Optimizing the Crisp Representation of the Fuzzy Description Logic $\mathcal{SROIQ}$

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Abstract. Classical ontologies are not suitable to represent imprecise nor uncertain pieces of information. Fuzzy Description Logics were born to represent the former type of knowledge, but they require an appropriate fuzzy language to be agreed and an important number of available resources to be adapted. This paper faces these problems by presenting a reasoning preserving procedure to obtain a crisp representation for a fuzzy extension of the logic $\mathcal{SROIQ}$ which uses Gödel implication in the semantics of fuzzy concept and role subsumption. This reduction allows to reuse a crisp representation language as well as currently available reasoners. Our procedure is optimized with respect to the related work, reducing the size of the resulting knowledge base, and is implemented in DeLorean, the first reasoner supporting fuzzy OWL DL.

1 Introduction

Description Logics (DLs for short) [1] are a family of logics for representing structured knowledge which have proved to be very useful as ontology languages. For instance, $\mathcal{SROIQ}(\mathbf{D})$ [2] is the subjacent DL of OWL 1.1., a recent extension of the standard language OWL[1] which is its most likely immediate successor.

Nevertheless, it has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real-world domains. Since fuzzy logic is a suitable formalism to handle these types of knowledge, several fuzzy extensions of DLs can be found in the literature (see [3] for an overview).

Defining a fuzzy DL brings about that crisp standard languages are no longer appropriate, new fuzzy languages need to be used, and hence the large number of resources available need to be adapted to the new framework, requiring an important effort. An additional problem is that reasoning within (crisp) expressive DLs has a very high worst-case complexity (e.g. NExpTime in $\mathcal{SHOIN}$) and, consequently, there exists a significant gap between the design of a decision procedure and the achievement of a practical implementation [4].

An alternative is to represent fuzzy DLs using crisp DLs and to reduce reasoning within fuzzy DLs to reasoning within crisp ones. This has several advantages:

[1] http://www.w3.org/TR/owl-features

– There would be no need to agree a new standard fuzzy language, but every developer could use its own language expressing fuzzy $\mathcal{SROIQ}$, as long as he implements the reduction that we describe.
– We will continue using standard languages with a lot of resources available, so the need (and cost) of adapting them to the new fuzzy language is avoided.
– We will continue using the existing crisp reasoners. We do not claim that reasoning will be more efficient, but it supposes an easy alternative to support early reasoning in future fuzzy languages. In fact, nowadays there is no reasoner fully supporting a fuzzy extension of OWL DL.

Under this approach an immediate practical application of fuzzy ontologies is feasible, because of its tight relation with already existing languages and tools which have proved their validity.

Although there has been a relatively significant amount of works in extending DLs with fuzzy set theory ([3] is a good survey), the representation of them using crisp description logics has not received such attention. The first efforts in this direction are due to U. Straccia, who considered fuzzy $\mathcal{ALCH}$ [5] and fuzzy $\mathcal{ALC}$ with truth values taken from an uncertainty lattice [6]. F. Bobillo et al. extended Straccia's work to $\mathcal{SHOIN}$, including fuzzy nominals and fuzzy General Concept Inclusions (GCIs) with a semantics given by Kleene-Dienes (KD) implication [7]. Finally, G. Stoilos et al. extended this work to $\mathcal{SROIQ}$ [8]. This paper improves the latter work providing the following contributions:

– We provide a full representation, differently from [8] which do not show how to reduce qualified cardinality restrictions, local reflexivity concepts in expressions of the form $\rho(\exists S.Self, \lhd\gamma)$ nor negative role assertions.
– [5, 8] force GCIs and Role Inclusion Axioms (RIAs) to be either true or false, but we will allow them to be verified up to some degree by using Gödel implication in the semantics.
– We improve one of their starting points (the reduction presented in [5]) by reducing the number of new atomic elements and their corresponding axioms.
– We show how to optimize some important GCIs.
– We present DeLorean, our implementation of the reduction and the first implemented reasoner supporting fuzzy $\mathcal{SHOIN}$.

The remainder is organized as follows. Section 2 describes a fuzzy extension of $\mathcal{SROIQ}$ and discusses some logical properties. Section 3 depicts a reduction into crisp $\mathcal{SROIQ}$, whereas Section 4 presents our implementation of the procedure. Finally, in Section 5 we set out some conclusions and ideas for future work.

2 Fuzzy $\mathcal{SROIQ}$

In this section we define $f\mathcal{SROIQ}$, which extend $\mathcal{SROIQ}$ to the fuzzy case by letting *(i)* concepts denote fuzzy sets of individuals and *(ii)* roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. The following definition combines [7–9], but we will use Gödel implication in the semantics of GCIs and RIAs.

Syntax. $f\mathcal{SROIQ}$ assumes three alphabets of symbols, for concepts, roles and individuals. The concepts of the language (denoted C or D) can be built inductively from atomic concepts (A), atomic roles (R), top concept $\top$, bottom concept $\bot$, named individuals (o_i), universal role (U) and simple roles (S, which will be defined below) according to the following syntax rule, where n, m are natural numbers ($n \geq 0, m > 0$) and $\alpha_i \in [0,1]$: $C, D \rightarrow A \mid \top \mid \bot \mid C \sqcap D \mid C \sqcup D \mid \neg C \mid \forall R.C \mid \exists R.C \mid \{\alpha_1/o_1, \ldots, \alpha_m/o_m\} \mid (\geq n\ S.C) \mid (\leq n\ S.C) \mid \exists S.Self$. Notice that the only difference with the crisp case is the presence of fuzzy nominals [7]. Complex roles are built using the syntax rule $R \rightarrow R_A \mid R^- \mid U$.

A fuzzy Knowledge Base (fKB) comprises two parts: the extensional knowledge, i.e. particular knowledge about some specific situation (a fuzzy Assertional Box or ABox K_A with statements about individuals) and the intensional knowledge, i.e. general knowledge about the application domain (a fuzzy Terminological Box or TBox K_T and a fuzzy Role Box or RBox K_R).

In the rest of the paper we will assume $\bowtie \in \{\geq, <, \leq, >\}$, $\alpha \in (0,1]$, $\beta \in [0,1)$ and $\gamma \in [0,1]$. Moreover, for every operator $\bowtie$ we define *(i)* its symmetric operator $\bowtie^-$ defined as $\geq^- = \leq, >^- = <, \leq^- = \geq, <^- = >$, and *(ii)* its negation operator $\neg \bowtie$, defined as $\neg \geq = <, \neg > = \leq, \neg \leq = >, \neg < = \geq$.

A *fuzzy ABox* consists of a finite set of *fuzzy assertions*. A fuzzy assertion can be an inequality assertion $\langle a \neq b \rangle$, an equality assertion $\langle a = b \rangle$ or a constraint on the truth value of a concept or role assertion, i.e. an expression of the form $\langle \Psi \bowtie \alpha \rangle$, where Ψ is an assertion of the form $a : C$ or $(a, b) : R$.

A *fuzzy TBox* consists of *fuzzy GCIs*, which constrain the truth value of a GCI i.e. they are expressions of the form $\langle \Omega \geq \alpha \rangle$ or $\langle \Omega > \beta \rangle$, where $\Omega = C \sqsubseteq D$.

A *fuzzy RBox* consists of a finite set of role axioms, which can be *fuzzy RIAs* $\langle w \sqsubseteq R \geq \alpha \rangle$ or $\langle w \sqsubseteq R > \beta \rangle$ for a role chain $w = R_1 R_2 \ldots R_n$, or any other of the role axioms from the crisp case: *transitive* $trans(R)$, *disjoint* $dis(S_1, S_2)$, *reflexive* $ref(R)$, *irreflexive* $irr(S)$, *symmetric* $sym(R)$ or *asymmetric* $asy(S)$. As in the crisp case, role axioms cannot contain U and every RIA should be $\prec$-regular for a regular order $\prec$. A RIA $\langle w \sqsubseteq R \rhd \gamma \rangle$ is *$\prec$-regular* if $R = R_A$ and *(i)* $w = RR$, or *(ii)* $w = R^-$, or *(iii)* $w = S_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or *(iv)* $w = RS_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or *(v)* $w = S_1 \ldots S_n R$ and $S_i \prec R$ for all $i = 1, \ldots, n$.

Simple roles are inductively defined: *(i)* R_A is simple if does not occur on the right side of a RIA, *(ii)* R^- is simple if R is, *(iii)* if R occurs on the right side of a RIA, R is simple if, for each $\langle w \sqsubseteq R \rhd \gamma \rangle$, $w = S$ for a simple role S.

A fuzzy axiom τ is *positive* (denoted $\langle \tau \rhd \alpha \rangle$) if it is of the form $\langle \tau \geq \alpha \rangle$ or $\langle \tau > \beta \rangle$, and *negative* (denoted $\langle \tau \lhd \alpha \rangle$) if it is of the form $\langle \tau \leq \beta \rangle$ or $\langle \tau < \alpha \rangle$. $\langle \tau = \alpha \rangle$ is equivalent to the pair of axioms $\langle \tau \geq \alpha \rangle$ and $\langle \tau \leq \alpha \rangle$.

Notice that negative GCIs or RIAs are not allowed, because they correspond to negated GCIs and RIAs respectively, which are not part of crisp $\mathcal{SROIQ}$.

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping *(i)* every individual onto an element of $\Delta^{\mathcal{I}}$, *(ii)* every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1]$, *(iii)* every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1]$.

$C^{\mathcal{I}}$ (resp. $R^{\mathcal{I}}$) denotes the membership function of the fuzzy concept C (resp. fuzzy role R) w.r.t. $\mathcal{I}$. $C^{\mathcal{I}}(a)$ (resp. $R^{\mathcal{I}}(a,b)$) gives us the degree of being the individual a an element of the fuzzy concept C (resp. the degree of being (a,b) an element of the fuzzy role R) under the fuzzy interpretation $\mathcal{I}$. We do not impose unique name assumption, i.e. two nominals might refer to the same individual. For a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles as follows:

$$
\begin{aligned}
\top^{\mathcal{I}}(a) &= 1 \\
\bot^{\mathcal{I}}(a) &= 0 \\
(C \sqcap D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \otimes D^{\mathcal{I}}(a) \\
(C \sqcup D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \oplus D^{\mathcal{I}}(a) \\
(\neg C)^{\mathcal{I}}(a) &= \ominus C^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) &= \inf_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a,b) \rightarrow C^{\mathcal{I}}(b)\} \\
(\exists R.C)^{\mathcal{I}}(a) &= \sup_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a,b) \otimes C^{\mathcal{I}}(b)\} \\
\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}^{\mathcal{I}}(a) &= \sup_{i \mid a \in \{o_i^{\mathcal{I}}\}} \alpha_i \\
(\geq 0\ S.C)^{\mathcal{I}}(a) &= \top^{\mathcal{I}}(a) = 1 \\
(\geq m\ S.C)^{\mathcal{I}}(a) &= \sup_{b_1, \ldots, b_m \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{n} \{S^{\mathcal{I}}(a,b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes (\otimes_{j<k} \{b_j \neq b_k\})] \\
(\leq n\ S.C)^{\mathcal{I}}(a) &= \inf_{b_1, \ldots, b_{n+1} \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{n+1} \{S^{\mathcal{I}}(a,b_i) \otimes C^{\mathcal{I}}(b_i)\}) \rightarrow (\oplus_{j<k} \{b_j = b_k\})] \\
(\exists S.Self)^{\mathcal{I}}(a) &= S^{\mathcal{I}}(a,a) \\
(R^{-})^{\mathcal{I}}(a,b) &= R^{\mathcal{I}}(b,a) \\
U^{\mathcal{I}}(a,b) &= 1
\end{aligned}
$$

A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- *(i)* $\langle a : C \geq \alpha \rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$,
- *(ii)* $\langle (a,b) : R \geq \alpha \rangle$ iff $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$,
- *(iii)* $\langle a \neq b \rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- *(iv)* $\langle a = b \rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- *(v)* $\langle C \sqsubseteq D \geq \alpha \rangle$ iff $\inf_{a \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(a) \rightarrow D^{\mathcal{I}}(a)\} \geq \alpha$,
- *(vi)* $\langle R_1 \ldots R_n \sqsubseteq R \geq \alpha \rangle$ iff $\sup_{b_1 \ldots b_{n+1} \in \Delta^{\mathcal{I}}} \bigotimes [R_1^{\mathcal{I}}(b_1, b_2), \ldots, R_n^{\mathcal{I}}(b_n, b_{n+1})] \rightarrow R^{\mathcal{I}}(b_1, b_{n+1}) \geq \alpha$,
- *(vii)* $trans(R)$ iff $\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a,c) \otimes R^{\mathcal{I}}(c,b)$,
- *(viii)* $dis(S_1, S_2)$ iff $\forall a, b \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(a,b) \otimes S_2^{\mathcal{I}}(a,b) = 0$,
- *(ix)* $ref(R)$ iff $\forall a \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,a) = 1$,
- *(x)* $irr(S)$ iff $\forall a \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(a,a) = 0$,
- *(xi)* $sym(R)$ iff $\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(b,a)$,
- *(xii)* $asy(S)$ iff $\forall a, b \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(a,b) > 0$ then $S^{\mathcal{I}}(b,a) = 0$,
- *(xii)* a fKB iff it satisfies each element in fK_A, fK_T and fK_R.

In cases $(i), (ii)$ similar definitions can be given for $> \beta$, $\leq \beta$ and $< \alpha$, whereas in cases $(v), (vi)$ a similar definition can be given for $> \beta$.

Notice that individual assertions are considered to be crisp

In the rest of the paper we will only consider fKB satisfiability, since (as in the crisp case) most inference problems can be reduced to it [10].

Some logical properties. It can be easily shown that $f\mathcal{SROIQ}$ is a sound extension of crisp $\mathcal{SROIQ}$, because fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{0, 1\}$.

In the fuzzy DLs literature, the notation $f_i\mathcal{DL}$ has been proposed [11], where i is the fuzzy implication function considered. Here in after we will concentrate on $f_{KD}\mathcal{SROIQ}$, restricting ourselves to the Zadeh family: minimum t-norm, maximum t-conorm, Łukasiewicz negation and KD implication, with the exception of GCIs and RIAs, where we will consider Gödel implication. This choice comes from the fact that KD implication specifies a t-norm, a t-conorm and a negation which make possible the reduction to a crisp KB, as we will see in Section 3 (other fuzzy operators are not suitable for a similar reduction).

However, the use of KD implication in the semantics of GCIs and RIAs brings about two counter-intuitive effects: *(i)* in general concepts (and roles) do not fully subsume themselves and *(ii)* crisp subsumption (holding to degree 1) forces some fuzzy concepts and roles to be interpreted as crisp [7].

Another common semantics which could be considered is the one based on Zadeh's set inclusion ($C \sqsubseteq D = \forall x \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$) as in [10, 12], but it forces the axioms to be either true or false. For example, under this semantics it is not possible that concept $Hotel$ subsumes concept Inn with degree 0.5.

Gödel implication solves the afore-mentioned problems and is suitable for a classical representation as we will see in Section 3. Moreover, for GCIs of the form $\langle C \sqsubseteq D \geq 1\rangle$, the semantics is equivalent to that of Zadeh's set inclusion.

It is possible to transform concept expressions into a semantically equivalent *Negation Normal Form* (NNF), which is obtained by pushing in the usual manner negation in front of atomic concepts, fuzzy nominals and local reflexivity concepts. In the case of $\neg(\geq 0\ S)$, it could be replaced by $\perp$ since it is an inconsistent concept. In the following, we will assume that concepts are in NNF.

Irreflexive, transitive and symmetric role axioms are syntactic sugar for every R-implication (and consequently it can be assumed that they do not appear in fKBs) due to the following equivalences:

- $irr(S) \equiv \langle \top \sqsubseteq \neg\exists S.Self \geq 1\rangle$,
- $trans(R) \equiv \langle RR \sqsubseteq R \geq 1\rangle$,
- $sym(R) \equiv \langle R \sqsubseteq R^{-} \geq 1\rangle$.

3 An Optimized Crisp Representation for Fuzzy $\mathcal{SROIQ}$

In this section we show how to reduce a $f_{KD}\mathcal{SROIQ}$ fKB into a crisp KB, similarly as in [5, 7, 8]. The procedure preserves reasoning, so existing $\mathcal{SROIQ}$ reasoners could be applied to the resulting KB. First we will describe the reduction and then we will provide an illustrating example. The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new crisp elements.

Adding (an optimized number of) new elements. Let $\mathcal{A}^{fK}$ and $\mathcal{R}^{fK}$ be the set of atomic concepts and atomic roles occurring in a fKB $fK = \langle fK_A, fK_T, fK_R \rangle$. In [5] it is shown that the set of the degrees which must be considered for any reasoning task is defined as $N^{fK} = X^{fK} \cup \{1 - \alpha \mid \alpha \in X^{fK}\}$, where $X^{fK} = \{0, 0.5, 1\} \cup \{\gamma \mid \langle \tau \bowtie \gamma \rangle \in fK\}$. This also holds in $f_{KD}\mathcal{SROIQ}$, but note that it is not necessarily true when other fuzzy operators are considered. Without loss of generality, it can be assumed that $N^{fK} = \{\gamma_1, \ldots, \gamma_{|N^{fK}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leq i \leq |N^{fK}| - 1$. It is easy to see that $\gamma_1 = 0$ and $\gamma_{|N^{fK}|} = 1$.

Now, for each $\alpha, \beta \in N^{fK}$ with $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, for each $A \in \mathcal{A}^{fK}$ and for each $R_A \in \mathcal{R}^{fK}$, two new atomic concepts $A_{\geq\alpha}, A_{>\beta}$ and two new atomic roles $R_{\geq\alpha}, R_{>\beta}$ are introduced. $A_{\geq\alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e the α-cut of A. The other new elements are defined in a similar way. The atomic elements $A_{>1}, R_{>1}, A_{\geq 0}$ and $R_{\geq 0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fKB (e.g. we do not allow GCIs of the form $C \sqsubseteq D \geq 0$). Note that [5, 7] consider $A_{\geq 0}$ and $R_{\geq 0}$.

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq |N^{fK}| - 1, 2 \leq j \leq |N^{fK}| - 1$ and for each $A \in \mathcal{A}^{fK}$, $T(N^{fK})$ is the smallest terminology containing these two axioms: $A_{\geq\gamma_{i+1}} \sqsubseteq A_{>\gamma_i}$, $A_{>\gamma_j} \sqsubseteq A_{\geq\gamma_j}$. Similarly, for each $R_A \in \mathcal{R}^{fK}$, $R(N^{fK})$ is the smallest terminology containing $R_{\geq\gamma_{i+1}} \sqsubseteq R_{>\gamma_i}$ and $R_{>\gamma_i} \sqsubseteq R_{\geq\gamma_i}$.

In contrast to previous works, which use two more atomic concepts $A_{\leq\beta}, A_{<\alpha}$ and some additional axioms ($2 \leq k \leq |N^{fK}|$) [5, 7]:

$$\begin{array}{rlrl} A_{<\gamma_k} \sqsubseteq A_{\leq\gamma_k}, & & A_{\leq\gamma_i} \sqsubseteq A_{<\gamma_{i+1}} \\ A_{\geq\gamma_k} \sqcap A_{<\gamma_k} \sqsubseteq \bot, & & A_{>\gamma_i} \sqcap A_{\leq\gamma_i} \sqsubseteq \bot \\ \top \sqsubseteq A_{\geq\gamma_k} \sqcup A_{<\gamma_k}, & & \top \sqsubseteq A_{>\gamma_i} \sqcup A_{\leq\gamma_i} \end{array}$$

we use $\neg A_{>\gamma_k}$ rather than $A_{\leq\gamma_k}$ and $\neg A_{\geq\gamma_k}$ instead of $A_{<\gamma_k}$, since the six axioms above follow immediately from the semantics of the crisp concepts as Proposition 1 shows:

Proposition 1. *If $A_{\geq\gamma_{i+1}} \sqsubseteq A_{>\gamma_i}$ and $A_{>\gamma_k} \sqsubseteq A_{\geq\gamma_k}$ hold, then the followings axioms are verified:*

$$\begin{array}{ll} (1)\ \neg A_{\geq\gamma_k} \sqsubseteq \neg A_{>\gamma_k} & (2)\ \neg A_{>\gamma_i} \sqsubseteq \neg A_{\geq\gamma_{i+1}} \\ (3)\ A_{\geq\gamma_k} \sqcap \neg A_{\geq\gamma_k} \sqsubseteq \bot & (4)\ A_{>\gamma_i} \sqcap \neg A_{>\gamma_i} \sqsubseteq \bot \\ (5)\ \top \sqsubseteq A_{\geq\gamma_k} \sqcup \neg A_{\geq\gamma_k} & (6)\ \top \sqsubseteq A_{>\gamma_i} \sqcup \neg A_{>\gamma_i} \end{array}$$

(1) and (2) derive from the fact that in crisp DLs $A \sqsubseteq B \equiv \neg B \sqsubseteq \neg A$. (3) and (4) come from the law of contradiction $A \sqcap \neg A \sqsubseteq \bot$, while (5) and (6) derive from the law of excluded middle $\top \sqsubseteq A \sqcup \neg A$. Moreover, we do not introduce the axiom $A_{>0} \sqsubseteq A_{\geq 0}$; since $A_{\geq 0}$ is equivalent to $\top$ the axiom trivially holds.

Mapping fuzzy concepts, roles and axioms. Concept and role expressions are reduced using mapping ρ, as shown in Table 1. Axioms are reduced as in Table 2, where σ maps fuzzy axioms into crisp assertions and κ maps fuzzy TBox (resp. RBox) axioms into crisp TBox (resp. RBox) axioms.

Notice that $\rho(R, \lhd\gamma)$ can only appear in a (crisp) negated role assertion. Notice also that expressions of the form $\rho(A, \geq 0), \rho(A, > 1), \rho(A, \leq 1), \rho(A, < 0)$ cannot appear, because there exist some restrictions on the degree of the axioms in the fKB. The same also holds for $\top$, $\bot$ and R_A. Besides, expressions of the form $\rho(U, \lhd\gamma)$ cannot appear either. Observe that the reduction preserves simplicity of the roles and regularity of the RIAs.

Our reduction of a fuzzy GCI $\langle C \sqsubseteq D \geq 1\rangle$ is equivalent to the reduction of a GCI under a semantics based on Zadeh's set inclusion proposed in [5], although it introduces some unnecessary axioms: $C_{\geq 0} \sqsubseteq D_{\geq 0}$ and $C_{>1} \sqsubseteq D_{>1}$.

Summing up, a fKB $fK = \langle fK_A, fK_T, fK_R\rangle$ is reduced into a KB $\mathcal{K}(fK) = \langle \sigma(fK_A), T(N^{fK}) \cup \kappa(fK, fK_T), R(N^{fK}) \cup \kappa(fK, fK_R)\rangle$.

Example 1. Let us consider the following fKB: $\{\langle sym(isCloseTo)\rangle, \langle (h_1, h_2) : isCloseTo \leq 0.75\rangle\}$. Firstly, $\langle sym(isCloseTo)\rangle$ is represented as the fuzzy RIA $\langle isCloseTo \sqsubseteq isCloseTo^- \geq 1\rangle$. Now, we have to compute the number of truth values which have to be considered: $X^{fK} = \{0, 0.5, 1, 0.75\}$, so $N^{fK} = \{0, 0.25, 0.5, 0.75, 1\}$.

Next, we create some new atomic concepts and roles, as well as some axioms preserving their semantics. $T(N^{fK}) = \emptyset$ and $R(N^{fK})$ will contain the following axioms: $isCloseTo_{\geq 1} \sqsubseteq isCloseTo_{>0.75}$, $isCloseTo_{>0.75} \sqsubseteq isCloseTo_{\geq 0.75}$, $isCloseTo_{\geq 0.75} \sqsubseteq isCloseTo_{>0.5}$, $isCloseTo_{>0.5} \sqsubseteq isCloseTo_{\geq 0.5}$, $isCloseTo_{\geq 0.5} \sqsubseteq isCloseTo_{>0.25}$, $isCloseTo_{>0.25} \sqsubseteq isCloseTo_{\geq 0.25}$ and $isCloseTo_{\geq 0.25} \sqsubseteq isClose\ To_{>0}$.

Finally, we map axioms in the ABox, TBox and RBox. Firstly, $\sigma(\langle (h_1, h_2) : isCloseTo \leq 0.75\rangle) = (h_1, h_2) : \neg isCloseTo_{>0.75}$. Then, $\kappa(\langle isCloseTo \sqsubseteq isCloseTo^- \geq 1\rangle) = \{isCloseTo_{>0} \sqsubseteq isCloseTo^-_{>0}, isCloseTo_{\geq 0.25} \sqsubseteq isCloseTo^-_{\geq 0.25}, isCloseTo_{>0.25} \sqsubseteq isCloseTo^-_{>0.25}, isCloseTo_{\geq 0.5} \sqsubseteq isCloseTo^-_{\geq 0.5}, isCloseTo_{>0.5} \sqsubseteq isCloseTo^-_{>0.5}, isCloseTo_{\geq 0.75} \sqsubseteq isCloseTo^-_{\geq 0.75}, isCloseTo_{>0.75} \sqsubseteq isCloseTo^-_{>0.75}, isCloseTo_{\geq 1} \sqsubseteq isCloseTo^-_{\geq 1}\}$. □

Optimizing GCI reductions. GCI reductions can be optimized in several cases:

- $\langle C \sqsubseteq \top \bowtie \gamma\rangle$ and $\langle \bot \sqsubseteq D \bowtie \gamma\rangle$ are tautologies, so their reductions are unnecessary in the resulting KB.
- $\kappa(\top \sqsubseteq D \bowtie \gamma) = \top \sqsubseteq \rho(D, \bowtie \gamma)$. Note that this kind of axiom appears in role range axioms i.e. C is the range of R iff $\top \sqsubseteq \forall R.C$ holds with degree 1.
- $\kappa(C \sqsubseteq \bot \bowtie \gamma) = \rho(C, > 0) \sqsubseteq \bot$. This appears when two concepts are disjoint i.e. C and D are disjoint iff $C \sqcap D \sqsubseteq \bot$ holds with degree 1.

Another optimization involving GCIs follows from the following observation. If the resulting TBox contains $A \sqsubseteq B$, $A \sqsubseteq C$ and $B \sqsubseteq C$, then $A \sqsubseteq C$ is unnecessary. This is very useful in concept definitions involving the nominal constructor. For example, the reduction of the definition $\kappa(C \sqsubseteq \{1/o_1, 0.5/o_2\}) = \{C_{>0} \sqsubseteq \{o_1, o_2\}, C_{\geq 0.5} \sqsubseteq \{o_1, o_2\}, C_{>0.5} \sqsubseteq \{o_1\}, C_{\geq 1} \sqsubseteq \{o_1\}\}$ can be optimized to: $\{C_{>0} \sqsubseteq \{o_1, o_2\}, C_{\geq 0.5} \sqsubseteq \{o_1\}\}$.

Table 1. Mapping of concept and role expressions.

x	y	$\rho(x,y)$
$\top$	$\rhd\gamma$	$\top$
$\top$	$\lhd\gamma$	$\bot$
$\bot$	$\rhd\gamma$	$\bot$
$\bot$	$\lhd\gamma$	$\top$
A	$\geq \alpha$	$A_{\geq\alpha}$
A	$> \beta$	$A_{>\beta}$
A	$\leq \beta$	$\neg A_{>\beta}$
A	$< \alpha$	$\neg A_{\geq\alpha}$
$\neg A$	$\bowtie \gamma$	$\rho(A, \bowtie^{-} 1-\gamma)$
$C \sqcap D$	$\rhd\gamma$	$\rho(C,\rhd\gamma) \sqcap \rho(D,\rhd\gamma)$
$C \sqcap D$	$\lhd\gamma$	$\rho(C,\lhd\gamma) \sqcup \rho(D,\lhd\gamma)$
$C \sqcup D$	$\rhd\gamma$	$\rho(C,\rhd\gamma) \sqcup \rho(D,\rhd\gamma)$
$C \sqcup D$	$\lhd\gamma$	$\rho(C,\lhd\gamma) \sqcap \rho(D,\lhd\gamma)$
$\exists R.C$	$\rhd\gamma$	$\exists\rho(R,\rhd\gamma).\rho(C,\rhd\gamma)$
$\exists R.C$	$\lhd\gamma$	$\forall\rho(R,\neg\lhd\gamma).\rho(C,\lhd\gamma)$
$\forall R.C$	$\geq \alpha$	$\forall\rho(R,> 1-\alpha).\rho(C,\geq \alpha)$
$\forall R.C$	$> \beta$	$\forall\rho(R,\geq 1-\beta).\rho(C,> \beta)$
$\forall R.C$	$\leq \beta$	$\exists\rho(R,\geq 1-\beta).\rho(C,\leq \beta)$
$\forall R.C$	$< \alpha$	$\exists\rho(R,> 1-\alpha).\rho(C,< \alpha)$
$\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}$	$\bowtie \gamma$	$\{o_i \mid \alpha_i \bowtie \gamma, 1 \leq i \leq n\}$
$\neg\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}$	$\bowtie \gamma$	$\rho(\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}, \bowtie^{-} 1-\gamma)$
$\geq 0\ S.C$	$\bowtie \gamma$	$\rho(\top, \bowtie \gamma)$
$\geq m\ S.C$	$\rhd\gamma$	$\geq m\ \rho(S,\rhd\gamma).\rho(C,\rhd\gamma)$
$\geq m\ S.C$	$\lhd\gamma$	$\leq m{-}1\ \rho(S,\neg\lhd\ \gamma).\rho(C,\neg\lhd\ \gamma)$
$\leq n\ S.C$	$\geq \alpha$	$\leq n\ \rho(S,> 1-\alpha).\rho(C,> 1-\alpha)$
$\leq n\ S.C$	$> \beta$	$\leq n\ \rho(S,\geq 1-\beta).\rho(C,\geq 1-\beta)$
$\leq n\ S.C$	$\leq \beta$	$\geq n{+}1\ \rho(S,\geq 1-\beta).\rho(C,\geq 1-\beta)$
$\leq n\ S.C$	$< \alpha$	$\geq n{+}1\ \rho(S,> 1-\alpha).\rho(C,> 1-\alpha)$
$\exists S.Self$	$\rhd\gamma$	$\exists\rho(S,\rhd\gamma).Self$
$\exists S.Self$	$\lhd\gamma$	$\neg\exists\rho(S,\neg\lhd\gamma).Self$
R_A	$\geq \alpha$	$R_{\geq\alpha}$
R_A	$> \beta$	$R_{>\beta}$
R_A	$\leq \beta$	$\neg R_{>\beta}$
R_A	$< \alpha$	$\neg R_{\geq\alpha}$
R^{-}	$\rhd\gamma$	$\rho(R,\rhd\gamma)^{-}$
U	$\geq \alpha$	U
U	$> \beta$	U

Table 2. Reduction of the axioms.

$\sigma(\langle a\!:\!C \bowtie \gamma\rangle)$	$a\!:\!\rho(C, \bowtie \gamma)$
$\sigma(\langle (a,b)\!:\!R \bowtie \gamma\rangle)$	$(a,b)\!:\!\rho(R, \bowtie \gamma)$
$\sigma(\langle a \neq b\rangle)$	$a \neq b$
$\sigma(\langle a = b\rangle)$	$a = b$
$\kappa(C \sqsubseteq D \geq \alpha)$	$\bigcup_{\gamma \in N^{fK}-\{0\} \mid \gamma \leq \alpha} \{\rho(C, \geq \gamma) \sqsubseteq \rho(D, \geq \gamma)\} \bigcup_{\gamma \in N^{fK} \mid \gamma < \alpha} \{\rho(C, > \gamma) \sqsubseteq \rho(D, > \gamma)\}$
$\kappa(C \sqsubseteq D > \beta)$	$\kappa(C \sqsubseteq D \geq \beta) \cup \{\rho(C, > \beta) \sqsubseteq \rho(D, > \beta)\}$
$\kappa(\langle R_1 \dots R_n \sqsubseteq R \geq \alpha\rangle)$	$\bigcup_{\gamma \in N^{fK}-\{0\} \mid \gamma \leq \alpha} \{\rho(R_1, \geq \gamma) \dots \rho(R_n, \geq \gamma) \sqsubseteq \rho(R, \geq \gamma)\} \bigcup_{\gamma \in N^{fK} \mid \gamma < \alpha}$ $\{\rho(R_1, > \gamma) \dots \rho(R_n, > \gamma) \sqsubseteq \rho(R, > \gamma)\}$
$\kappa(\langle R_1 \dots R_n \sqsubseteq R > \beta\rangle)$	$\kappa(\langle R_1 \dots R_n \sqsubseteq R \geq \beta\rangle) \cup \{\rho(R_1, > \beta) \dots \rho(R_n, > \beta) \sqsubseteq \rho(R, > \beta)\}$
$\kappa(dis(S_1, S_2))$	$dis(\rho(S_1, > 0), \rho(S_2, > 0))$
$\kappa(ref(R))$	$ref(\rho(R, \geq 1))$
$\kappa(asy(S))$	$asy(\rho(S, > 0)$

Theorem 1. *A $f_{KD}\mathcal{SROIQ}$ fKB fK is satisfiable iff $\mathcal{K}(fK)$ is satisfiable.*

Complexity. $|\mathcal{K}(fK)|$ is $O(|fK|^2)$ i.e. the resulting knowledge base is quadratic. The ABox is actually linear while the TBox and the RBox are both quadratic:

- $|N^{fK}|$ is linearly bounded by $|fK_A| + |fK_T| + |fK_R|$.
- $|\sigma(fK_A)| = |fK_A|$.
- $|T(N^{fK})| = 2 \cdot (|N^{fK}| - 1) \cdot |A^{fK}|$.
- $|\kappa(fK, \mathcal{T})| \leq 2 \cdot (|N^{fK}| - 1) \cdot |\mathcal{T}|$.
- $|R(N^{fK})| = 2 \cdot (|N^{fK}| - 1) \cdot |R^{fK}|$.
- $|\kappa(fK, \mathcal{R})| \leq 2 \cdot (|N^{fK}| - 1) \cdot |\mathcal{R}|$.

The resulting KB is quadratic because it depends on the number of relevant degrees $|N^{fK}|$. An immediate solution to obtain a KB which is linear in complexity is to fix the number of degrees which can appear in the knowledge base. From a practical point of view, in most of the applications it is sufficient to consider a small number of degrees, e.g. $\{0, 0.25, 0.5, 0.75, 1\}$.

It is easy to see that the complexity of the crisp representation is caused by fuzzy concepts and roles. Fortunately, in real applications not all concepts and roles will be fuzzy. Another optimization would be allowing to specify that a concept (resp. a role) is crisp. For instance, suppose that A is a fuzzy concept. Then, we need $N^{fK}-1$ concepts of the form $A_{\geq\alpha}$ and another $N^{fK}-1$ concepts of the form $A_{>\beta}$ to represent it, as well as $2 \cdot |N^{fK}| - 3$ axioms to preserve their semantics. On the other hand, if A is declared to be crisp, we just need one concept to represent it and no new axioms. The case for crisp roles is similar.

An interesting property of the procedure is that the reduction of an ontology can be reused when adding new axioms. In fact, for every new axiom τ, the reduction procedure generates only one new axiom or a (linear in size) set of axioms if τ does not introduce new atomic concepts nor new atomic roles and, in case τ is a fuzzy axiom, if it does not introduce a new degree of truth.

4 Implementation: DeLorean

Our prototype implementation of the reduction process is called DeLorean (DEscription LOgic REasoner with vAgueNess). It has been developed in Java with Jena API[2], the parser generator JavaCC[3], and using DIG 1.1 interface[4] to communicate with crisp DL reasoners. Currently the logic supported is $f_{KD}\mathcal{SHOIN}$ (OWL DL), since DIG interface does not yet support full $\mathcal{SROIQ}$.

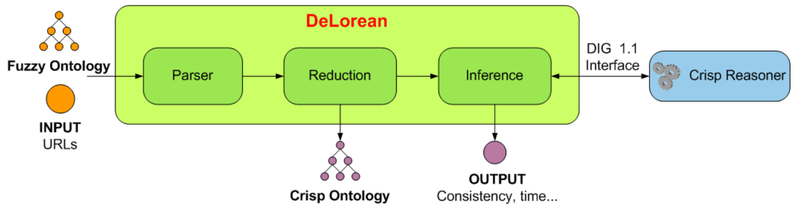

Fig. 1. Architecture of DeLorean reasoner.

Figure 1 illustrates the architecture of the system. The *Parser* reads an input file with a fuzzy ontology and translates it into an internal representation. As we have remarked in the Introduction, we could use any language to encode the fuzzy ontology, as long as the Parser can understand the representation and the reduction is properly implemented; consequently we will not get into details of our particular choice. In the next step, the *Reduction* module implements the procedure described in Section 3, building a Jena model from which an OWL file with an equivalent crisp ontology is created. Finally, the *Inference* module tests this ontology for consistency, using any crisp reasoner through the DIG interface. The *User interface* allows the user to introduce the inputs and shows the result of the reasoning and the elapsed time.

We have carried out some experiments in order to evaluate our approach in terms of reasoning, that is, in order to check that the results of the reasoning tasks over the crisp ontology were the expected. The aim of this section is not to perform a full benchmark, which could be the topic of a forthcoming work. Nevertheless, we will show some performance examples to show that our approach is feasible and the increment of time for small ontologies when using a limited number of degrees of truth is acceptable. In any case, optimizations are crucial.

We considered the Koala ontology[5], a sample $\mathcal{ALCON}(\mathcal{D})$ ontology with 20 named classes, 15 anonymous classes, 4 object properties, 1 datatype property (which we have omitted) and 6 individuals. We extended its axioms with random (lower bound) degrees and we used Pellet reasoner through the DIG interface.

[2] `http://jena.sourceforge.net/`
[3] `https://javacc.dev.java.net`
[4] `http://dl.kr.org/dig/`
[5] `http://http://protege.cim3.net/file/pub/ontologies/koala/koala.owl`

Table 3 shows the influence of the number of degrees on the time of the reduction process as well as on the time (in seconds) of a classification test over the resulting crisp ontology.

Table 3. Influence of the number of degrees in the performance of DeLorean.

Number of degrees	crisp	3	5	7	9	11
Reduction time	-	1.18	6.28	23.5	64.94	148.25
Reasoning time	0.56	0.98	1.343	2.88	4.28	6.47

It can be observed that the reduction time is quite large with respect to the reasoning time. We recall that DeLorean is currently just a prototype, so the implementation of the reduction process should be optimized. Moreover, as already discussed in the previous section, the reduction can be reused and hence needs to be computed just once. Regarding the reasoning time, the increment of complexity when the fuzzy ontology contains 3 or 5 degrees can be assumed.

5 Conclusions and Future Work

In this paper we have shown how to reduce a fuzzy extension of $\mathcal{SROIQ}$ with fuzzy GCIs and RIAs (under a novel semantics using Gödel implication) into $\mathcal{SROIQ}$. We have enhanced previous works by reducing the number of new elements and axioms. We have also presented DeLorean, our implementation of this reduction procedure which is, to the very best of our knowledge, the first reasoner supporting fuzzy $\mathcal{SHOIN}$ (and hence and eventually fuzzy OWL DL). The very preliminary experimentation shows that our approach is feasible in practice when the number of truth degrees is small, even for our non-optimized prototype. This work means an important step towards the possibility of dealing with imprecise and vague knowledge in DLs, since it relies on existing languages and tools.

In general, Gödel implication provides better logical properties than KD, but KD for example allows to reason with *modus tolens* [7]. A representation language could allow the use of two types of GCIs and RIAs $\sqsubseteq_{KD}$ y $\sqsubseteq_{G}$ (with semantics based on KD and Gödel implications respectively) similarly as [13] which allows three types of subsumption. This way, the ontology developer would be free to choose the better option for his own needs. [7] shows how to reduce GCIs under KD semantics, and RIAs can be reduced similarly.

Future work could include to compare DeLorean with other fuzzy DL reasoners, although they support different languages and features and, as far as we know, there does not exist any significant fuzzy knowledge base. We will also allow the definition of crisp concepts and roles in the fuzzy language. Finally, the reasoner will be extended to $f_{KD}\mathcal{SROIQ}$ (and hence OWL 1.1) as soon as DIG 2.0 interface is available, so it is independent of any concrete crisp reasoner.

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