

# Fast Classical and Quantum Algorithms for Online $k$ -server Problem on Trees <sup>\*</sup> <sup>\*\*</sup>

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**Abstract.** We consider online algorithms for the  $k$ -server problem on trees. Chrobak and Larmore proposed a  $k$ -competitive algorithm for this problem that has the optimal competitive ratio. However, a naive implementation of their algorithm has time complexity  $O(n)$  to process each request, where  $n$  is the number of nodes. We propose a new time-efficient implementation of this algorithm that has  $O(n \log n)$  time complexity for preprocessing and  $O(k^2 + k \cdot \log n)$  time for processing a request. We also propose a quantum algorithm for the case where the nodes of the tree are presented using string paths. In this case, no preprocessing is needed, and the time complexity for each request is  $O(k^2 \sqrt{n} \log n)$ . When the number of requests is  $o\left(\frac{\sqrt{n}}{k^2}\right)$ , we obtain a quantum speed-up on the total runtime compared to our classical algorithm.

We also give a simple quantum algorithm to find the first marked element in a collection of  $m$  objects, that works even in the presence of two-sided bounded errors on the input oracle. It has worst-case query complexity  $O(\sqrt{m})$ . In the particular case of one-sided errors on the input, it has expected query complexity  $O(\sqrt{x})$  where  $x$  is the position of the first marked element. Compared with previous work, our algorithm can handle errors in the input oracle.

**Keywords:** online algorithms ·  $k$ -server problem on trees · quantum computing · binary search.

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## 1 Introduction

Online optimization is a field of optimization theory that deals with optimization problems having no knowledge of the future [27]. An online algorithm reads an input piece by piece and returns an answer piece by piece immediately, even if the answer can depend on future pieces of the input. The goal is to return an answer that minimizes an objective function (the cost of the output). The most standard method to define the effectiveness of an online algorithm is the competitive ratio [20]. The competitive ratio is the approximation ratio achieved by the algorithm. That is the worst-case ratio between the cost of the solution found by the algorithm and the cost of an optimal solution. If the ratio is  $c$ , then the online algorithm is called  $c$ -competitive. In the general setting, online algorithms have unlimited computational power. Nevertheless, many papers consider them with different restrictions. Some of them are restrictions on memory [6, 9, 24, 2, 5, 18, 23, 21, 26, 25, 22], others are restrictions on time complexity [15, 31].

In this paper, we focus on efficient online algorithms in terms of time complexity. We consider the  $k$ -server problem on trees. Chrobak and Larmore [11] proposed a  $k$ -competitive algorithm for this problem where the competitive ratio  $k$  is the best possible for deterministic algorithms for this problem. The existing implementation of their algorithm has  $O(n)$  time complexity for each request, where  $n$  is the number of nodes in the tree. For general graphs, there exists a time-efficient algorithm for the  $k$ -server problem [31] that uses min-cost-max-flow algorithms. However, in the special case of a tree, this algorithm is not optimal. We propose a new time-efficient implementation of the algorithm from [11]. It has  $O(n \log n)$  time complexity for preprocessing and  $O(k^2 + k \log n)$  for processing a request. It is based on fast algorithms for computing Lowest Common Ancestor (LCA) [7] and the binary lifting technique [8].

We also revisit the problem of finding the first marked element in a collection of  $m$  objects. It is well-known that it can be solved in expected  $O(\sqrt{m})$  queries when given quantum oracle access to the input, and expected  $O(\sqrt{x})$  queries where  $x$  is the position of the first marked element [13],[29, Theorem 10] and [28, Theorem 6]. We give a  $O(\sqrt{m})$  queries algorithm that works even in the presence of *two-sided bounded errors* in the input. We also provide an expected  $O(\sqrt{x})$  queries algorithm in the case where the input has *one-sided errors* only. Note that contrary to the previous results, we assume that the oracle can have errors whereas [13, 29, 28] assume that the oracle is perfect.

We then consider the  $k$ -server problem in the case where the description of the tree is given by string paths. The string path of a node in a rooted tree is a sequence of length  $h$ , where  $h$  is the height of the node, describing the path from the root to the node. Such a way of representing the trees is useful, for example, as a path to a file in file systems. Assuming oracle access to the length of the string path of each node and to each element on the string path with time complexity  $O(1)$ , we obtain a quantum algorithm for this problem with  $O(k^2 \sqrt{n} \log(n))$  running time to process each request and without preprocessing. This algorithm is based on our improved quantum search algorithm. When the

number of requests is  $o\left(\frac{\sqrt{n}}{k^2}\right)$ , the total runtime of our quantum algorithm is smaller than the classical one.

The structure of the paper is the following. Section 2 contains preliminaries. The classical algorithm is described in Section 3. Section 4 contains our improved quantum search algorithm. The quantum algorithm for the  $k$ -server problem is described in Section 5.

## 2 Preliminaries

### 2.1 Online algorithms

**An online minimization problem** consists of a set  $\mathcal{I}$  of inputs and a cost function. Each input  $I = (x_1, \dots, x_n)$  is a sequence of requests, where  $n$  is the length of the input  $|I| = n$ . Furthermore, a set of feasible outputs (or solutions)  $\tilde{O}(I)$  is associated with each  $I$ ; an output is a sequence of answers  $O = (y_1, \dots, y_n)$ . The cost function assigns a positive real value  $cost(I, O)$  to  $I \in \mathcal{I}$  and  $O \in \tilde{O}(I)$ . An optimal solution for  $I \in \mathcal{I}$  is  $O_{opt}(I) = \arg \min_{O \in \tilde{O}(I)} cost(I, O)$ .

Let us define an online algorithm for this problem. **A deterministic online algorithm**  $A$  computes the output sequence  $A(I) = (y_1, \dots, y_n)$  such that  $y_i$  is computed based on  $x_1, \dots, x_i$ . We say that  $A$  is  $c$ -competitive if there exists a constant  $\alpha \geq 0$  such that, for every  $n$  and for any input  $I$  of size  $n$ , we have:  $cost(I, A(I)) \leq c \cdot cost(I, O_{opt}(I)) + \alpha$ . The minimal  $c$  that satisfies the previous condition is called the **competitive ratio** of  $A$ .

### 2.2 Rooted Trees

Consider a rooted tree  $G = (V, E)$ , where  $V$  is the set of nodes/vertices, and  $E$  is the set of edges. Let  $n = |V|$  be the number of nodes, or equivalently the size of the tree. We denote by 1 the root of the tree. A path  $P$  is a sequence of nodes  $(v_1, \dots, v_h)$  that are connected by edges, i.e.  $(v_i, v_{i+1}) \in E$  for all  $i \in \{1, \dots, h-1\}$ , such that there are no duplicates among  $v_1, \dots, v_h$ . Here  $h$  is a length of the path. Between any two nodes  $v$  and  $u$  on the tree, there is a unique path. The distance  $dist(v, u)$  is the length of this path. For each node  $v$  we can define a parent node  $PARENT(v)$  which is the first node on the unique path from  $v$  to root 1. We have  $dist(1, PARENT(v)) + 1 = dist(1, v)$ . Additionally, we can define the set of children  $CHILDREN(v) = \{u : PARENT(u) = v\}$ . Any node  $y$  on the unique path from root 1 to node  $v$  is an ancestor of node  $v$ .

**Lowest Common Ancestor (LCA)**. Given two nodes  $u$  and  $v$  of a rooted tree, the Lowest Common Ancestor is the node  $w$  such that  $w$  is an ancestor of both  $u$  and  $v$ , and  $w$  is the closest one to  $u$  and  $v$  among all such ancestors. The following result is well-known.

**Lemma 1 ([7]).** *There is an algorithm for the LCA problem with the following properties: (i) The time complexity of the preprocessing step is  $O(n)$ ; (ii) The time complexity of computing LCA for two vertices is  $O(1)$ .*

We call `LCA_PREPROCESSING()` the subroutine that does the preprocessing for the algorithm and `LCA( $u, v$ )` that computes the LCA of two nodes  $u$  and  $v$ .

**Binary Lifting Technique.** This technique from [8] allows us to obtain a vertex  $v'$  that is at distance  $z$  from a vertex  $v$  with  $O(\log n)$  time complexity. There are two procedures:

- `BL_PREPROCESSING()` prepares the required data structures. The time complexity is  $O(n \log n)$ .
- `MOVEUP( $v, z$ )` returns a vertex  $v'$  on the path from  $v$  to the root and at distance  $\text{dist}(v', v) = z$ . The time complexity is  $O(\log n)$ .

The technique is well documented in the literature. We present an implementation in the arxiv version [19] for completeness.

### 2.3 $k$ -server Problem on Trees

Let  $G = (V, E)$  be a rooted tree, and we are given  $k$  servers that can move among nodes of  $G$ . At each time slot, a request  $q \in V$  appears. We have to “serve” this request, that is, to choose one of the  $k$  servers and move it to  $q$ . The other servers are also allowed to move. The cost function is the distance by which we move the servers. In other words, if before the request, the servers are at positions  $v_1, \dots, v_k$  and after the request they are at  $v'_1, \dots, v'_k$ , then  $q \in \{v'_1, \dots, v'_k\}$  and the cost of the move is  $\sum_{i=1}^k \text{dist}(v_i, v'_i)$ . The cost of a sequence of requests is the sum of the costs of serving each requests. The problem is to design a strategy that minimizes the cost of servicing a sequence of requests given online.

### 2.4 Quantum query model

We use the standard form of the quantum query model. Let  $f : D \rightarrow \{0, 1\}$ ,  $D \subseteq \{0, 1\}^m$  be an  $m$  variable function. We wish to compute on an input  $x \in D$ . We are given an oracle access to the input  $x$ , i.e. it is realized by a specific unitary transformation usually defined as  $|i\rangle |z\rangle |w\rangle \rightarrow |i\rangle |z + x_i \pmod{2}\rangle |w\rangle$  where the  $|i\rangle$  register indicates the index of the variable we are querying,  $|z\rangle$  is the output register, and  $|w\rangle$  is some auxiliary work-space. An algorithm in the query model consists of alternating applications of arbitrary unitaries independent of the input and the query unitary, and a measurement in the end. The smallest number of queries for an algorithm that outputs  $f(x)$  with probability  $\geq \frac{2}{3}$  on all  $x$  is called the quantum query complexity of the function  $f$  and is denoted by  $Q(f)$ . We refer the readers to [30, 3, 1] for more details on quantum computing.

**Definition 1 (Search problem).** *Suppose we have a set of objects named  $\{1, 2, \dots, m\}$ , of which some are targets. Suppose  $\mathcal{O}$  is an oracle that identifies the targets. The goal of a search problem is to find a target  $i \in \{1, 2, \dots, m\}$  by making queries to the oracle  $\mathcal{O}$ .*

In search problems, one will try to minimize the number of queries to the oracle. In the classical setting, one needs  $O(m)$  queries to solve such a problem. Grover, on the other hand, constructed a quantum algorithm that solves the

search problem with only  $O(\sqrt{m})$  queries [16], provided that there is a unique target. When the number of targets is unknown, Brassard *et al.* designed a modified Grover algorithm that solves the search problem with  $O(\sqrt{m/\lambda})$  queries [10], where  $\lambda$  is the number of targets, which is of the same order as the query complexity of the Grover search.

### 3 A Fast Online Algorithm for $k$ -server Problem on Trees with Preprocessing

We first describe Chrobak-Larmore’s  $k$ -competitive algorithm for  $k$ -server problem on trees from [11]. Assume that we have a request on a vertex  $q$ , and the servers are on the vertices  $v_1, \dots, v_k$ . We say that a server  $i$  is *active* if there are no other servers on the path from  $v_i$  to  $q$ . In each phase, we move every *active* server one step towards the vertex  $q$ . After each phase, the set of *active* servers can change. We repeat this phase (moving of the active servers) until one of the servers reaches the queried vertex  $q$ .

The naive implementation of this algorithm has time complexity  $O(n)$  for each request. First, we run a depth-first search with time labels [12], whose result allows us to check in constant time whether a vertex  $u$  is an ancestor of a vertex  $v$ . Recall that time labels record the first timestamp  $f(v)$  when the depth-first search enters a node  $v$ , and the last timestamp  $\ell(v)$  when the depth-first search finishes processing the last child of node  $v$ ; the timestamp increases every time a new node is visited. A node  $u$  is then an ancestor of  $v$  if and only if the interval  $[f(v), \ell(v)]$  is contained in  $[f(u), \ell(u)]$ . After that, we can move each active server towards the queried vertex, step by step. Together all active servers cannot visit more than  $O(n)$  vertices.

In the following, we present an effective implementation of Chrobak-Larmore’s algorithm with preprocessing. The preprocessing part is done once and has  $O(n \log n)$  time complexity (Theorem 1). The request processing part is done for each request and has  $O(k^2 + k \cdot \log n)$  time complexity (Theorem 2).

#### 3.1 Preprocessing

We do the following steps for the preprocessing :

1. We do required preprocessing for LCA algorithm (Section 2.2)
2. We do required preprocessing for Binary lifting technique (Section 2.2)
3. Additionally, for each vertex  $v$  we compute the distance from the root to  $v$ , i.e.  $\text{dist}(1, v)$ . This can be done using a depth-first search algorithm [12]. We store all the distances in an array. See the arxiv version [19] for an implementation of  $\text{COMPUTEDISTANCE}(u)$ .

**Theorem 1.** *The preprocessing has time complexity  $O(n \log n)$ .*

*Proof.* The time complexity of the preprocessing phase is  $O(n)$  for LCA,  $O(n \log n)$  for the binary lifting technique and  $O(n)$  for  $\text{COMPUTEDISTANCE}(1)$ . Therefore, the total time complexity is  $O(n \log n)$ .  $\square$

---

**Algorithm 1** PREPROCESSING. The preprocessing procedure.

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```

LCA_PREPROCESSING()
BL_PREPROCESSING()
dist(1, 1) ← 0
COMPUTEDISTANCE(1)

```

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### 3.2 Request Processing

Assume that we have a request on a vertex  $q$ , and the servers are on the vertices  $v_1, \dots, v_k$ . We do the following steps, implemented in Algorithms 2 and 4.

**Step 1.** We sort all the servers by their distance to the node  $q$ . The distance  $\text{dist}(v, q)$  between a node  $v$  and the node  $q$  can be computed in the following way. Let  $l = \text{LCA}(v, q)$  be the lowest common ancestor of  $v$  and  $q$ , then  $\text{dist}(v, q) = \text{dist}(1, q) + \text{dist}(1, v) - 2 \cdot \text{dist}(1, l)$ . Using the preprocessing, this quantity can be computed in constant time. We denote by  $\text{SORT}(q, v_1, \dots, v_k)$  this sorting procedure. In the following steps we assume that  $\text{dist}(v_i, q) \leq \text{dist}(v_{i+1}, q)$  for  $i \in \{1, \dots, k-1\}$ .

**Step 2.** The first server on  $v_1$  processes the request. We move it to the node  $q$ .

**Step 3.** For  $i \in \{2, \dots, k\}$  we consider the server on  $v_i$ . It will be inactive when some other server with a smaller index arrives on the path between  $v_i$  and  $q$ . Section 3.3 contains the different cases that can happen and how to compute the distance  $d$  traveled by  $v_i$  before it becomes inactive. We then move the  $i$ -th server  $d$  steps towards the request  $q$ . The new position of the  $i$ -th server is  $v'_i$ .

---

**Algorithm 2** REQUEST( $q$ ). Request processing procedure.

---

```

SORT( $q, v_1, \dots, v_k$ )
 $v'_1 \leftarrow q$ 
for  $i \in \{2, \dots, k\}$  do
     $d \leftarrow \text{DISTANCETOINACTIVE}(q, i)$            ▷ see Algorithm 3
     $v'_i \leftarrow \text{MOVE}(v_i, d)$                    ▷ see Algorithm 4

```

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### 3.3 Distance to Inactive State

When processing a request, all servers except one will eventually become inactive. The crucial part of the optimization is to compute when a server becomes inactive quickly. For the purpose of computing this time, we claim that we can pretend that servers “never go inactive”. Formally, let  $q$  be a request,  $i$  be a server, and  $j$  another server with smaller index. We know that  $i$  will become inactive because it is not the closest to the target. However it is possible that this particular server  $j$  is not the one that will render  $i$  inactive. Nevertheless, we can pretend that  $j$  will never become inactive and compute the distance  $i$  will travel before going inactive because of  $j$ , call this distance  $d_{i,j}^q$  (the index  $i$  is fixed in this reasoning). We claim the following:

**Lemma 2.** *For any request  $q$  and server  $i > 1$  (i.e. a server that will become inactive), the distance  $D_i^q$  travelled by  $i$  before it becomes inactive is equal to  $\min_{j < i} d_{i,j}^q$ .*

*Proof.* Let  $j_0$  be one of the servers that renders  $i$  inactive, then  $D_i^q = d_{i,j_0}^q$  because  $j_0$  will not become inactive before it makes  $i$  inactive, hence for the purpose of computing  $D_i$ , it makes no difference whether  $j_0$  eventually becomes inactive or not. Therefore, we only need to prove no other  $d_{i,j}^q$  is strictly smaller. Assume for contradiction that  $d_{i,j}^q < D_i^q$  for some  $j < i$ , and pick  $j$  so that  $d_{i,j}^q$  is minimum among all  $j < i$  (and in case of equality, pick  $j$  the smallest possible). Then, it means there exists a vertex  $t$  such that  $d_{i,j}^q = \text{dist}(v_j, t) \leq \text{dist}(v_i, t)$  and  $t$  is on the paths from  $v_i$  and  $v_j$  to  $q$ . Now we claim that  $j$  must become inactive before it reaches  $t$ . Indeed, if not, it would reach  $t$  and makes  $i$  inactive after a distance  $d_{i,j}^q < D_i^q$ , which is impossible by definition of  $D_i^q$ . Therefore,  $j$  is rendered inactive before reaching  $t$  by another server  $\ell$  reaching some vertex  $u$  on the path from  $v_j$  to  $t$ . In particular, we must have  $D_j^q = \text{dist}(v_\ell, u) \leq \text{dist}(v_j, u)$  and  $\text{dist}(v_j, u) < \text{dist}(v_j, t)$ . But now observe that if we pretend that  $\ell$  never goes inactive, it will reach  $t$  after travelling a distance  $\text{dist}(v_\ell, u) + \text{dist}(u, t) \leq \text{dist}(v_j, u) + \text{dist}(u, t) = \text{dist}(v_j, t)$  hence  $d_{i,\ell}^q = \text{dist}(v_\ell, t) \leq \text{dist}(v_j, t) = d_{i,j}^q$ . But we chose  $j$  so that  $d_{i,j}^q$  is minimal so we must have  $d_{i,\ell}^q = d_{i,j}^q$  and therefore,  $j < \ell$  (we sort by index in case of tie). Going back to the computation, we see that  $d_{i,\ell}^q = d_{i,j}^q$  implies that  $\text{dist}(v_i, u) = \text{dist}(v_\ell, u)$ , *i.e.*  $i$  and  $\ell$  reach  $u$  at the same time. But when two servers reach the same vertex simultaneously, the greater index goes inactive, *i.e.*  $\ell$  would go inactive because of  $j$ . This is a contradiction because we assumed that  $\ell$  is the one making  $j$  inactive.  $\square$

We have now reduced the problem to the following question: given a server  $i$  and another server  $j$  with smaller index, compute  $d_{i,j}^q$ , the distance until  $i$  becomes inactive because of  $j$ , pretending that  $j$  never goes inactive. There are several cases to consider, depicted in Figure 1, depending on the relationship of  $v_i$ ,  $v_j$  and  $q$  in the tree. Let  $t$  be the vertex where the paths from  $v_i$  to  $q$  and  $v_j$  to  $q$  intersect the first time, then  $d_{i,j}^q = \text{dist}(v_j, t)$  and

1. if  $q$  is an ancestor of  $v_i$  and  $v_j$ , then  $t = \text{LCA}(v_i, v_j)$ ;
2. if  $q$  is an ancestor of  $v_i$  but not of  $v_j$ , then  $t = q$ ;
3. if  $v_i$  is an ancestor of  $q$ , then  $t = \text{LCA}(q, v_j)$  because  $v_i$  must also be ancestor of  $v_j$  since  $v_j$  is closer to  $q$  than  $v_i$ ;
4. if the LCA of  $v_j$  and  $q$  is not an ancestor of  $v_i$ , then  $t = \text{LCA}(v_j, q)$ ;
5. if the LCA of  $v_i$  and  $v_j$  is not an ancestor of  $q$ , then  $t = \text{LCA}(v_j, v_i)$ ;
6. otherwise  $t = \text{LCA}(v_i, q)$ .

Note that in this case distinction, the order of the cases is important: if cases 1 to 3 do not apply for example, then we know that  $v_i$  is not an ancestor of  $q$  and  $q$  is not an ancestor of  $v_i$ .

**Lemma 3.** `DISTANCETOINACTIVE` has time complexity  $O(k)$ .

*Proof.* Since a vertex  $u$  is ancestor of  $v$  if  $\text{LCA}(u, v) = u$ , we can check this condition in  $O(1)$  due to results from Section 2.2. It follows that we can compute  $d_{i,j}^q$  for every  $i, j, q$  in  $O(1)$  and there are at most  $k$  other servers to consider.  $\square$

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**Algorithm 3** DISTANCETOINACTIVE( $q, i$ ). Compute the distance travelled before going inactive.

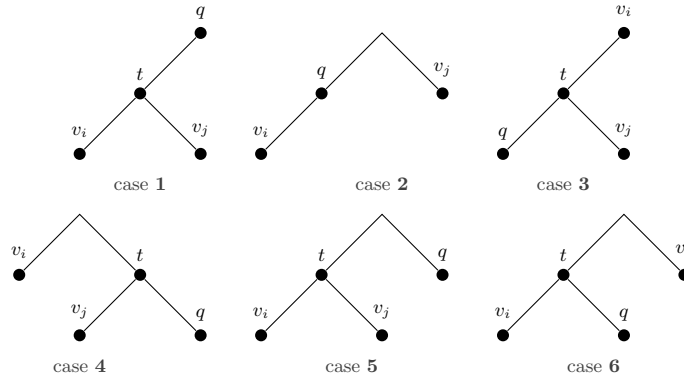
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```

 $d \leftarrow \infty$ 
for  $j \in \{1, \dots, i - 1\}$  do
     $t \leftarrow$  do case analysis as above
     $d \leftarrow \min(d, \text{dist}(t, v_j))$ 
return  $d$ 

```

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**Fig. 1.** List of the various cases to consider when computing the distance before a server  $i$  is rendered inactive by a (closer to the request) server  $j$ .

### 3.4 How to Move a Server

We now consider the following problem: given a server  $v$  and a distance  $z$ , how to efficiently compute the new position of the server after moving it  $z$  steps towards  $q$ . We use the binary lifting technique for this procedure.

Let  $l = \text{LCA}(v, q)$ . If  $\text{dist}(l, v) \geq z$ , then the result node is on the path between  $v$  and  $l$ . We can thus invoke  $\text{MOVEUP}(v, z)$  from Section 2.2. Otherwise, we should move the server first to  $l$ . We then move it  $z - \text{dist}(l, v)$  steps down towards  $q$  from  $l$ . Moving down from  $l$  is the same as moving up  $\text{dist}(l, q) - (z - \text{dist}(l, v))$  steps from  $q$ . The algorithm is presented in Algorithm 4.

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**Algorithm 4** MOVE( $v, z$ ). Moves of a server from  $v$  to distance  $z$  on a path from  $v$  to  $q$ .

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```

 $l = \text{LCA}(v, q)$ 
if  $z \leq \text{dist}(l, v)$  then
     $\text{Result} \leftarrow \text{MOVEUP}(v, z)$ 
if  $z > \text{dist}(l, v)$  then
     $z \leftarrow z - \text{dist}(l, v)$ 
     $\text{Result} \leftarrow \text{MOVEUP}(q, \text{dist}(l, q) - z)$ 
return  $\text{Result}$ 

```

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**Lemma 4.** *The time complexity of MOVE is  $O(\log n)$ .*

*Proof.* The time complexity of MOVEUP is  $O(\log n)$  using the binary lifting technique from Section 2.2 and LCA is in  $O(1)$  by Section 2.2. Furthermore, we can compute the distance between any two nodes in  $O(1)$  thanks to the preprocessing. Therefore, the total complexity is  $O(\log n)$ .  $\square$

**Theorem 2.** *The time complexity of the request processing phase is  $O(k^2 + k \log n)$ .*

*Proof.* The complexity of sorting the servers by distance is  $O(k \log k)$ . For each server, we compute the distance traveled before being inactive in  $O(1)$  by Lemma 3. We then move each server by that distance in time  $O(\log n)$  by Lemma 4. Therefore, the complexity of processing one server is  $O(k + \log n)$ , and there are  $k$  servers.  $\square$

## 4 Binary Search for a Function with Errors

Consider a search space  $S = \{1, \dots, m\}$  and a subset  $M \subseteq S$  of marked elements. Define the indicator function  $g_M : S^2 \rightarrow \{0, 1\}$  by  $g_M(\ell, r) = 1$  if  $\{\ell, \dots, r\} \cap M \neq \emptyset$ , and 0 otherwise. In other words,  $g_M(\ell, r)$  indicates whether there is a marked element from  $M$  in the interval  $[\ell, r]$ . Now assume that we do not know  $M$  but have access to a two-sided probabilistic approximation  $\tilde{g}$  of  $g_M$ . Formally, there is a probability  $p < 1/2$  such that for any  $\ell, r \in S$ ,  $\tilde{g}(\ell, r) = g_M(\ell, r)$  with probability at least  $1 - p$  and  $1 - g_M(\ell, r)$  otherwise. Intuitively,  $\tilde{g}$  behaves like  $g_M$  with probability at least  $1 - p$ . However, sometimes it makes mistakes and returns a completely wrong answer. Note that  $\tilde{g}$  has *two-sided* error: it can return 0 even if the interval  $[\ell, r]$  contains a marked element, but more importantly, it can also return 1 even though the interval does *not* contain any marked element. We further assume that a call to  $\tilde{g}(\ell, r)$  takes time  $T(r - \ell)$  where  $T$  is some nondecreasing function. Typically, we assume that  $T(m) = o(m)$ , *i.e.*  $T$  is strictly better than a linear search.

We now consider the problem of finding the first marked element in  $S$ , with probability at least, say,  $1/2$ . A trivial algorithm is to perform a linear search in  $O(m)$  until  $\tilde{g}$  returns 1. If  $\tilde{g}$  had no errors, we could perform a binary search in  $O(T(m))$ . This does not work very well in the presence of errors because decisions made are irreversible, and errors accumulate quickly. Our observation is that if we modify the binary search to boost the success probability of certain calls to  $\tilde{g}$ , we can still solve the problem in time in  $O(T(m))$ .

### 4.1 Algorithm

The idea is inspired by [4] and very similar to [14] except that the calls to  $\tilde{g}$  (the “comparisons”) do not necessarily have unit cost. The difference with [14] is apparent in the complexity since we can avoid the extra log factor when the cost of each comparison becomes high enough. For reasons that become clear in the proof, we need to boost some calls’ success probability. We do so by repeating

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**Algorithm 5** Binary search for a function with two-sided errors

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$\ell \leftarrow 1, r \leftarrow m + 1$  ▷ search interval  
 $d \leftarrow 1$  ▷ depth of the search  
**while**  $\ell < r$  **do**  
     $mid \leftarrow \lfloor (\ell + r)/2 \rfloor$   
     $v_l \leftarrow \tilde{g}(\ell, mid)$  ▷ repeat  $d$  times and take the majority  
    **if**  $v_l = 0$  **then**  
         $\ell \leftarrow mid + 1$   
    **else**  
         $r \leftarrow mid$   
     $d \leftarrow d + 1$

---

them several times and taking the majority: by this we mean that we take the most common answer, and return an error in the case of a tie.

**Proposition 1.** *Assume that  $T$  satisfies  $T(m/k) = O(T(m)/k^\alpha)$  for some  $\alpha > 0$  and every  $m$  and  $k$ , then with probability more than 0.5, Algorithm 5 returns the position of the first marked element, or  $m + 1$  if none exists. The running time is  $O(T(m))$ .*

*Proof.* The correctness of the algorithm, when there are no errors, is clear. We need to argue about the complexity and error probability.

At the  $u^{th}$  iteration of the loop, the algorithm considers a segment  $[\ell, r]$  of length at most  $m \cdot 2^{-(u-1)}$ . The complexity of  $\tilde{g}(\ell, mid)$  is at most  $O(T(r - \ell)) = O(T(m \cdot 2^{-(u-1)}))$  but we repeat it  $2u$  times, so the total complexity of the  $u^{th}$  iteration is  $O(uT(m \cdot 2^{-(u-1)}))$ . The number of iterations is at most  $\log_2 m$ . Hence, the total complexity is

$$\begin{aligned} O\left(\sum_{u=1}^{\log_2 m} uT(m \cdot 2^{-(u-1)})\right) &= O\left(\sum_{u=1}^{\log_2 m} T(m) 2^{-\alpha u} u\right) = O\left(T(m) \sum_{u=1}^{\infty} 2^{-\alpha u} u\right) \\ &= O\left(T(m) \frac{2^\alpha}{(2^\alpha - 1)^2}\right) = O(T(m)). \end{aligned}$$

Finally, we need to analyze the success probability of the algorithm: at the  $u^{th}$  iteration, the algorithm will run each test  $2u$  times and each test has a constant probability of failure  $p$ . Hence for the algorithm to fail at iteration  $u$ , at least half of the  $2u$  runs must fail: this happens with probability at most  $\binom{2u}{u} p^u \leq \left(\frac{2ue}{u}\right)^u p^u \leq (2ep)^u$ , where  $e = \exp(1)$ . Hence, the probability that the algorithm fails is bounded by  $\sum_{u=1}^{\log_2 m} (2ep)^u \leq \sum_{u=1}^{\infty} (2ep)^u \leq \frac{2ep}{1-2ep}$ . By taking  $p$  small enough (say  $2ep < \frac{1}{3}$ ), which is always possible by repeating the calls to  $\tilde{g}$  a constant number of times to boost the probability, we can ensure that the algorithm fails less than half of the time.  $\square$

*Remark 1.* The condition  $T(m/k) = O(T(m)/k^\alpha)$  for some  $\alpha > 0$  and every  $m$  and  $k$  is clearly satisfied by any function of the form  $T(m) = m^\alpha \log^\beta m \log^\gamma \log m$ .

## 4.2 Application to Quantum Search

A particularly useful application of the previous section is for quantum search, particularly when  $\tilde{g}$  is a Grover-like search. Indeed, Grover's search can decide in  $O(\sqrt{m})$  queries if a marked element exists in an array of size  $m$ , with a constant probability of error.

More precisely, assume that we have a function  $f : \{1, \dots, m\} \rightarrow \{0, 1\}$  and the task is to find the minimal  $x \in \{1, \dots, m\}$  such that  $f(x) = 1$ . If we let  $\tilde{g}(\ell, r) = \text{GROVER}(\ell, r, f)$  then  $\tilde{g}$  has query complexity  $T(m) = \sqrt{m}$  and fails with constant probability. Hence, we can apply Proposition 1 and obtain an algorithm to find the first marked element with query complexity  $T(m) = O(\sqrt{m})$  and constant probability of error. In fact, note that we are not making use of Proposition 1 to its full strength because  $\tilde{g}$  really has one-sided error: it will never return 1 if there are no marked element. We will make use of this observation later.

**Proposition 2.** *There is a quantum algorithm that finds the first marked element in an array of size  $m$  in  $O(\sqrt{m})$  queries and error probability  $< 0.5$ .*

As observed above, we are not really using Proposition 1 to its full strength because Grover's search has one-sided error. This suggests that there is room for improvement. Suppose that we now only have access to a two-sided probabilistic approximation  $\tilde{f}$  of  $f$ . In other words,  $f$  can now make mistakes: it can return 1 for an unmarked element or 0 for a marked element with some small probability. Formally,  $\tilde{f}(x) = f(x)$  with probability at least  $1 - p$  and  $1 - f(x)$  otherwise, for some probability  $p < 1/2$ . We cannot apply Grover's search directly in this case, but some variants have been developed that can handle bounded errors [17]. Using this result, we can build a two-sided error function  $\tilde{g}$  with high probability of success and time complexity  $O(\sqrt{m})$ . Applying Proposition 1 again, we obtain the following improvement:

**Proposition 3.** *There exists a quantum algorithm `FINDFIRST` that finds the first marked element in an array of size  $m$  in  $O(\sqrt{m})$  queries and error probability less than 0.5; even when the oracle access to the array has a two-sided error.*

In quantum computing,  $f$  rarely has two-sided errors. For instance, Grover's search has a one-sided error only. If we assume that  $\tilde{f}$  has one-sided error only, we can obtain a slightly better version of Proposition 3. Formally, we assume that  $\tilde{f}(x) = f(x)$  with probability at least  $1 - p$  and 0 otherwise.

**Proposition 4.** *There exists a quantum algorithm that finds the first marked element in an array of size  $m$  in expected  $O(\sqrt{x})$  queries and with error probability less than 0.5, where  $x$  is the position of the first marked element, or  $O(\sqrt{m})$  queries if none is marked. Furthermore, it works even when the oracle access to the array has one-sided error. Additionally, it has a worst-case query complexity of  $O(\sqrt{m})$  in all cases.*

*Proof.* Let FINDFIRST denote the algorithm from Proposition 3 and GROVERTWOSIDED denote the variant of Grover’s algorithm of [17] that works with two-sided error oracles. Recall that we assume that  $\tilde{f}$  has one-sided error, *i.e.* it may return 0 instead of 1 with small probability but not the other way around. Consider the following algorithm:

---

**Algorithm 6** FINDFIRSTADVANCED( $m, f$ ). Find the first marked element in an array.

---

```

 $r \leftarrow 1$  ▷ size of the search space
while  $r \leq n$  and  $\text{GROVERTWOSIDED}(1, r, \tilde{f}) = 0$  do
     $r \leftarrow \min(m, 2r)$ 
return FINDFIRST( $r, \tilde{f}$ )

```

---

We now show that this algorithm satisfies the requirements of Proposition 4. To simplify the proof, we assume that the array always contains a marked element; this is without loss of generality because we can add an extra object at the end that is always marked. Furthermore, we assume that  $n$  is a power 2, this is again without loss of generality because we can add dummy object at the end at the cost of doubling the array size at most.

Recall that  $\tilde{f}$  has a one-sided error, and the same applies to GROVERTWOSIDED in this case. Therefore the test  $\text{GROVERTWOSIDED}(1, r, \tilde{f}) = 0$  can only fail if there actually is a marked element in the interval  $[1, r]$ . Of course, the problem is that it can succeed even though there is a marked element in this interval. Let  $p$  be the probability that this happens (*i.e.* GROVERTWOSIDED fails), we know that this is  $< 1/2$  by [17, Theorem 10]. Let  $x$  be the position of the first marked element and let  $\ell_x$  be such that  $2^{\ell_x} \leq x < 2^{\ell_x+1}$ . Let  $R$  be the value of  $r$  after the loop, it is a random variable and always a power of 2. By the above reasoning, it is always the case that  $R \geq x$ . Furthermore, for any  $\ell_x \leq \ell < \log_2 n$ , the probability that  $R = 2^\ell$  is at most  $p^{\ell-\ell_x}(1-p)$ . The call to FINDFIRST takes time  $O(\sqrt{R})$  by Proposition 3. Hence, the expected time complexity of this algorithm is

$$\begin{aligned}
 O\left(\sum_{\ell=\ell_x}^{\log n} p^{\ell-\ell_x}(1-p)\sqrt{2^\ell}\right) &= O\left(\sqrt{2^{\ell_x}} \sum_{\ell=0}^{\infty} p^\ell \sqrt{2^\ell}\right) \\
 &= O\left(\sqrt{2^{\ell_x}} \frac{1}{1-\sqrt{2}p}\right) \\
 &= O(\sqrt{x})
 \end{aligned}$$

where we assume that  $p$  is small enough. This is always possible by repeating the calls to FINDFIRST a constant number of times to reduce the failure probability  $p$ . Finally, we note that the only way this algorithm can fail is if the (unique) call to FINDFIRST fails and this only happen with constant probability.  $\square$

## 5 A Fast Quantum Implementation of Online Algorithm for $k$ -server Problem on Trees

We consider a special way of storing a rooted tree. Assume that for each vertex  $v$  we have access to a sequence  $a^v = (a_1^v, \dots, a_d^v)$  for  $d = \text{dist}(1, v) + 1$ . Here  $a^v$  is a path from the root (the vertex 1) to the vertex  $v$ ,  $a_1^v = 1, a_d^v = v$ . Such a way of describing a tree is not uncommon, for example when the tree represents a file system. A file path “c:\Users\MyUser\newdoc.txt” is exactly such a path in the file system tree. Here “c”, “Users”, “MyUser” are ancestors of “newdoc.txt”, “c” is the root and “newdoc.txt” is the node itself.

We assume that we have access to the following two oracles in time  $O(1)$ : (i) given a vertex  $u$ , a (classical) oracle that returns the length of the string path  $a^u$ ; (ii) given a vertex  $u$  and an index  $i$ , a quantum oracle that returns the  $i^{\text{th}}$  vertex  $a_i^u$  of the sequence  $a^u$ .

We can solve the  $k$ -server problem on trees using the same algorithm as in Section 3 with the following modifications: (i) The function  $\text{LCA}(u, v)$  becomes  $\text{LCP}(a^u, a^v)$  where  $\text{LCP}(a^u, a^v)$  is a longest common prefix of two sequences  $a^u$  and  $a^v$ . (ii)  $\text{MOVEUP}(v, z)$  is the vertex  $a_{d-z}^v$  where  $a^v$  is the sequence for  $v$  and  $d = |a^v|$ ; (iii) We can compute  $\text{dist}(u, v)$  if  $u$  is the ancestor of  $v$ : it is  $d' - d''$ , where  $d'$  is a length of  $a^u$  and  $d''$  is a length of  $a^v$ . Note that the invocations of  $\text{dist}$  in Algorithms 4 and 2 are always this form. The only exception is  $\text{dist}(u, v)$  in  $\text{SORT}$  in which the function uses  $\text{LCA}$  as a subroutine. The complexity of  $\text{SORT}$  is thus the same as the complexity of  $\text{LCA}$  or  $\text{LCP}$  in our case.

By doing so, we do not need any preprocessing. We now replace the  $\text{LCP}(a^u, a^v)$  function by a quantum subroutine  $\text{QLCP}(a^u, a^v)$ , presented in Section 5.1, and keep everything else as is. This subroutine runs in time  $O(\sqrt{n} \log n)$  with  $O(\frac{1}{n^3})$  error probability. This allows us to obtain the following result.

**Theorem 3.** *There is a quantum algorithm that processes a request in time  $O(k^2 \sqrt{n} \log n)$  with probability of error  $O(\frac{1}{n})$ . It does not require any preprocessing.*

*Proof.* The complexity  $\text{MOVE}$  is the complexity of  $\text{LCA}$  that is  $\text{QLCP}$  in our implementation, plus the complexity of  $\text{MOVEUP}$ . The former has complexity is  $O(\sqrt{n} \log n)$  by Lemma 5, and the latter  $O(1)$  by the oracle. Therefore, the total running time of  $\text{MOVE}$  is  $O(\sqrt{n} \log n)$ .

The complexity of  $\text{REQUEST}$  is  $O(k^2)$  times the cost of  $\text{LCA}$  that is  $\text{QLCP}$  in our implementation, and then a call to  $\text{MOVE}$ . Additionally, the  $\text{SORT}$  function invokes  $\text{LCA}$  to compute distances. Hence, the complexity of  $\text{SORT}$  is  $O(k \log k \cdot \sqrt{n} \cdot \log n)$ , and the total complexity is  $O(k^2 \sqrt{n} \cdot \log n)$ .

We invoke,  $\text{QLCP}$  at most  $4k^2$  times so the success probability is at least  $(1 - \frac{1}{n^3})^{4k^2} \geq (1 - \frac{1}{n^3})^{4n^2} \geq 1 - \Omega(\frac{1}{n})$  for enough big  $n$ . Therefore, the error probability is  $O(\frac{1}{n})$ . Note that we do not need any preprocessing.  $\square$

## 5.1 Quantum Algorithm for Longest Common Prefix of Two Sequences

Let us consider the Longest Common Prefix (LCP) problem. Given two sequences  $(q_1, \dots, q_d)$  and  $(b_1, \dots, b_s)$ , the problem is to find  $t$  such that  $q_1 = b_1, \dots, q_t = b_t$  and  $q_i \neq b_i$  for  $t + 1 \leq i \leq m$ , where  $m = \min(d, s)$ . Consider a function  $f : \{1, \dots, m\} \rightarrow \{0, 1\}$  such that  $f(i) = 1$  iff  $q_i \neq b_i$ . Assume that  $x$  is the minimal argument such that  $f(x) = 1$ , then  $t = x - 1$ . The LCP problem is equivalent to finding the first marked element. It can be solved as follows.

---

**Algorithm 7** QLCP( $q, b$ ). Quantum algorithm for the longest common prefix.

---

```

 $m \leftarrow \min(d, s)$ 
 $x \leftarrow \text{FINDFIRST}(m, f)$             $\triangleright$  Repeat  $3 \log m$  times and take the majority vote
if  $x = \text{NULL}$  then
     $x \leftarrow m + 1$ 
return  $x - 1$ 

```

---

**Lemma 5.** *Algorithm 7 finds the LCP of two sequences of length  $m$  in time  $O(\sqrt{m} \log m)$  and with probability of error  $O(\frac{1}{m^3})$ .*

*Proof.* The correctness of the algorithm follows from the definition of  $f$ . The complexity of FINDFIRST is  $O(\sqrt{m})$  by Proposition 2. The total running time is  $O(\sqrt{m} \log m)$  because of the repetitions.  $\square$

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