Reasoning in the Description Logic ALC under Category Semantics

Extended Abstract

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1. Introduction

Languages based on Description Logics (DLs) [1] such as OWL [2], OWL2 [3], are widely used to represent ontologies in semantics-based applications. $\mathcal{ALC}$ is the smallest DL involving roles which is closed under negation. It is a suitable logic for a first attempt to replace the usual set-theoretical semantics by another one. We present in this paper a reformulation of the usual set-theoretical semantics of the description logic $\mathcal{ALC}$ with general TBoxes by using categorical language. In this setting, $\mathcal{ALC}$ concepts are represented as objects, concept subsumptions as arrows, and memberships as logical quantifiers over objects and arrows of categories. Such a category-based semantics provides a more modular representation of the semantics of $\mathcal{ALC}$. This feature allows us to define a sublogic of $\mathcal{ALC}$ by dropping the interaction between existential and universal restrictions, which would be responsible for an exponential complexity in space. Such a sublogic is undefinable in the usual set-theoretical semantics, We show that this sublogic is PSPACE by proposing a deterministic algorithm for checking concept satisfiability which runs in polynomial space.

It is well known that $\mathcal{ALC}$ with general TBoxes is EXPTIME-complete [4, 5] while $\mathcal{ALC}$ without TBox is PSPACE-complete [6]. Hence, the interaction between existential and universal restrictions with general TBoxes would be responsible for an intractable space complexity if PSPACE$\subsetneq$EXPTIME. The main motivation of this work consists in identifying a sublogic of $\mathcal{ALC}$ with general TBoxes which allows to reduce the reasoning complexity without losing too much of the expressive power.

To reduce space complexity when reasoning with $\mathcal{ALC}$ under the usual set-theoretical semantics, one can restrict expressiveness of TBoxes by preventing them from having cyclic axioms [7]. This restriction may be too strong for those who wish to express simple knowledge of cyclic nature such as Human $\sqsubseteq$ $\exists$hasParent.Human.

However, we can find a sublogic of $\mathcal{ALC}$ under category-theoretical semantics such that it allows to fully express general TBoxes and only needs to drop a part of the semantics of universal restrictions. Such a sublogic is undefinable in the usual set-theorectical semantics. Indeed, a

universal restriction $\forall R.C$ in $\mathcal{ALC}$ can be defined under category-theoretical semantics by using the following two informal properties (that will be developed in more detail below, in Definition 3): (i) $\forall R.C$ is "very close" to $\neg\exists R.\neg C$; and (ii) $\exists R.C \sqcap \forall R.D$ is "very close" to $\exists R.(C \sqcap D)$. We can observe that Property (ii) is a (weak) representation of the interaction between universal and existential restrictions. If we may just remove this interaction from the categorical semantics of universal restriction, we obtain a new logic, namely $\mathcal{ALC}_{\overline{\forall}}$, in which reasoning will be tractable in space. The semantic loss caused by this removal might be tolerable in certain cases. We illustrate it with the example below.

Example 1. *Consider for instance the following $\mathcal{ALC}$ TBox:*
  HappyChild $\sqsubseteq$ $\exists$eatsFood.Dessert $\sqcap$ $\forall$eatsFood.HotMeal
  Dessert $\sqsubseteq$ $\neg$HotMeal
As in the usual set-theoretical semantics, nobody can be a HappyChild *under category-theoretical semantics: the concept is unsatisfiable in this world. Indeed, according to the first axiom if somebody is a* HappyChild, *then they must simultaneously (1) have some* Dessert *to eat, and (2) eat only* HotMeal, *which contradicts the second axiom. Now, if the second axiom is removed from the TBox, under set-theoretical semantics, the first axiom entails that if somebody is a* HappyChild, *then they eat* HotMeal. *In fact, the first subconcept (*$\exists$eatsFood.Dessert*) ensures that there exists at least one food item that they eat, and the second one (*$\forall$eatsFood.HotMeal*) that this food is necessarily* HotMeal.

  Under the category-theoretical semantics that we will define in this paper ($\mathcal{ALC}_{\overline{\forall}}$*), the first axiom alone does not allow to entail that if somebody is a* HappyChild *then they eat* HotMeal. *This is due to the fact that the element of the definition of universal quantification that was there to represent Property (ii) has been dropped, and that unlike set-theoretical semantics, we no longer have a set of individuals providing an extensional "support" for the second subconcept.*

2. Category-Theoretical Semantics of $\mathcal{ALC}$

We can observe that the set-theoretical semantics of $\mathcal{ALC}$ is based on set membership relationships, while ontology inferences, such as consistency or concept subsumption, involve set inclusions. This explains why inference algorithms developed in this setting often have to build sets of individuals connected in some way for representing a model. In this section, we use some basic notions in category theory to characterize the semantics of $\mathcal{ALC}$. Instead of set membership, in this categorical language, we use "objects" and "arrows" to define semantics of a given object.

Definition 1 (Syntax categories). *Let $\mathbf{R}$ and $\mathbf{C}$ be non-empty sets of* role names *and* concept names *respectively. We define a* concept syntax category $\mathscr{C}_c$ *and a* role syntax category $\mathscr{C}_r$ *from the signature $\langle \mathbf{C}, \mathbf{R} \rangle$ as follows:*

  1. *Each role name R in $\mathbf{R}$ is an object R of $\mathscr{C}_r$. In particular, there are initial and terminal objects $R_\perp$ and $R_\top$ in $\mathscr{C}_r$ with arrows $R \longrightarrow R_\top$ and $R_\perp \longrightarrow R$ for all object R of $\mathscr{C}_r$. There is also an identity arrow $R \longrightarrow R$ for each object R of $\mathscr{C}_r$.*

2. *Each concept name in* $\mathbf{C}$ *is an object of* $\mathscr{C}_c$. *In particular,* $\bot$ *and* $\top$ *are respectively initial and terminal objects, i.e. there are arrows* $C \longrightarrow \top$ *and* $\bot \longrightarrow C$ *for each object* C *of* $\mathscr{C}_c$. *Furthermore, for each object* C *of* $\mathscr{C}_c$ *there is an identity arrow* $C \longrightarrow C$, *for each object* R *of* $\mathscr{C}_r$ *there are two objects of* $\mathscr{C}_c$, *namely* $\mathsf{dom}(R)$ *and* $\mathsf{cod}(R)$, *and for each object* $\exists R.C$ *of* $\mathscr{C}_c$ *there is an object* $R_{(\exists R.C)}$ *of* $\mathscr{C}_r$.

3. *If there are arrows* $E \longrightarrow F$ *and* $F \longrightarrow G$ *in* $\mathscr{C}_c$ *(resp.* $\mathscr{C}_r$*), then there is an arrow* $E \longrightarrow G$ *in* $\mathscr{C}_c$ *(resp.* $\mathscr{C}_r$*).*

4. *There are two functors* dom *and* cod *from* $\mathscr{C}_r$ *to* $\mathscr{C}_c$, *i.e. they associate two objects* $\mathsf{dom}(R)$ *and* $\mathsf{cod}(R)$ *of* $\mathscr{C}_c$ *to each object* R *of* $\mathscr{C}_r$ *such that*

   a) $\mathsf{dom}(R_\top) = \top$, $\mathsf{cod}(R_\top) = \top$, $\mathsf{dom}(R_\bot) = \bot$ *and* $\mathsf{cod}(R_\bot) = \bot$.

   b) *if there is an arrow* $R \longrightarrow R'$ *in* $\mathscr{C}_r$ *then there are arrows* $\mathsf{dom}(R) \longrightarrow \mathsf{dom}(R')$ *and* $\mathsf{cod}(R) \longrightarrow \mathsf{cod}(R')$.

   c) *if there are arrows* $R \longrightarrow R' \longrightarrow R''$ *in* $\mathscr{C}_r$ *then there are arrows* $\mathsf{dom}(R) \longrightarrow \mathsf{dom}(R'')$ *and* $\mathsf{cod}(R) \longrightarrow \mathsf{cod}(R'')$.

   d) *if there is an arrow* $\mathsf{dom}(R) \longrightarrow \bot$ *or* $\mathsf{cod}(R) \longrightarrow \bot$ *in* $\mathscr{C}_c$, *then there is an arrow* $R \longrightarrow R_\bot$ *in* $\mathscr{C}_r$.

For each arrow $E \longrightarrow F$ *in* $\mathscr{C}_c$ *or* $\mathscr{C}_r$, E *and* F *are respectively called* domain *and* codomain *of the arrow. We use also* $\mathsf{Ob}(\mathscr{C})$ *and* $\mathsf{Hom}(\mathscr{C})$ *to denote the collections of objects and arrows of a category* $\mathscr{C}$.

Definition 1 provides a general framework with syntax elements and necessary properties from category theory. We need to "instantiate" it to obtain categories which capture semantic constraints coming from axioms.

Definition 2 (Ontology categories). *Let* C *be an* $\mathcal{ALC}$ *concept and* $\mathcal{O}$ *an* $\mathcal{ALC}$ *ontology from a signature* $\langle \mathbf{C}, \mathbf{R} \rangle$. *We define a* concept ontology category $\mathscr{C}_c \langle C, \mathcal{O} \rangle$ *and a* role ontology category $\mathscr{C}_r \langle C, \mathcal{O} \rangle$ *as follows:*

   1. $\mathscr{C}_c \langle C, \mathcal{O} \rangle$ *and* $\mathscr{C}_r \langle C, \mathcal{O} \rangle$ *are syntax categories from* $\langle \mathbf{C}, \mathbf{R} \rangle$.

   2. C *is an object of* $\mathscr{C}_c \langle C, \mathcal{O} \rangle$.

   3. *If* $E \sqsubseteq F$ *is an axiom of* $\mathcal{O}$, *then* E, F *are objects and* $E \longrightarrow F$ *is an arrow of* $\mathscr{C}_c \langle C, \mathcal{O} \rangle$.

In the sequel, we introduce category-theoretical semantics of disjunction, conjunction, negation, existential and universal restriction objects if they appear in $\mathscr{C}_c \langle C, \mathcal{O} \rangle$. Some of them require more *explicit* properties than those needed for the set-theoretical semantics.

Definition 3 (Category-theoretical semantics). *Assume that* $\mathscr{C}_c \langle C, \mathcal{O} \rangle$ *and* $\mathscr{C}_r \langle C, \mathcal{O} \rangle$ *have all concept and role objects used in the following properties. Category-theoretical semantics of disjunction, conjunction, negation, existential and univeral restrictions are defined by using arrows in* $\mathscr{C}_c \langle C, \mathcal{O} \rangle$ *as follows.*

Disjunction:

$$C \longrightarrow C \sqcup D \text{ and } D \longrightarrow C \sqcup D \text{ are arrows of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{1}$$

$$\text{For all object } X \text{ of } \mathscr{C}_c\langle C, \mathcal{O}\rangle, C \longrightarrow X \text{ and } D \longrightarrow X \text{ imply } C \sqcup D \longrightarrow X \tag{2}$$

Conjunction:

$$C \sqcap D \longrightarrow C \text{ and } C \sqcap D \longrightarrow D \text{ are arrows of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{3}$$

$$\text{For all object } X \text{ of } \mathscr{C}_c\langle C, \mathcal{O}\rangle, X \longrightarrow C \text{ and } X \longrightarrow D \text{ imply } X \longrightarrow C \sqcap D \tag{4}$$

$$C \sqcap (D \sqcup E) \longrightarrow (C \sqcap D) \sqcup (C \sqcap E) \text{ is an arrow of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{5}$$

Negation:

$$C \sqcap \neg C \longrightarrow \bot \text{ and } \top \longrightarrow C \sqcup \neg C \text{ are arrows of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{6}$$

$$\text{For all object } X \text{ of } \mathscr{C}_c\langle C, \mathcal{O}\rangle, C \sqcap X \longrightarrow \bot \text{ implies } X \longrightarrow \neg C \tag{7}$$

$$\text{For all object } X \text{ of } \mathscr{C}_c\langle C, \mathcal{O}\rangle, \top \longrightarrow C \sqcup X \text{ implies } \neg C \longrightarrow X \tag{8}$$

Existential restriction:

$$R_{(\exists R.C)} \longrightarrow R, \mathsf{cod}(R_{(\exists R.C)}) \longrightarrow C \text{ are arrows of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{9}$$

$$\mathsf{dom}(R_{(\exists R.C)}) \Longleftrightarrow \exists R.C \text{ are arrows of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{10}$$

$$\text{For all object } R' \text{ of } \mathscr{C}_r\langle C, \mathcal{O}\rangle, R' \longrightarrow R, \mathsf{cod}(R') \longrightarrow C \text{ imply } \mathsf{dom}(R') \longrightarrow \mathsf{dom}(R_{(\exists R.C)}) \tag{11}$$

Universal restriction:

$$\forall R.C \Longleftrightarrow \neg \exists R.\neg C \text{ are arrows of } \mathscr{C}_c\langle C, \mathcal{O}\rangle \tag{12}$$

$$\text{For all object } R' \text{ of } \mathscr{C}_r\langle C, \mathcal{O}\rangle, R' \longrightarrow R, \mathsf{dom}(R') \longrightarrow \forall R.C \text{ imply } \mathsf{cod}(R') \longrightarrow C \tag{13}$$

Note that both $\mathscr{C}_c\langle C, \mathcal{O}\rangle$ and $\mathscr{C}_r\langle C, \mathcal{O}\rangle$ may consist of more objects. However, new arrows should be derived from those existing or the properties given in Definitions (1-3). Adding to $\mathscr{C}_c\langle C, \mathcal{O}\rangle$ a new arrow that is *independent* from those existing leads to a semantic change of the ontology. We now introduce category-theoretical satisfiability of an $\mathcal{ALC}$ concept with respect to an $\mathcal{ALC}$ ontology.

Definition 4. *Let C_0 be an $\mathcal{ALC}$ concept, $\mathcal{O}$ an $\mathcal{ALC}$ ontology. C is category-theoretically unsatifiable with respect to $\mathcal{O}$ if there is an ontology category $\mathscr{C}_c\langle C_0, \mathcal{O}\rangle$ which has an arrow $C_0 \longrightarrow \bot$.*

Theorem 1. *Let $\mathcal{O}$ be an $\mathcal{ALC}$ ontology and C_0 an $\mathcal{ALC}$ concept. C_0 is category-theoretically unsatifiable with respect to $\mathcal{O}$ iff C_0 is set-theoretically unsatifiable.*

3. Reasoning in a Sublogic of $\mathcal{ALC}$

In this section, we identify a new sublogic of $\mathcal{ALC}$, namely $\mathcal{ALC}_{\overline{\forall}}$, obtained from $\mathcal{ALC}$ with all the properties introduced in Definition 3, except for Property (13). This sublogic cannot be defined under the usual set-theoretical semantics since Property (13) is *not independent* from the properties defined for existential restriction and Property (12) in this setting. Indeed, for every interpretation $\mathcal{I}$ it holds that if $R'^{\mathcal{I}} \subseteq R^{\mathcal{I}}$, $\mathsf{dom}(R')^{\mathcal{I}} \subseteq \forall R.C^{\mathcal{I}}$ and $x \in \mathsf{cod}(R')^{\mathcal{I}}$, then $x \in C^{\mathcal{I}}$.

To show the independence of Property (13) under the category semantics, we can build a category that verifies Properties (1-12) but Property (13) cannot be derived from this category.

Definition 5 (Category-theoretical unsatisfiability in $\mathcal{ALC}_{\overline{\forall}}$). *Let C be an $\mathcal{ALC}_{\overline{\forall}}$ concept, $\mathcal{O}$ an $\mathcal{ALC}_{\overline{\forall}}$ ontology. C is category-theoretically unsatifiable with respect to $\mathcal{O}$ if there is an ontology category $\mathscr{C}_c\langle C, \mathcal{O} \rangle$ which has an arrow $C \longrightarrow \bot$.*

To check satisfiability of an $\mathcal{ALC}_{\overline{\forall}}$ concept C_0 with respect to an ontology $\mathcal{O}$, we define a set of saturation rules each of which represents a property introduced for the category semantics except for Property (5) (distributivity). We initialize an ontology category $\mathscr{C}\langle C_0, \mathcal{O} \rangle$ and saturate it by applying the saturation rules to $\mathscr{C}_c\langle C_0, \mathcal{O} \rangle$ until no rule is applicable.

Theorem 2. *Satisfiability of an $\mathcal{ALC}_{\overline{\forall}}$ concept with respect to an ontology can be decided in polynomial space.*

As mentioned above, we do not introduce a saturation rule to directly capture Property (5) since this may lead to adding an exponential number of objects. The polynomial complexity in space results from constructing a category $\mathscr{C}_c$ by using a rule, namely [$\sqcup_{\mathsf{dis}}$-**rule**], that call an algorithm check instead of applying directly Property (5). This algorithm traverses at most all binary trees T_C where C is an object of the form $C = (C_1 \sqcup D_1) \sqcap \cdots \sqcap (C_n \sqcup D_n)$ that must be included in $\mathscr{C}_c$. Such a tree T_C is composed of nodes of the form $X_1 \sqcap \cdots \sqcap X_n$ where $X_i \in \{C_i, D_i, C_i \sqcup D_i\}$ for $1 \leq i \leq n$. Such nodes $X_1 \sqcap \cdots \sqcap X_n$ are not necessarily included in $\mathscr{C}_c$ but each conjunct X_i is an object of $\mathscr{C}_c$. This allows the algorithm check to discover all arrows $X \longrightarrow Y$ where X, Y are objects of $\mathscr{C}_c$ by using arrows of $\mathscr{C}_c$ and those resulting from distributivity (Property (5)) applied to nodes of trees T_C. Since the nodes of a tree T_C can be encoded as binary numbers between 0 and $2^n - 1$, we can use an algorithm that runs in polynomial space to traverse all its nodes.

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