# General Admissibly Ordered Interval-valued Overlap Functions

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**Abstract**

Overlap functions are a class of aggregation functions that measure the overlapping degree between two values. They have been successfully applied in several problems in which associativity is not required, such as classification and image processing. Some generalizations of overlap functions were proposed for applications in problems with more than two classes, such as $n$-dimensional and general overlap functions. To measure the overlapping of interval data, interval-valued overlap functions were defined, and, later, they were also generalized in the form of $n$-dimensional and general iv-overlap functions. In order to apply some of those concepts in problems with interval data considering the use of admissible orders, which are total orders that refine the most used partial order for intervals, $n$-dimensional admissibly ordered iv-overlap functions were recently introduced, proving to be suitable to be applied in classification problems. However, the sole construction method presented for this kind of function do not allow the use of the well known lexicographical orders. So, in this work we combine previous developments to introduce general admissibly ordered iv-overlap functions, present different construction methods for them and how to combine such methods, showcasing the flexibility of this approach, while also being compatible with the lexicographical orders.



## 1. Introduction

Overlap functions are aggregation functions, introduced in the context of image processing problems, to measure the overlapping between classes [1]. They have been studied in the literature by many authors, mainly because of either the advantages they present over t-norms [2, 3] or their great applicability, as in: fuzzy rule-based classification [4, 5] and decision making [6].

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The concept of $n$-dimensional overlap functions was introduced [7] to allow the application of overlap functions in problems with multiple classes. By relaxing the boundary conditions, general overlap functions were defined, showing good behaviour in classification problems [8].

When working with fuzzy systems, one may face the problem regarding the uncertainty in assigning the values of the membership degrees or defining the membership functions that are adopted in the system. In the literature, a proposed solution is given by the use of interval-valued (iv) fuzzy sets (IVFSs) [9], where membership degrees are represented by intervals, whose widths express such uncertainty [10]. IVFSs have been successfully applied in different fields, such as classification [11], image processing [12], game theory [13] and pest control [14].

To avoid a stalemate when comparing interval data, Bustince et al. [15] introduced the concept of admissible orders for intervals, that is, total order relations that refine the usual product order [16], which is a partial order. Since their introduction, several works were developed taking admissible orders into account, such as [17].

Qiao and Hu [18] and Bedregal et al. [19] defined, independently, the concept of iv-overlap functions. By extending and generalizing iv-overlap functions, Asmus et al. [11] introduced the concepts of n-dimensional iv-overlap functions and general iv-overlap functions, both notions taking into account the usual increasingness with respect to the product order.

Allowing for a broader practical application of ($n$-dimensional) iv-overlap functions, Asmus et al. [17] introduced the concept of n-dimensional admissibly ordered iv-overlap functions, which are n-dimensional iv-overlap functions that are increasing with respect to an admissible order. They also presented a construction method, which, however, cannot generate n-dimensional iv-overlap functions that are increasing with respect to the well known lexicographical orders [15]. Although this is not a serious problem, with the initial motivation to overcome this drawback, in this present work we combine the recent developed concepts on ($n$-dimensional, general) iv-overlap functions and admissible orders to introduce general admissibly ordered iv-overlap function. The resulting definition proved to be much more flexible and adaptable, allowing for the development of different construction methods, and even the composition of functions constructed through those methods.

The paper is organized as follows. Section 2 presents some preliminary concepts. In Section 3, we introduce the concept of general admissibly ordered iv-overlap functions, studying its representation and some construction methods. Section 4 is the Conclusion.

## 2. Preliminaries

In this section, we recall some basic concepts used in this work.

An aggregation function [20] is a mapping $A : [0,1]^n \to [0,1]$ that is increasing in each argument and satisfying: **(A1)** $A(0,\dots,0) = 0$; **(A2)** $A(1,\dots,1) = 1$. A function $On : [0,1]^n \to [0,1]$ is said to be an n-dimensional overlap function [7] if, for all $\vec{x} \in [0,1]^n$: **(On1)** $On$ is commutative; **(On2)** $On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^{n} x_i = 0$; **(On3)** $On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1$; **(On4)** $On$ is increasing; **(On5)** $On$ is continuous. When $On$ is strictly increasing in $(0,1]$, it is called a strict n-dimensional overlap function. A 2-dimensional overlap function is just called overlap function [1, 21].

**Definition 1.** *[8] A function $GO : [0,1]^n \to [0,1]$ is said to be a general overlap function if, for all $\vec{x} \in [0,1]^n$: **(GO1)** $On$ is commutative; **(GO2)** $\prod_{i=1}^n x_i = 0 \Rightarrow GO(\vec{x}) = 0$ **(GO3)** $\prod_{i=1}^n x_i = 1 \Rightarrow GO(\vec{x}) = 1$; **(GO4)** $GO$ is increasing; **(GO5)** $GO$ is continuous.*

**Example 1.** *The following are all examples of general overlap functions, defined for all $\vec{x} \in [0,1]^n$:*

***a)** The product $GO_P$, given by $GO_P(\vec{x}) = \prod_{i=1}^n x_i$;*

***b)** The function $GO_L$, given by $GO_L(\vec{x}) = \max\{(\sum_{i=1}^n x_i) - (n-1), 0\}$;*

***c)** The geometric mean $GO_{Gm}$, given by $GO_{Gm}(\vec{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$.*

Now, let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$. Denote $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$. Given any $X = [x_1, x_2] \in L([0,1])$, $\underline{X} = x_1$ and $\overline{X} = x_2$ denote, respectively, the left and right projections of $X$, and $w(X) = \overline{X} - \underline{X}$ denotes the width of $X$. The interval product order is defined for all $X, Y \in L([0,1])$ by: $X \leq_{Pr} Y \Leftrightarrow \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}$. We call as $\leq_{Pr}$-increasing a function that is increasing with respect to the product order $\leq_{Pr}$.

Given $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ as

$$\widehat{f,g}(\vec{X}) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]. \tag{1}$$

**Definition 2.** *[10] Let $IF : L([0,1])^n \to L([0,1])$ be an $\leq_{Pr}$-increasing interval function. $IF$ is said to be representable if there exist increasing functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$.*

The functions $f$ and $g$ are the *representatives* of the interval function $F$. When $F = \widehat{f,f}$, we denote simply as $\widehat{f}$.

The notion of admissible orders for intervals came from the interest in refining the product order $\leq_{Pr}$ to a total order.

**Definition 3.** *[15] Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is an admissible order if: (i) $\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$; (ii) For all $X, Y \in L([0,1])$, $X \leq_{AD} Y$ whenever $X \leq_{Pr} Y$.*

**Example 2.** *Examples of admissible orders on $L([0,1])$ are the lexicographical orders with respect to the first and second coordinate, defined, respectively, by:*

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y}), \quad X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

**Definition 4.** *[15] For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X}, \overline{X}) < K_\alpha(\underline{Y}, \overline{Y}) \text{ or } (K_\alpha(\underline{X}, \overline{X}) = K_\alpha(\underline{Y}, \overline{Y}) \text{ and } K_\beta(\underline{X}, \overline{X}) \leq K_\beta(\underline{Y}, \overline{Y})),$$

*where $K_\alpha, K_\beta : [0,1]^2 \to [0,1]$ are aggregation functions defined, respectively, by*

$$K_\alpha(x,y) = x + \alpha \cdot (y - x), \quad K_\beta(x,y) = x + \beta \cdot (y - x). \tag{2}$$

Then, the relation $\leq_{\alpha,\beta}$ is an admissible order.

**Remark 1.** *By varying the values of $\alpha$ and $\beta$ one can recover some of the known admissible orders, e.g., the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$ can be recovered by $\leq_{0,1}$ and $\leq_{1,0}$, respectively.*

For simplicity, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$. Also, we denote an iv-function that is increasing with respect to an admissible order $\leq_{AD}$ as $\leq_{AD}$-increasing. Every $\leq_{AD}$-increasing function is also $\leq_{Pr}$-increasing, since every admissible order $\leq_{AD}$ refines $\leq_{Pr}$.

The interval-product is defined, for all $X, Y \in L([0,1])$, by $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$.

A function $IA : L([0,1])^n \to L([0,1])$ is an iv-aggregation function [22] if: **(IA1)** $IA$ is $\leq_{Pr}$-increasing; **(IA2)** $IA([0,0], \ldots, [0,0]) = [0,0]$ and $IA([1,1], \ldots, [1,1]) = [1,1]$.

**Definition 5.** *[11] A function $IOn : L([0,1])^n \to L([0,1])$ is an n-dimensional iv-overlap function if, for all $\vec{X} \in L([0,1])^n$: **(IOn1)** $IOn$ is commutative; **(IOn2)** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0]$; **(IOn3)** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1]$; **(IOn4)** $IOn$ is $\leq_{Pr}$-increasing; **(IOn5)** $IOn$ is Moore continuous.*

For $n = 2$, $IOn$ is just called iv-overlap function [18, 19].

Let $On_1, On_2 : [0,1]^n \to [0,1]$ be $n$-dimensional overlap functions such that $On_1 \leq On_2$. By [11], the function $IOn : L([0,1])^n \to L([0,1])$ given, for all $\vec{X} \in L([0,1])^n$, by $IOn(\vec{X}) = \widehat{On_1, On_2}(\vec{X})$, is a representable $n$-dimensional iv-overlap function. As both its representatives are $n$-dimensional overlap functions, it is said to be $o$-representable [11].

**Definition 6.** *[11] A function $IGO : L([0,1])^n \to L([0,1])$ is a general iv-overlap function if, for all $\vec{X} \in L([0,1])^n$: **(IGO1)** $IGO$ is commutative; **(IGO2)** $\prod_{i=1}^n X_i = [0,0] \Rightarrow IGO(\vec{X}) = [0,0]$; **(IGO3)** $\prod_{i=1}^n X_i = [1,1] \Rightarrow IGO(\vec{X}) = [1,1]$; **(IGO4)** $IGO$ is $\leq_{Pr}$-increasing; **(IGO5)** $IGO$ is Moore continuous.*

**Definition 7.** *[17] A function $AOn : L([0,1])^n \to L([0,1])$ is an n-dimensional admissibly ordered iv-overlap function for an admissible order $\leq_{AD}$ (n-dimensional $\leq_{AD}$-overlap function) if it satisfies **(IOn1)**, **(IOn2)** and **(IOn3)** from Def. 5, and **(AOn4)**: $AOn$ is $\leq_{AD}$-increasing.*

**Remark 2.** *Observe that condition **(IOn5)** was not considered in Def. 7, as the continuity condition of overlap functions was only a requirement in order for them to be applied in image processing problems, which was not the case in [17].*

The following Theorem presents a construction method for $n$-dimensional $\leq_{\alpha,\beta}$-overlap functions:

**Theorem 1.** *[17] Let $On$ be a strict n-dimensional overlap function, $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AOn^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$AOn^\alpha(\vec{X}) = [On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) - \alpha m, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) + (1-\alpha)m], \text{ where}$$
$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), 1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\},$$

*is an $n$-dimensional $\leq_{\alpha,\beta}$-overlap function.*

**Remark 3.** *Notice that **(IOn2)** and **(IOn3)** are both necessary and sufficient conditions. For that reason, the construction method presented in Theo. 1 must consider $\alpha \in (0,1)$ and, consequently, cannot be applied to obtain neither an $n$-dimensional $\leq_{0,1}$-overlap function nor an $n$-dimensional $\leq_{1,0}$-overlap function, that is, $n$-dimensional admissibly ordered iv-overlap functions that are increasing with respect to the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, respectively. This drawback is going to be addressed in our developments in this work. Furthermore, the chosen $n$-dimensional overlap function $On$ must be strict, to ensure that the constructed function is $\leq_{\alpha,\beta}$-increasing.*

Let $c \in [0,1]$ and $\alpha \in [0,1]$. Denote by $d_\alpha(c)$ the maximal possible width of an interval $Z \in L([0,1])$ such that $K_\alpha(Z) = c$. For any $X \in L([0,1])$, define $\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))}$. In [23], it was shown that

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha}\right\},$$

where it is set that $\frac{r}{0} = 1$, for all $r \in [0,1]$.

**Theorem 2.** *[23] Let $\alpha, \beta \in [0,1]$, such that, $\alpha \neq \beta$. Let $A_1, A_2 : [0,1]^n \to [0,1]$ be two aggregation functions where $A_1$ is strictly increasing. Then $IF^\alpha : L([0,1])^n \to L([0,1])$ defined by:*

$$IF^\alpha_{A1,A2}(\vec{X}) = R, \ \text{where,} \begin{cases} K_\alpha(R) = A_1(K_\alpha(X_1), \dots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X_1), \dots, \lambda_\alpha(X_n)), \end{cases}$$

*for all $\vec{X} \in L([0,1])^n$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.*

**Corollary 1.** *Let $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict $n$-dimensional overlap function and $A : [0,1]^n \to [0,1]$ be an aggregation function. Then $IF^\alpha_{O,A} : L([0,1])^n \to L([0,1])$ defined by:*

$$IF^\alpha_{On,A}(\vec{X}) = R, \ \text{where,} \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \dots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \dots, \lambda_\alpha(X_n)), \end{cases}$$

*for all $\vec{X} \in L([0,1])^n$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.*

## 3. General admissibly ordered iv-overlap functions

By combining the concepts of general iv-overlap functions and $n$-dimensional admissibly ordered iv-overlap functions, we introduce the following definition:

**Definition 8.** *A function $AGO : L([0,1])^n \to L([0,1])$ is a general admissibly ordered iv- overlap function for an admissible order $\leq_{AD}$ (general $\leq_{AD}$-overlap function) if it satisfies the conditions **(IGO1)**, **(IGO2)** and **(IGO3)** of Def. 1, and **(AGO4)**: $AGO$ is $\leq_{AD}$-increasing.*

The following result is immediate:

**Proposition 1.** *If $AOn : L([0,1])^n \to L([0,1])$ is an $n$-dimensional $\leq_{AD}$-overlap function, then it is also a general $\leq_{AD}$-overlap function, but the converse may not hold.*

Here we present some results regarding representable general iv-overlap functions and their increasingness with respect to a particular admissible order. In the following result, consider that a strict general overlap function is a general overlap function that is strictly increasing in $(0,1]$.

**Lemma 1.** *Let $GO : [0,1]^n \to [0,1]$ be a strict general overlap function. Then, $GO$ is an $n$-dimensional overlap function.*

*Proof. It is immediate that $GO$ satisfies **(On1)**, **(On4)** and **(On5)** and, by **(GO2)** and **(GO3)**, it respects the necessary conditions ($\Leftarrow$) of **(On2)** and **(On3)**. Then, we prove the sufficient conditions.*

***(On2)** ($\Rightarrow$) Suppose that $GO$ is strict and does not respect **(On2)** ($\Rightarrow$). Take $\vec{y} \in (0,1]^n$ such that $GO(\vec{y}) = 0$. Then, there exist $\vec{x} \in (0,1]^n$ such that $\vec{x} < \vec{y}$ and, by **(GO4)**, $GO(\vec{x}) = GO(\vec{y}) = 0$, which is a contradiction since $GO$ is strict. Thus, $GO$ respects **(On2)**.*

***(On3)** ($\Rightarrow$) Suppose that $GO$ is strict and does not respect **(On3)** ($\Rightarrow$). By **(GO2)**, one has that $\vec{x} = (1,\dots,1) \Rightarrow GO(\vec{x}) = 1$. Now, take $\vec{y} \in [0,1]^n$ such that $y_i \neq 1$ for some $i \in \{1,\dots,n\}$ and $GO(\vec{y}) = 1$. Then, one has that $\vec{y} < \vec{x}$ and $GO(\vec{y}) = GO(\vec{x}) = 1$, which is a contradiction since $GO$ is strict. Thus, $GO$ respects **(On3)**.*

***Theorem 3.*** *Let $IGO : L([0,1])^n \to L([0,1])$ be a representable general iv-overlap function such that $IGO = \widehat{f,g}$, with $f, g : [0,1]^n \to [0,1]$ and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, $IGO$ is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and $g$ is a strict n-dimensional overlap function.*

*Proof.* *Analogous to the proof of Theo. 3 in [17], taking into account Lem. 1.*

*Then, the following result is immediate:*

***Corollary 2.*** *Let $On : [0,1]^n \to [0,1]$ be an $n$-dimensional overlap function and $IGO : L([0,1])^n \to L([0,1])$ be a general iv-overlap function such that $IGO = \widehat{On}$, and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, $IGO$ is a general $\leq_{\alpha,\beta}$-overlap if and only if $\alpha = 1$ and $On$ is a strict n-dimensional overlap function.*

***Example 3.*** *Consider the general overlap function $GO_P$ as defined in Ex. 1 for $n = 2$. As it is a strict general overlap function, then, by Lem. 1, it is also a strict overlap function. Then, the iv-function $AGO_P : L([0,1])^2 \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^2$, by $AGO_P(\vec{X}) = \widehat{GO_P}(\vec{X})$ is a general $\leq_{1,0}$-overlap function, and also an $2$-dimensional $\leq_{1,0}$-overlap function.*

*The first construction method for general $\leq_{AD}$-overlap functions is an adaptation of Theo. 1, by taking $\alpha \in [0,1]$, obtaining a general $\leq_{\alpha,\beta}$-overlap function.*

***Theorem 4.*** *Let $On$ be a strict n-dimensional overlap function, $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AGO^{\alpha} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$AGO^{\alpha}(\vec{X}) = [On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) - \alpha m, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) + (1-\alpha)m], \text{ where}$$
$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), 1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\},$$

*is a general $\leq_{\alpha,\beta}$-overlap function.*

*Proof.* *Analogous to the proof of Theo. 4 in [17].*

***Remark 4.*** *Observe that **(IGO2)** and **(IGO3)** are only sufficient conditions, allowing for $\alpha \in [0,1]$ in the construction method presented in Theo. 4, differently than in Theo. 1, in which $\alpha \in (0,1)$. This means that, through Theo. 4, one can obtain general $\leq_{AD}$-overlap functions that are increasing with respect to either one of the lexicographical orders.*

***Remark 5.*** *Regarding Theo. 4, one could think that it could be based on a general overlap function $GO$ instead of a $n$-dimensional overlap function $On$, for it to be even more broad of a method. However, as the base function needs to be strictly increasing in order to the constructed iv-function $AGO^{\alpha}$ to be $\leq_{\alpha,\beta}$-increasing, by Lem. 1, one has that every strict general overlap function is also an $n$-dimensional overlap function, and that is why we chose to maintain $On$ in Theo. 4 to reinforce this fact.*

***Example 4.*** *Consider the general overlap function $GO_P$ as defined in Ex. 1. Then, for $\alpha = 1$ and $\beta = 0$, the iv-function $AGO_P^1 : L([0,1])^n \to L([0,1])$ defined for all $\vec{X} \in L([0,1])^n$, by*

$$AGO_P^1(\vec{X}) = [GO_P(\overline{X_1}, \ldots, \overline{X_n}) - m, GO_P(\overline{X_1}, \ldots, \overline{X_n})], \text{ where}$$
$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, GO_P(\overline{X_1}, \ldots, \overline{X_n}), 1 - GO_P(\overline{X_1}, \ldots, \overline{X_n})\},$$

*is a general $\leq_{1,0}$-overlap function, or in other words, a general $\leq_{Lex2}$-overlap function. It is noteworthy that $AGO_P^1$ is not an $n$-dimensional $\leq_{1,0}$-overlap function.*

*The next construction methods are inspired on Theo. 2. First, we will present a more restrictive construction method for $n$-dimensional $\leq_{\alpha,\beta}$-overlap functions:*

***Theorem 5.*** *Let $\alpha, \beta \in (0,1)$, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict $n$-dimensional overlap function and $A : [0,1]^n \to [0,1]$ a commutative aggregation function. $AOn_A^{\alpha} : L([0,1])^n \to L([0,1])$ defined by*

$$AOn_A^{\alpha}(\vec{X}) = R, \text{ where, } \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

*for all $\vec{X} \in L([0,1])^n$, is an $n$-dimensional $\leq_{\alpha,\beta}$-overlap function.*

*Proof. From Theo. 2, it is immediate that $AOn_A^{\alpha}$ is well defined and $\leq_{\alpha,\beta}$-increasing, thus, respecting condition **(AOn4)**. Now, let us verify if $AOn_A^{\alpha}$ respects the remainder conditions from Def. 7:*

***(IOn1)*** *Immediate, since $On$ and $A$ are commutative.*

***(IOn2)*** *($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ and suppose that $AOn_A^{\alpha}(\vec{X}) = R = [0,0]$. Then, we have that $K_\alpha(R) = K_\alpha([0,0]) = 0 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n))$, for all $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_\alpha(X_i) = 0$ for some $i \in \{1, \ldots, n\}$, for all $\alpha \in (0,1)$, and, therefore, $\prod_{i=1}^n X_i = [0,0]$. ($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n = [0,0]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 0$, for all $\alpha \in (0,1)$. Then, by **(On2)**, one has that $K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 0$, for all $\alpha \in (0,1)$, meaning that $AOn_A^{\alpha}(\vec{X}) = R = [0,0]$;*

***(IOn3)*** *($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ such that $AOn_A^{\alpha}(\vec{X}) = R = [1,1]$. Then, one has that $K_\alpha(R) = K_\alpha([1,1]) = 1 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n))$. By **(On3)**, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, for all $\alpha \in (0,1)$, meaning that $\prod_{i=1}^n X_i = [1,1]$. ($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, for all $\alpha \in (0,1)$. Then, by **(i)** and **(O3)**, one has that $K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 1$, for all $\alpha \in (0,1)$, meaning that $AOn_A^{\alpha}(\vec{X}) = R = [1,1]$.*

*The following result is immediate, as it derives from a similar situation as discussed in Remarks 4 and 5.*

***Theorem 6.*** *Let $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict $n$-dimensional overlap function and $A : [0,1]^n \to [0,1]$ a commutative aggregation function. $AGO_A^{\alpha} : L([0,1])^n \to L([0,1])$ defined by*

$$AGO_A^{\alpha}(\vec{X}) = R, \text{ where, } \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

*for all $\vec{X} \in L([0,1])^n$, is an general $\leq_{\alpha,\beta}$-overlap function.*

***Example 5.*** *Consider the general overlap functions $GO_L$ and $GO_{Gm}$ as defined in Ex. 1. For $\alpha = 1$ and $\beta = 0$, the iv- function $AGm_{GO_L}^1 : L([0,1])^n \to L([0,1])$ defined for all $\vec{X} \in L([0,1])^n$, by*

$$AGm_{GO_L}^1(\vec{X}) = R, \text{ where, } \begin{cases} K_1(R) = GO_{Gm}(\overline{X_1}, \ldots, \overline{X_n}), \\ \lambda_1(R) = GO_L(\lambda_1(X_1), \ldots, \lambda_1(X_n)), \end{cases}$$

*is a general $\leq_{1,0}$-overlap function, but not an $n$-dimensional $\leq_{1,0}$-overlap function.*

*The following method allow the construction of general $\leq_{AD}$-overlap functions by the generalized composition of general $\leq_{AD}$-overlap functions by an $\leq_{AD}$-increasing iv-aggregation function.*

***Theorem 7.*** *Consider $IM : L([0,1])^m \to L([0,1])$. For a tuple $\overrightarrow{AGO} = (AGO_1, \ldots, AGO_m)$ of general $\leq_{AD}$-overlap functions, define the mapping $IM_{\overrightarrow{AGO}} : L([0,1])^n \to L([0,1])$, for all $\vec{X} \in L([0,1])^n$, by: $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X}))$. Then, $IM_{\overrightarrow{AGO}}$ is a general $\leq_{AD}$-overlap function if and only if $IM$ is an $\leq_{AD}$-increasing iv-aggregation function.*

*Proof.* (⇒) *Suppose that* $IM_{\overrightarrow{AGO}}$ *is a general* $\leq_{AD}$*-overlap function. Then it is immediate that* $IM$ $\leq_{AD}$*-increasing, and, also,* $\leq_{Pr}$*-increasing* ***(IA2)****. Now consider* $\vec{X} \in L([0,1])^n$ *such that* $\prod_{i=1}^n X_i = [0,0]$*. Then, by* ***(IGO2)****, one has that:* $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = [0,0]$ *and* $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [0,0]$*. Thus, it holds that* $IM([0,0], \ldots, [0,0]) = [0,0]$*. Now, consider* $\vec{X} \in L([0,1])^n$*, such that* $X_i = [1,1]$ *for all* $i \in \{1, \ldots, n\}$*. Then, by* ***(IGO3)****, one has that:* $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = [1,1]$ *and* $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [1,1]$*. Therefore, it holds that* $IM([1,1], \ldots, [1,1]) = [1,1]$*. This proves that* $IM$ *also satisfies condition* ***(IA1)****, and, thus, an* $\leq_{AD}$*-increasing iv-aggregation function.* (⇐) *Suppose that* $IM$ *is an* $\leq_{AD}$*-increasing iv-aggregation function. Then it is immediate that* $IM_{\overrightarrow{AGO}}$ *is commutative (by* ***(IGO1)****), and respects* ***(AGO4)****.*

***(IGO2)*** *Consider* $\vec{X} \in L([0,1])^n$ *such that* $\prod_{i=1}^n X_i = [0,0]$*. Then, by* ***(IGO2)****, one has that* $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [0,0]$*. It follows that:* $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = IM([0,0], \ldots, [0,0]) = [0,0]$*, by condition* ***(IA1)****, since* $IM$ *is an iv-aggregation function.*

***(IGO3)*** *Take* $\vec{X} \in L([0,1])^n$ *such that* $X_i = [1,1]$ *for all* $i \in \{1, \ldots, n\}$*. Then, by* ***(IGO3)****, it holds that* $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [1,1]$*. Thus,* $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = IM([1,1], \ldots, [1,1]) = [1,1]$*, by* ***(IA1)****. Then,* $IM_{\overrightarrow{AGO}}(\vec{X})$ *is a general* $\leq_{AD}$*-overlap function.*

***Example 6.*** *Consider the general* $\leq_{1,0}$*-overlap functions* $AGO_P$*,* $AGO_P^1$ *and* $AGm_{GO_L}^1$*, from Ex.s 3, 4 and 5. Then, the iv-function* $AGO : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by* $AGO(\vec{X}) = AGO_P(AGO_P^1(\vec{X}), AGm_{GO_L}^1(\vec{X}))$*, is a general* $\leq_{1,0}$*-overlap function.*

## 4. Conclusion

*In this paper we presented the concept of general admissibly ordered iv-overlap functions, a more flexible definition of* $n$*-dimensional iv-overlap functions that are increasing with respect to an admissible order. This new definition allowed us to construct several iv-overlap operations taking into account different admissible orders, in particular,* $\leq_{\alpha,\beta}$ *orders with any* $\alpha, \beta \in [0,1]$ *such that* $\alpha \neq \beta$*, which includes the lexicographical orders. Finally, those constructed functions can be combined by generalized composition to obtain new general admissibly ordered iv-overlap functions, showcasing their adaptability.*

*Most construction methods for* $\leq_{\alpha,\beta}$*-increasing functions are based on the aggregation of the* $K_\alpha$ *values of the inputs by strictly increasing aggregation functions, which is a restriction that could be interesting to overcome in our future work. We also intend to apply the developed functions (with different combination of construction methods) in classification problems with interval-valued data.*

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