\nabla\nabla} \subseteq X^{\nabla\nabla}$ and by definition of π_1, on a $o \in \pi_1$ (In a completely analogous way, we show $\pi_2 = Y_2'$).

Finally, we have shown the existence of $C_3 = (Y_1', Y_1'')$ such that $Y_1' = \pi_1$ and $\pi_1 \subseteq Y_1''$ as well as of $C_4 = (Y_2', Y_2'')$ such that $Y_2' = \pi_2$ and $\pi_2 \subseteq Y_2''$. □

In conclusion, the propositions 1, 2 and 3 show that for any concept $C = (X^{\nabla\nabla}, X^{\nabla})$ we find on K_1 a concept (X_1, Y_1) such that $X_1 = \pi_1$ and $\pi_1' \subseteq Y_1$ and there exists on K_2 a concept (X_2, Y_2) such that $X_2 = \pi_2$ and $\pi_2' \subseteq Y_2$. It is to note that if $\perp \in \pi_1$ (respectively π_2) we have $\pi_1 = \{x \mid \exists y, (x, y) \in O_{\bowtie}\}$ (respectively $\pi_2 = \{x \mid \exists x, (x, y) \in O_{\bowtie}\}$). Example 3 illustrates these properties.

Example 3. Let us consider the relational family as well as the semi-join context defined in the Example 2.

On the joined context, we find the concept $C = (\{(La, t3), (La, f5)\}, \{\text{Adult}, \text{Female}, \text{IT}, pw\})$. Here, $\pi_1 = \{La\}$ and $\pi_2 = \{t3, f5\}$. We check that there exists on K_P a concept (π_1, π_1'), namely the concept $C_1 = (\{La\}, \{Adult, Female, IT\})$, and on K_C a concept (π_2, π_2'), the concept $C_2 = (\{t3, f5\}, \{pw\})$. After an iteration, RCA extends these concepts' intents to $\{Adult, Female, IT, \exists pos : C_2\}$ and $\{pw, \exists pos^{-1} : C_1\}$, respectively.

3.3 Aggregation operation in mono-relational case

Assume again the RCF in Figure 2 and let us consider FCA is applied on the context schematically visualized in Figure 4. Intuitively, this amounts to extending the domain context of the relation by appending some new attributes. These are derived from the range context attributes by a technique akin to relational scaling, i.e. one basically simulating a one-shot RCA-like context refinement.

Formally speaking, we design the context $K_{\lessdot} = (O_{\lessdot}, A_{\lessdot}, I_{\lessdot})$ where $O_{\lessdot} = O_1$, $A_{\lessdot} = A_1 \cup \rho(A_2)$, and $\rho(A_2)$ are the attributes resulting from the application of the scaling operator ρ to the attribute concept of $a \in A_2$. We will denote such an attribute $\overline{\rho r : a}$ to avoid confusion with RCA's own relational attributes. Notice that $A_1 \cap \rho(A_2) = \emptyset$ holds. Next, we introduce $Constraint(\rho, r, o_p, (X, Y))$, a predicate verifying whether o_p and (X, Y), from the domain and the range of r, respectively, jointly comply to the semantic of ρ. Thus, $Constraint(\forall, r, o_p, (X, Y))$ is *true* iff $r(o_p) \subseteq X$. The predicate is a compact expression of the incidence $I_{\lessdot}$:

- if $a_p \in A_1$ then $(o_p, a_p) \in I_{\lessdot}$ iff $(o_p, a_p) \in I_1$,
- if $a_p \in \rho(A_2)$ then $(o_p, a_p) \in I_{\lessdot}$ iff $Constraint(\rho, r, o_p, (a'_p, a''_p))$ is true.

Example 4 illustrates the $\forall\exists$ case (reasoning with other operators is similar). For the O_2 perspective, it is enough to swap K_1 and K_2 and replace r by r^{-1}.

Example 4. Consider again the RCF in Example 2. We aggregate the family via $\forall\exists$: For an $o \in O_P$ and $a \in A_C$ s.t. $\overline{\forall\exists pos : a} \in \rho(A_C)$ it holds $(o, \overline{\forall\exists pos : a}) \in I$ iff 1) $\exists o_C \in O_C$ s.t. $(o, o_C) \in pos$ and 2) $\forall o_C \in O_C$, $(o, o_C) \in pos$ entails $o_v \in a'$ (there is at least one image of o by pos and all such images carry a. Table 5 depicts the resulting aggregated context $K_{\lessdot}$.

$K_{\lessdot}$	Senior	Adult	Male	Female	I.T.	Sport	$\overline{\forall\exists pos : el}$	$\overline{\forall\exists pos : pu}$	$\overline{\forall\exists pos : cp}$	$\overline{\forall\exists pos : ch}$
Fa	×		×						×	×
La		×		×	×			×		
Sh		×		×					×	
Tr		×	×		×	×				

Table 5: $K_{\lessdot}$ for Example 4

$K_{\lessdot}$	Senior	Adult	Male	Female	I.T.	Sport	$\overline{\exists pos : el}$	$\overline{\exists pos : pu}$	$\overline{\exists pos : cp}$	$\overline{\exists pos : ch}$
Fa	×		×						×	×
La		×		×	×		×	×	×	
Sh		×		×				×	×	×
Tr		×	×		×	×				

Table 6: $K_{\lessdot}$ for Example 6

To define a formal concept on the aggregated table, we first identify the component of the intent on the part $\rho(A_2)$. Again, we denote the derivations in K_1 and K_2 by $'$, and in the aggregated context by $^{\nabla}$.

Definition 1. *The* relational deviation *of $X \subseteq O_{\lessdot}$, denoted $\delta(X)$, is the set of its attributes from $\rho(A_2)$, i.e. $\delta(X) = X^{\nabla} \cap \rho(A_2)$.*

Proposition 4 *Given a $X \subseteq O_{\lessdot}$, $\delta(X) = \cap_{o \in X}\{\overline{\rho r : a} \mid Constraint(\rho, r, o, (a', a''))\}$.*

Proof. Let $o \in X$. By construction, for any $\overline{\rho r : a} \in \rho(A_2)$, holds $\overline{\rho r : a} \in o^{\nabla}$ iff $Constraint(\rho, r, o, (a', a''))$. Thus $o^{\nabla} \cap \rho(A_2) = \{\overline{\rho r : a} \mid Constraint(\rho, r, o, (a', a''))\}$. As $X' = \bigcap_{o \in X} o'$, we have $X' \cap \rho(A_2) = \bigcap_{o \in X} o^{\nabla} \cap \rho(A_2)$, hence $\delta(X) = \cap_{o \in X}\{\overline{\rho r : a} \mid Constraint(\rho, r, o, (a', a''))\}$. □

A formal concept on the aggregated context is then characterized by:

Proposition 5 *Let $X \subseteq O_{\lessdot}$, then the concept $C = (X^{\nabla\nabla}, X^{\nabla})$ of the aggregated context satisfies $X^{\nabla} = X' \cup \delta(X)$ and $X^{\nabla\nabla} = X'' \cap \{o \mid \delta(X) \subseteq o^{\nabla}\}$.*

Proof. By definition, we have $A_{\lessdot} = A_1 \cup \rho(A_2)$ and $A_1 \cap \rho(A_2) = \emptyset$. Thus we can write $X^{\nabla} = (X^{\nabla} \cap A_1) \cup (X^{\nabla} \cap \rho(A_2))$, that is $X^{\nabla} = X' \cup \delta(X)$.

By deriving X^{∇}, we determine that $X^{\nabla\nabla} = \{o \mid o \in O_1 \wedge X' \subseteq o^{\nabla} \wedge \delta(X) \subseteq o^{\nabla}\}$. Now, as $X' \subseteq A_1$, we have $X' \subseteq o^{\nabla}$ iff $X' \subseteq o^{\nabla} \cap A_1$, that is $X' \subseteq o'$. Finally, $X^{\nabla\nabla} = X'' \cap \{o \mid \delta(X) \subseteq o^{\nabla}\}$. □

Now, let us assume we have the result of RCA using the same relational scaling with ρ along r on the simple RCF in Figure 2. Let X be the extent of a concept from $K_{\lessdot}$. The set $\delta(X)$ is well-defined, hence we can denote its i-th member by $\overline{\rho r : a_{\delta,i}}$ (where $a_{\delta,i} \in A_2$). As every concept $C_{\delta(X),i} = (a'_{\delta,i}, a''_{\delta,i})$ is well defined on K_2, in RCA, K_1 will be refined with all the attributes $\rho r : C_{\delta(X),i}$ at the first relational scaling step. Let $Y_\delta = X' \cup_{i \in 1..|\delta(X)|} \rho r : C_{\delta(X),i}$, we claim that $C = (X^{\nabla\nabla}, X^{\nabla})$ and $C_\delta = (Y'_\delta, Y''_\delta)$ have the same extent:

Proposition 6 $Y'_\delta = X^{\nabla\nabla}$.

Proof. $Y'_\delta \subseteq X^{\nabla\nabla}$: Let $o \in Y'_\delta$. First, o carries all the attributes of X', thus $X' \subseteq o^{\nabla}$. Moreover, for each attribute $\overline{\rho r : a_{\delta,i}} \in \delta(X)$ a concept $C_i = (a'_{\delta,i}, a''_{\delta,i})$ exists such that o carries the attribute $\rho r : C_i$ (for which $Constraint(\rho, r, o, C_i)$ is true). Since $a_{\delta,i}$ is in the intent of C_i, we can verify that $\overline{\rho r : a_{\delta,i}} \in o^{\nabla}$. Since for all i, we have $\overline{\rho r : a_{\delta,i}} \in o^{\nabla}$, then we have $\delta(X) \subseteq o^{\nabla}$. Finally, since $\delta(X) \subseteq o^{\nabla}$ and $X' \subseteq o^{\nabla}$, we have $X^{\nabla} \subseteq o^{\nabla}$. By derivation, we have $o^{\nabla\nabla} \subseteq X^{\nabla\nabla}$. Finally, $Y'_\delta \subseteq X^{\nabla\nabla}$. □

$Y'_\delta \supseteq X^{\nabla\nabla}$: The 1st relational scaling step will necessarily produce $\rho r : (a'_{\delta,i}, a''_{\delta,i})$ for each $\overline{\rho r : a_{\delta,i}} \in \delta(X)$. Let $o \in X^{\nabla\nabla}$, then o carries all the attributes of X'. Moreover, after the scaling step, o gets incident to each attribute $\overline{\rho r : a_{\delta,i}} \in \delta(X)$ ($Constraint(\rho, r, o, (a'_{\delta,i}, a''_{\delta,i}))$ is necessarily satisfied). Thus, we have $Y_\delta \subseteq o'$ and therefore $o'' \subseteq Y'_\delta$. Finally, since $o \in o''$ we conclude that $X^{\nabla\nabla} \subseteq Y'_\delta$. □

Proposition 6 states that for any concept from the aggregated context, an RCA concept with the same extent exists. Definition 2 introduces the notion of relational weakening (illustrated by Example 5) to enable the mapping between both intents. The latter is given by proposition 7.

Definition 2. *Let a concept C be produced by RCA and let Y_r be the set of relational attributes of the intent of C. We call* relational weakening *of C, noted $\Omega(C)$, the set $\Omega(C) = \bigcup_{\rho r:(U,V) \in Y_r} \{\overline{\rho r : v} \mid v \in V\}$*

Example 5. Assume the contexts of Example 2: Context K_C gives rise to the concepts $C_1 = (\{t3\}, \{el, pw\})$, $C_2 = (\{f5\}, \{pw, cp\})$, $C_3 = (\{t3, zo\}, \{el\})$ and $C_4 = (\{zo, f5\}, \{cp\})$. After a scaling with $\exists$, K_P yields the concept $C = (\{La\}, \{Adult, Female, I.T., \exists pos : C_1, \exists pos : C_2, \exists pos : C_3, \exists pos : C_4, \exists pos : \top\})$. Then $\Omega(C) = \{\exists pos : el, \exists pos : pu, \exists pos : cp\}$.

Proposition 7 $\delta(X) \subseteq \Omega(C_\delta)$

Proof. Let's denote by Y_r the set of relational attributes in the intent of C_δ. Let $\overline{\rho r : a} \in \delta(X)$, then by scaling and construction of C_δ, it holds $\rho r : (a', a'') \in Y_r$ and as $a \in a''$, one concludes $\delta(X) \subseteq \Omega(C_\delta)$. □

While we've just shown that the extents of C and C_δ are equal, their intents might differ: As proposition 7 states, the intent of the aggregate table concept is a subset of the weakening of the RCA concept with the same extent (see Example 6 below). In this sense, we see the RCA concept as *more informative.*

Example 6. Assume the relational family defined by Example 2 with the $\exists$ operator. The aggregated context is presented in table 6. Now, after one iteration, RCA discovers the concept $(\{Fa\}, \{Senior, Male, \exists pos : (cp, ch)\})$ whereas FCA finds $(\{Fa\}, \{Senior, Male, \exists pos : (cp), \exists pos : (nd)\})$. While $\exists pos : (cp, ch)$ implies $\exists pos : (cp)$ and $\exists pos : (ch)$, the reverse does not hold.

4 Discussion

We've shown that for any FCA concept from an encoded context, RCA would reveal a counterpart concept, or a pair of such, conveying the same semantics (equal extent). Moreover, the syntactic expression of the RCA concept(s) is clearer than the FCA one, whatever the encoding. With semi-join, since separate RCA concepts map to the 1st and 2nd projections of a FCA concept, the clarity gain is immediate. Indeed, no confusion is ever possible as to which attribute of the semi-join intent is incident to which object. Moreover, redundancy in FCA concepts, e.g. shared 1st or 2nd projection, is avoided in RCA.

With aggregation, RCA trivially produces a concept of the same extent, yet it is more precise: The FCA counterpart is readily obtained by relational weakening. Here, higher-order encoding schemata are conceivable that mimic RCA iterations by nesting the scaling operators. Yet the maximal depth of these nestings in the resulting (pseudo-)relational attributes must be fixed beforehand. This is a serous limitation since we know of no simple way to determine the number of iterations required till the fixed point for a given RCA task, i.e. a RCF and a vector of scaling operators. This means that, at least in the realistic cases, RCA will be revealing concepts that FCA –over the aggregated context with all possible nestings of a depth up till the limit– will miss. A relevant question here is whether knowing the fixed point contexts in RCA, there is an equivalent aggregation context that comprises only nested operator attributes referring exclusively to attribute concepts. This would mean that a static encoding, i.e. without the need for explicitly composing RCA concepts popping at iterations 2+, exists. The cost of constructing such an encoding is, though, a separate concern.

5 Conclusion

We tackled here the question of whether RCA brings some effective scope extension to the realm of FCA, given that FCA is at its core. We've examined two complementary principles of encoding a relation into a single augmented context and compared FCA output on each of the contexts to the output of RCA on the original RCF. It was shown that in both cases, RCA is able to find counterpart concepts (same extent) to those found by FCA, while the RCA intents at its 1st iteration are at least as expressive as the FCA ones.

A more systematic study should allow us to demonstrate similar results in the more complex cases of multiple relations in the RCF as well as multiple relations between the same pair of contexts.

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