# On Counting k-Convex Polyominoes (Short Paper)

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**Abstract**
We show that, for any fixed integer $k > 2$, the problem of computing the number of $k$-convex polyominoes of area $n$ is in P.

**Keywords**
polyominoes, counting problem, integer sequence

## 1. Introduction

A polyomino is a geometrical figure consisting of a finite set of connected unit squares (called cells) in the plane $\mathbb{Z} \times \mathbb{Z}$, considered up to translations. Polyominoes gained popularity after the paper of S. W. Golomb [1]. Nowadays they are widely studied by physicists, mathematicians, computer scientists and also by biologists. The problem of counting the number $c_n$ of polyominoes with $n$ cells (*i.e.* of area $n$) is probably one of the fundamental open problems in combinatorial geometry (see problem 37 in [2]). Due to the difficulty of the problem, simpler classes of polyominoes have been introduced. In particular, the class of convex polyominoes (polyominoes where the intersection with an infinite horizontal or vertical stripe is a finite segment) and some of its subclasses have been investigated [3, 4, 5, 6, 7]. In this paper, we consider the class $\mathsf{Conv}_k$ [8] containing all convex polyominoes $P$ with the property that any two cells of $P$ can be joined by a path in $P$ with at most $k$ changes of direction, where $k$ is a fixed integer greater than 2. We show that, for any fixed $k > 2$, the algorithm presented in [9] leads to a set of recurrence equations for computing the number of $k$-convex polyominoes of area $n$ in polynomial time, using $O(n^5)$ space.

## 2. Notation and preliminaries

Let $P$ be a polyomino with an $r \times c$ minimal bounding rectangle. The rows (resp., columns) of $P$ are numbered from bottom to top (resp., from left to right). The *area* $\mathrm{A}(P)$ of $P$ is the number of its cells. A cell of $P$ is identified by a pair of integers $(i, j)$, where $i$ (resp., $j$) is the row (resp., column) index. Two cells $a = (i, j)$ and $a' = (i', j')$ are *adjacent* if $|i - i'| + |j - j'| = 1$. Given two cells $a$ and $b$ of $P$, a *path* in $P$ from $a$ to $b$ is a sequence $q_1, q_2, \ldots, q_k$ of cells of $P$, with $q_1 = a$ and $q_k = b$, such that $q_i$ and $q_{i+1}$ are adjacent for all $i$ with $1 \leq i < k$. A

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*step* is a sequence of two adjacent cells $(i, j), (i', j')$. Steps are distinguished according to the directions $\mathsf{N}$ (North), $\mathsf{W}$ (West), $\mathsf{S}$ (South) and $\mathsf{E}$ (East). The number of *changes of direction* in a path $\beta \in \{\mathsf{N}, \mathsf{W}, \mathsf{S}, \mathsf{E}\}^+$ is defined as the number of indices $i$ such that $\beta_i \neq \beta_{i+1}$, with $1 \leq i < |\beta|$. A path is *monotone* if $\beta \in \{\mathsf{N}, \mathsf{W}\}^+$ (NW-path) or $\beta \in \{\mathsf{N}, \mathsf{E}\}^+$ (NE-path) or $\beta \in \{\mathsf{S}, \mathsf{E}\}^+$ (SE-path) or $\beta \in \{\mathsf{S}, \mathsf{W}\}^+$ (SW-path). A polyomino $P$ is *horizontally convex* (resp., *vertically convex*) if any row (resp. column) of $P$ consists of one segment. The class of *convex* polyominoes contains all polyominoes that are both horizontally and vertically convex. It has been proved [10, Proposition 1] that a polyomino $P$ is convex if and only if any two cells of $P$ are joined by a monotone path in $P$. The *degree of convexity* of $P$, denoted by $\deg_c(P)$, is defined as the least integer $k$ such that any two cells of $P$ can be joined by a monotone path in $P$ with at most $k$ changes of direction. A convex polyomino is called $k$-convex if its degree of convexity is at most $k$. Given a convex polyomino $P$ and its minimal bounding rectangle $B$, we say that $P$ is a *stack* (resp., *Ferrers diagram*, *parallelogram*, *rectangle*) if it shares exactly two adjacent (resp., three, two opposite, four) vertices with $B$. A stack $P$ is a *left* (resp., *right*) stack if the column with the largest area is the last (resp., first) one. Analogously, in a left (resp., right) Ferrers diagram the largest column is the last (resp., first) one. We denote by $\mathsf{L}$ (resp., $\mathsf{R}$) the set of left (resp., right) stacks. $\mathsf{F}_L$ (resp., $\mathsf{F}_R$) is the set of left (resp., right) Ferrers diagrams. Furthermore, we indicate by $\mathsf{C}$ (resp., $\mathsf{T}$) the set of parallelograms (resp., rectangles). For a class $\mathsf{A}$ of polyominoes, $\mathsf{A}(n)$ is the set of polyominoes in $\mathsf{A}$ of area $n$. By $\textsc{low}(j)$ (resp., $\textsc{high}(j)$) we denote the row index of the bottom (resp., top) cell of column $j$. Similarly, $\textsc{left}(i)$ denotes the column index of the leftmost cell of row $i$. Lastly, $\textsc{first}(P)$ (resp., $\textsc{last}(P)$) indicates the first (resp., last) column of $P$. Given two columns $i$ and $j$, we say that $i$ and $j$ are *overlapping* (resp., *disjoint*), denoted by $i \updownarrows j$ (resp., $i \asymp j$), if and only if $\textsc{low}(j) < \textsc{low}(i) \leq \textsc{high}(j) < \textsc{high}(i)$ or $\textsc{low}(i) < \textsc{low}(j) \leq \textsc{high}(i) < \textsc{high}(j)$ (resp., $\textsc{low}(i) > \textsc{high}(j)$ or $\textsc{low}(j) > \textsc{high}(i)$). Moreover, we say that $i$ *includes* $j$, denoted by $j \subseteq i$, if and only if $\textsc{low}(i) \leq \textsc{low}(j)$ and $\textsc{high}(i) \geq \textsc{high}(j)$. Given a convex polyomino $P$, let $e$ be the rightmost column of $P$ such that $c \subseteq e$ for $1 \leq c < e$. Then, $P$ is called *descending* (resp., *ascending*) if there exists a column $j$ such that $j > e$, $j \updownarrows e$ and $\textsc{low}(e) > \textsc{low}(j)$ (resp., $\textsc{low}(e) < \textsc{low}(j)$). The set of descending (resp., ascending) $k$-convex polyominoes is indicated by $\mathsf{DConv}_k$ (resp., $\mathsf{AConv}_k$). The set of all descening polyominoes is $\mathsf{DConv}$. If $P$ is neither descending nor ascending then $P \in \mathsf{T} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{L} \cup \mathsf{R}$ or it belongs to the class $\mathsf{LR}$ containing all convex polyominoes that are the concatenation of two polyominoes, $P = P_1 \cdot P_2$, where $P_1 \in \mathsf{L} \cup \mathsf{F}_L$, $P_2 \in \mathsf{T} \cup \mathsf{R} \cup \mathsf{F}_R$ and $\textsc{first}(P_2) \subsetneq \textsc{last}(P_1)$. Notice that any $P \in \mathsf{LR}$ contains a column $\bar{\jmath}$ such that $j \subseteq \bar{\jmath}$ for all columns $j$, hence $\deg_c(P) \leq 2$. Lastly, a cell $(i, j)$ of $P$ is a *NW-corner* if $(i, j-1)$ and $(i+1, j)$ are not cells of $P$. Analogously, we define *NE-corners*, *SE-corners* and *SW-corners*. Two corners are called *opposite* if either one is a NW-corner and the other is a SE-corner, or one is a NE-corner and the other is a SW-corner. Clearly, one has $\mathsf{Conv}_k = \mathsf{T} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{LR} \cup \mathsf{AConv}_k \cup \mathsf{DConv}_k$ and (because of symmetry) $|\mathsf{DConv}_k(n)| = |\mathsf{AConv}_k(n)|$, $|\mathsf{L}(n)| = |\mathsf{R}(n)|$, $|\mathsf{F}_L(n)| = |\mathsf{F}_R(n)|$. Since all unions are disjoint, it follows that

$$|\mathsf{Conv}_k(n)| = |\mathsf{T}(n)| + 2 \cdot |\mathsf{F}_L(n)| + 2 \cdot |\mathsf{L}(n)| + |\mathsf{LR}(n)| + 2 \cdot |\mathsf{DConv}_k(n)|. \tag{1}$$

Thus, the counting problem for $\mathsf{Conv}_k$ is reduced to computing $|\mathsf{DConv}_k(n)|$ and to some other counting problems that are immediately solved in polynomial time. In particular, we can

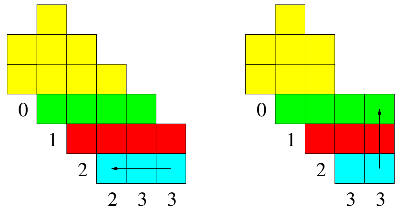


**Figure 1:** The arrows show the first steps of the paths determined by [9, Lemma 8] and associated with a SW corner. The integer below a column indicates its degree of convexity.

compute $|\mathsf{LR}(n)|$ in polynomial time as shown in [11]. In order to compute $|\mathsf{DConv}_k(n)|$ we define a decomposition for descending polyominoes.

**Definition 2.1** (standard decomposition)**.** A polyomino $P \in \mathsf{DConv}$ can be decomposed as $P = L \cdot F \cdot C \cdot R$ (with $F$, $R$ possibly empty) for suitable polyominoes $L \in \mathsf{L} \cup \mathsf{T} \cup \mathsf{F}_L$, $F \in \mathsf{F}_R$, $C \in \mathsf{C} \cup \mathsf{T} \cup \mathsf{F}_R$, and $R \in \mathsf{R} \cup \mathsf{T} \cup \mathsf{F}_R$ such that: $\textsc{first}(F) \subsetneq \textsc{last}(L)$, $\textsc{low}(\textsc{last}(L)) = \textsc{low}(\textsc{first}(F))$, $\textsc{last}(F) \upharpoonleft\downharpoonright \textsc{first}(C)$ (or $\textsc{last}(L) \upharpoonleft\downharpoonright \textsc{first}(C)$ if $F = \epsilon$), $\textsc{low}(\textsc{last}(L)) > \textsc{low}(\textsc{first}(C))$ and (if $R \neq \epsilon$) $\textsc{first}(R) \subsetneq \textsc{last}(C)$, $\textsc{low}(\textsc{last}(C)) < \textsc{low}(\textsc{first}(R))$.

We stress that the standard decomposition of $P$ is unique (*e.g.* $\textsc{last}(L)$ is the rightmost column $\bar{\jmath}$ of $P$ such that $j \subseteq \bar{\jmath}$ for $j < \bar{\jmath}$). The subset of $\mathsf{DConv}_2$ containing polyominoes decomposed as $L \cdot C \cdot R$ (resp., $L \cdot C$, $L \cdot F \cdot C \cdot R$, $L \cdot F \cdot C$) is $\mathsf{LCR}_2$ (resp., $\mathsf{LC}_2$, $\mathsf{LFCR}_2$, $\mathsf{LFC}_2$). Thus, for any $k > 2$ one has the partition $\mathsf{DConv}_k = \mathsf{LFCR}_k \cup \mathsf{LFC}_k \cup \mathsf{LCR}_k \cup \mathsf{LC}_k$.

By [9, Thm. 6], the degree of convexity of $P \in \mathsf{DConv}$ depends on the number $k$ of changes of direction in a particular NW-path that starts at a SE-corner $a$ and ends at a NW-corner $b$. This is a path where the first $k-1$ changes of direction always occur on the boundary of $P$, whereas the $k$th one possibly occurs on a cell that is not on the boundary. Furthermore, from [9, Thm. 6] it follows that the degree of convexity of the $j$th column of $P$ is $\deg_c(P, j) = \max\{D(a, b) | a = (\textsc{low}(j), j), b$ is a NW-corner of $P\}$, where $D(a, b)$, is the least integer k such that there exists a monotone path in $P$ from $a$ to $b$ with k changes of direction.

Given $P = L \cdot F \cdot C \cdot R$ and a column $j$ in $F \cdot C \cdot R$, we know from [9, Lemma 8] that $\deg_c(P, j) = \min(\deg_c(P, j') + 2, \deg_c(P, j'') + 1)$, where $j' = \textsc{left}(\textsc{high}(j))$ and $j'' = \textsc{left}(\textsc{low}(j))$. see Fig. 1 for an example. In the sequel, we consider each column $c$ of $P$ as the concatenation of vertical segments associated with different degrees of convexity. These segments are identified by considering the degrees of convexity of the leftmost columns that are reached by W-paths starting from cells of $c$.

**Definition 2.2** ($\beta$-segment)**.** A cell $(i, j)$ of $P \in \mathsf{DConv}$ belongs to a $\beta$-segment if and only if either $\beta > 0 \wedge i < \textsc{low}(\textsc{first}(P))$ and the cell $(i, \textsc{left}(i))$ belongs to a column of degree of convexity $\beta$, or $\beta = 0 \wedge i \geq \textsc{low}(\textsc{first}(P))$.

From [9, Lemma 8], if $\deg_c(P, j) = \alpha$ then column $j$ may contain only $\beta$-segments with $\alpha - 2 \leq \beta \leq \alpha$. Notice that $(i, j)$ belongs to a $\beta$-segment if and only if $(i, j-1)$ belongs to a $\beta$-segment or $(i, j-1)$ is not in $P$ and $\deg_c(P, j) = \beta$. Fig. 1 shows how the columns of a polyomino in $\mathsf{DConv}$ consist of at most three $\beta$-segments (depicted as vertical segments comprising cells of the same color).

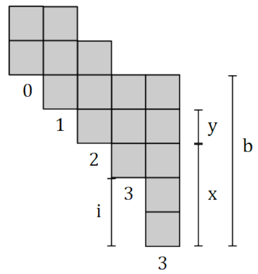


**Figure 2:** A polyomino counted by $C(16, 5, 3, 3, 1, 2)$.

## 3. Counting

We focus on computing $|\mathsf{DConv}_k(n)|$ since the remaining values in (1) are easily computed in polynomial time (see, for instance, [11]). Given $n, b, j, x, y, i \in \mathbb{N}$, with $n, b > 0$ and $x+y \leq b$, let $C(n, b, j, x, y, i)$ be the number of polyominoes $P$ in $\mathsf{LFC}_k(n) \cup \mathsf{LC}_k(n)$ such that $\mathsf{A}(\textsc{last}(P)) = b$, $\deg_c(P, \textsc{last}(P)) = j$, $\textsc{low}(\textsc{last}(P)-1) - \textsc{low}(\textsc{last}(P)) = i$, and where $\textsc{last}(P)$ contains a $j$-segment of area $x$, a $(j-1)$-segment of area $y$ and a $(j-2)$-segment of area $b-x-y$, see Fig. 2. Obviously, one has $|\mathsf{LFC}_k(n)| + |\mathsf{LC}_k(n)| = \sum_{b,j,x,y,i} C(n, b, j, x, y, i)$, and the problem boils down to compute $C(n, b, j, x, y, i)$ efficiently.

Consider the standard decomposition of a polyomino $P$ counted by $C(n, b, j, x, y, i)$, $P = L \cdot F \cdot C$ or $P = L \cdot C$. If the second to last column of $P$ belongs to $C$ then $C(n, b, j, x, y, i)$ depends on $C(n-b, b', j', x', y', i')$ for suitable $b', j', x', y', i'$. Otherwise, the second to last column of $P$ is in $F$ or in $L$, and $C(n, b, j, x, y, i)$ is computed using the values $L(m', p', q', y')$ and $F(m', p', q')$, where $L(m', p', q', y')$ (resp., $F(m', p', q')$) is the number of $P \in \mathsf{L}(m') \cup \mathsf{F}_L(m') \cup \mathsf{T}(m')$ (resp. $P \in \mathsf{F}_R(m')$) such that $\mathsf{A}(\textsc{first}(P)) = p'$, $\mathsf{A}(\textsc{last}(P)) = q'$ and $\textsc{low}(\textsc{first}(P)) - \textsc{low}(\textsc{last}(P)) = y'$. Here we consider only the case when column $\textsc{last}(P) - 1$ is in $C$ (we refer to the full paper for the remaining cases). This is the most complex case as it leads to four different equations, depending on the conditions $x > i$ (1), $x = i \wedge y = 0$ (2), $x = i \wedge y > 0 \wedge i > 0$ (3) and $x = i \wedge y > 0 \wedge i = 0$ (4). In case (1), column $\textsc{last}(P) - 1$ always has degree of convexity $j$, since the relation $x > i$ implies the existence of a column of degree of convexity $j$ to the left of $\textsc{last}(P)$, and the sequence of degrees of convexity of columns in $C$ is not decreasing by [9, Thm. 9]. Furthermore, one necessarily has $x + y < b$ because $\textsc{last}(P)$ contains a $(j-2)$-segment, see Fig. 3. So, it follows that $C(n, b, j, x, y, i) = \sum_{b'=b-i}^{n-b-2} \sum_{i'=0}^{x-i} C(n-b, b', j, x-i, y, i')$. In case (3) (see Fig. 4) the recurrence equation is $C(n, b, j, i, y, i) = \sum_{b'=b-i+1}^{n-b-2} \sum_{y'=b-i-y}^{b'-b+i-1} \sum_{i'=0}^{y} C(n-b, b', j-1, y, y', i') + \sum_{b'=b-i}^{n-b-2} C(n-b, b', j, 0, y, 0)$. In fact, the second to last column of a polyomino $P$ counted by $C(n, b, j, i, y, i)$ may have degree of convexity $j-1$ or $j$, but not $j-2$. Indeed, if one had $\deg_c(P, \textsc{last}(P)-1) = j-2$ then the sequence of degrees of convexity in $C$ would be decreasing, since $\textsc{last}(P)$ contains a $(j-1)$-segment and this implies the existence of a column $m$ to the left of $\textsc{last}(P)$ such that $\deg_c(P, m) = j-1$. Furthermore, $x = i \wedge i > 0$ implies that $\textsc{last}(P)$ contains a $(j-2)$-segment. As a consequence the $(j-1)$-segments in $\textsc{last}(P) - 1$ and in $\textsc{last}(P)$ have the same area. So, if $\deg_c(P, \textsc{last}(P) - 1) = j - 1$ we sum the values $C(n-b, b', j-1, y, y', i')$ on all $b', y', i'$ such that:

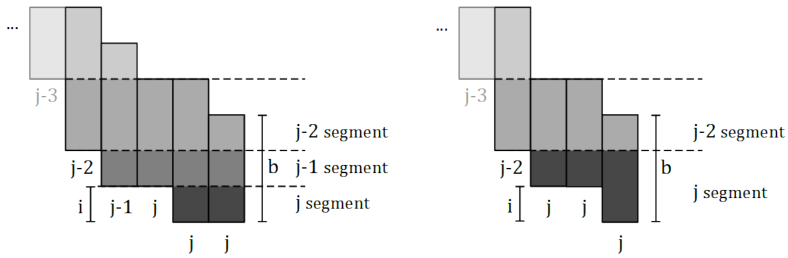


**Figure 3:** Possible configurations when $x > i$.

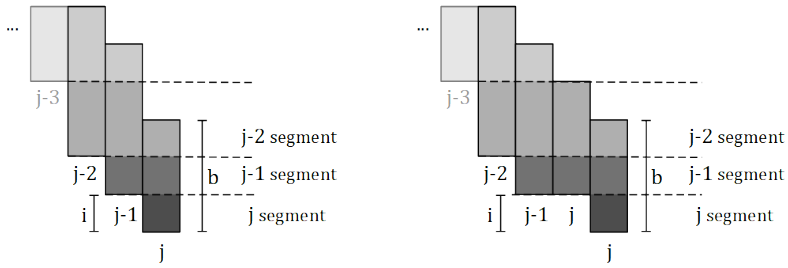


**Figure 4:** Possible configurations when $x = i$ and $x, y > 0$.

- the area $b'$ of column $\textsc{last}(P) - 1$ is at least $b - i + 1$ (since $b'$ must contain a $(j-3)$-segment) and at most $n - b - 2$ (since $\mathrm{A}(L) \geq 2$);
- the area $y'$ of the $(j-2)$-segment in $\textsc{last}(P) - 1$ is at least $b - i - y$ (the area of the $(j-2)$-segment in $\textsc{last}(P)$) and at most $b' - b + i - 1$ (because column $\textsc{last}(P) - 1$ must contain a $(j-3)$-segment);
- $i'$ is at least 0 and at most $y$ (the lower $i'$ cells of $\textsc{last}(P) - 1$ has degree $j - 1$ since $\textsc{left}(\textsc{low}(\textsc{last}(P) - 1) + e) = \textsc{last}(P) - 1$ for $0 \leq e < i'$).

Otherwise, if $\deg_c(P, \textsc{last}(P) - 1) = j$ we sum the values $C(n - b, b', j, 0, y, 0)$ on all $b'$ no smaller than $b - i$ and no greater than $n - b - 2$. From the existence of a $(j-1)$-segment in $\textsc{last}(P)$ and $\deg_c(P, \textsc{last}(P) - 1) = j$, it follows $\textsc{low}(\textsc{last}(P) - 2) - \textsc{low}(\textsc{last}(P) - 1) = 0$. Hence, $\textsc{last}(P) - 1$ cannot contain a $j$-segment. The recurrence equations for cases (2) and (4), as well as for the cases associated with the set $\mathsf{LCR}_k \cup \mathsf{LFCR}_k$, are obtained by a similar reasoning.

Based on these equations, we have developed a C++ program to compute $|\mathsf{Conv}_k(n)|$ for $k > 2$ and $n > 0$. This program has space complexity $O(n^5)$ (coming from the size of the table containing the values $C(n, b, j, x, y, i)$) and produced Table 1 in few minutes (on a Macbook Pro). We point out that this integer sequence does not appear in OEIS.

## References


[1] S. W. Golomb, Checker boards and polyominoes, Amer. Math. Monthly 61 (1954) 675–682.

[2] E. D. Demaine, J. S. B. Mitchell, J. O'Rourke, The open problems project, last update 2020. URL: http://cs.smith.edu/~jorourke/TOPP.

**Table 1**
The number of 3-convex polyominoes of area $n$ for $0 \leq n \leq 40$.

| |
|---|
| 0, 1, 2, 6, 19, 59, 172, 470, 1206, 2934, 6812, 15192, 32709, 68282, 138678, 274822, 532719, 1012144 |
| 1888226, 3464168, 6258249, 11146013, 19590450, 34011064, 58371083, 99103808, 166563604, |
| 277281796, 457451501, 748274488, 1214117566, 1954879052, 3124637754, 4959621329, |
| 7819943680, 12251575214, 19077932142, 29534613958, 45466767846, 69616951878, 106043316448 |


[3] M. Bousquet-Mélou, Convex polyominoes and heaps of segments, Journal of Physics A: Mathematical and General 25 (1992) 1925–1934. URL: https://doi.org/10.1088/0305-4470/25/7/031. doi:10.1088/0305-4470/25/7/031.

[4] E. Barcucci, R. Pinzani, R. Sprugnoli, Directed column-convex polyominoes by recurrence relations, in: M. C. Gaudel, J. P. Jouannaud (Eds.), TAPSOFT'93: Theory and Practice of Software Development, Springer Berlin Heidelberg, Berlin, Heidelberg, 1993, pp. 282–298.

[5] M. Bousquet-Mélou, A method for the enumeration of various classes of column-convex polygons., Discrete Math. 154 (1996) 1–25.

[6] A. Del Lungo, M. Nivat, R. Pinzani, S. Rinaldi, A bijection for the total area of parallelogram polyominoes, Discret. Appl. Math. 144 (2004) 291–302. URL: https://doi.org/10.1016/j.dam.2003.11.007. doi:10.1016/j.dam.2003.11.007.

[7] G. Castiglione, A. Restivo, Ordering and convex polyominoes, in: MCU 2004, volume 3354 of *Lecture Notes in Comput. Sci.*, Springer, 2005, pp. 128–139.

[8] A. Micheli, D. Rossin, Counting k-convex polyominoes, Electron. J. Comb. 20 (2013).

[9] S. Brocchi, G. Castiglione, P. Massazza, On the exhaustive generation of k-convex polyominoes, Theor. Comput. Sci. 664 (2017) 54–66.

[10] G. Castiglione, A. Restivo, Reconstruction of l-convex polyominoes, Electronic Notes in Discrete Mathematics 12 (2003) 290 – 301. URL: http://www.sciencedirect.com/science/article/pii/S1571065304004949. doi:https://doi.org/10.1016/S1571-0653(04)00494-9, 9th International Workshop on Combinatorial Image Analysis.

[11] V. Dorigatti, P. Massazza, On counting l-convex polyominoes, in: 22nd Italian Conference on Theoretical Computer Science, 2021, volume 3072 of *CEUR Proceedings*, 2021, pp. 193–198.