A Unifying Approach to Boundaries and Multidimensional Mereotopology

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Abstract

Mereotopologists generally come from either a point-set or algebraic topology perspective when developing boundary-based theories. Some involve dimensional aspects while others do not. Little work has studied whether the existing boundary characterizations convey the same meaning regardless of the mereotopology they build upon. To address this gap, we introduce a new mathematical theory that can be used as a unifying framework for boundaries in both multidimensional and nondimensional mereotopologies. With the proposed unifying framework, we are able to interpret both multidimensional- and nondimensional- based mereotoplogies.

Keywords

mereology, dimension, boundary, topology, manifold

1. Introduction

The Winograd Schema Challenge (WSC) [1] is an improvement to the Turing test, where a machine is asked to resolve the reference to a pronoun phrase (anaphora resolution). Many questions within WSC refer to spatial relations among parts, parts of parts and the boundaries between parts. For example [2]:

> There is a gap in the wall. You can see the garden [through/behind] it. You can see the garden [through/behind] what?
>
> **Answers:** The gap/the wall.

To approach this specific anaphora resolution, one must understand the intended semantics behind the spatial preposition, “through". In particular, the intended semantics can be captured through formalizing in terms of mereotopology: a theory manifesting mereological and topological classes and relations[3].

It is also worth characterizing two-dimensional lines, paths and cycles–exemplified in the following oil painting done by Wassily Kandinsky. Different from perfect geometric shapes, this composite of zero-, one-, and two-dimensional entities is what appears in reality. We want to investigate the capabilities of existing mereotopologies in characterizing such “imperfect" spaces.

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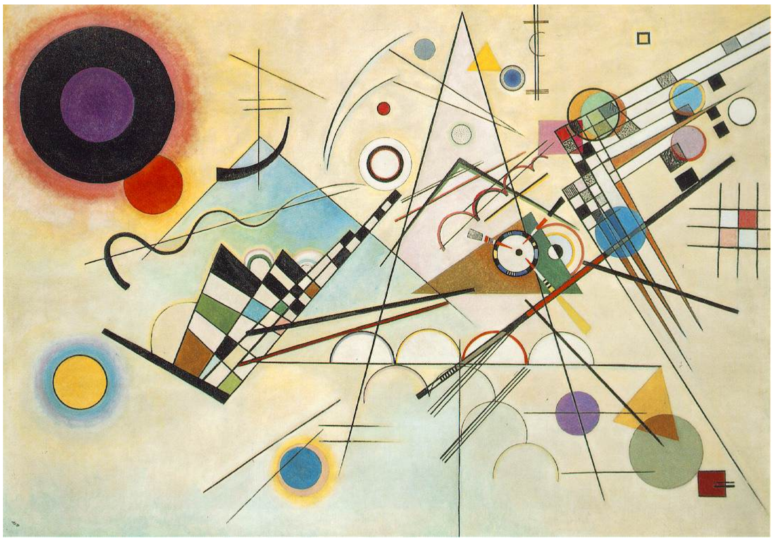

Figure 1: Wassily Kandinsky[4], Composition 8, 1923.

To answer the questions posed by these two motivating scenarios, we need to closely examine the notion of boundary. Little study has successfully formalized the notion of crossing a boundary, inclusive of all dimensions. In the WSC sentence pair, the preposition "through"[5] indicates a movement that goes into at one side and out at another. Hence, we need to study crossing the boundary in terms of a three-dimensional region. On the other hand, the depiction of overlapped points, lines, and cycles in Kandinsky's oil painting crosses both boundaries and dimensions. This brings our attention to the representation of boundaries across one and multiple dimensions.

The formalization of boundary has long been a topic of discussion in mereotopology. Boundary-agnostic theories have been excluded in this work for two reasons[3]. First, ignoring boundaries contradict the topological distinctions between open and closed entities. Next, in the absence of boundaries, an entity can be connected to its complement. The omission of boundary-less approaches leaves us with two directions for boundary-based approaches. Previous work has taken either a multi-dimensional or non-dimensional perspective. Smith's mereotopology[6] ignores dimension. His axiomatization arises from point-set topology in which he uses the relation $straddle(x, y)$, defined by a primitive relation $IP(x, y)$ (x is an interior part of y). He then takes the sum of all boundary points to form a complete set of boundary. The advocates for multidimensional characterizations come from an algebraic topology perspective; led by Gotts[7] and Galton[8], they take the boundary entity to be one dimension lower than the object it is attached to. Galton's theory defines the notion of boundary as a primitive relation. Gott's establishment in INCH Calculus is examined by Hahmann[9]. Hahmann's mereotopology – CODI – is definably equivalent to the corrected INCH Calculus.

1.1. Main Contributions

The various formalization regarding the notion of boundary best showcases disagreement behind characterizing spatial entities among mereotopologies. Few studies have focused on comparing and harmonizing different perspectives. The main problem we are solving is to find a minimal theory to axiomatize the notion of dimension for elements of a mereology. We introduce the notion of a multimereology, which amalgamates a mereology with an incidence structure to partition partial orderings.

The paper is organized as follows. Section 2 introduces the root theories of multimereology. Inspired by the notion of topological manifolds, Section 3 is initiated from a combinatorial topology standpoint to examine the comprehensiveness of possible multimereology candidates against certain use cases. Section 4 showcases a successful application of our proposed multimereology to be logical equivalent to CODI, a multidimensional mereotopology. Section 5 returns to the motivating question of boundary classification and unification.

2. Structures for Multidimensional Mereologies

In designing an ontology, our objective is twofold – first, to prove that the models of the ontology are actually the intended models, and second, to demonstrate that the intended models do indeed formalize the ontological commitments. Our strategy is to first specify a class of mathematical structures and show that the ontology axiomatizes this class of structures (that is, there is a one-to-one correspondence between the class of models of the ontology and the class of mathematical structures). We then specify a representation theorem for this class of mathematical structures to demonstrate that it formalizes the ontological commitments. The primary benefit of this strategy is that it makes explicit the modular organization of the subtheories of the ontology, thereby highlighting how other ontologies are reused.

2.1. Multimereologies

The first step is to specify the class of structures which capture the following intuitions and ontological commitments about multidimensional mereology:

1. Each entity has a unique dimension.
2. Dimensions are linearly ordered.
3. There is a parthood relation that is specified on entities with the same dimension.
4. Incidence corresponds to a multidimensional ordering.

To formalize these semantic requirements for a multidimensional mereology, we introduce the notion of a multimereology as the amalgamation of a partial ordering with an incidence structure. Incidence structures are a generalization of geometry, first introduced by Gino Fano and later used by Hilbert in his axiomatization of Euclidean geometry.

Definition 1. *A k-partite incidence structure is a tuple* $\mathbb{I} = \langle \Omega_1, ..., \Omega_k, \mathbf{I} \rangle$*, where* $\Omega_1, ..., \Omega_k$ *are sets such that* $\Omega_i \cap \Omega_j = \emptyset, i \neq j$ *and* $\mathbf{I} \subseteq (\bigcup_{i \neq j} \Omega_i \times \Omega_j)$*.*

Two elements of $\mathbb{I}$ *that are related by* $\mathbf{I}$ *are called incident. The neighbourhood of an element is the set of elements which are incident with it:* $N(\mathbf{x}) = \{\mathbf{y} \,:\, \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{I}\}$

Incidence structures are ideally suited to represent the dimensionality of different entities. In a tripartite incidence structure, points can be thought of as 0-dimensional, lines as 1-dimensional, and planes as 2-dimensional.

How is the incidence structure in a multimereology related to the partial ordering? The first approach to this question is to consider incidence to itself be a parthood relation, and enforce the transitivity of the incidence relation:

Definition 2. *A tripartite incidence structure* $\mathbb{I} = \langle P, L, Q, \mathbf{I} \rangle$ *is transitive iff*

$$\mathbf{I} \circ \mathbf{I} \subseteq \mathbf{I}$$

At first glance, this might seem unusual since the incidence relation is symmetric; we therefore use the following notion from graph theory which allows us to associate a transitive symmetric incidence relation with a partial ordering:

Definition 3. *The comparability graph for a partial ordering* $\mathbb{Q} = \langle M, \leq \rangle$ *is a graph* $\mathbb{G}^{\mathbb{Q}} = \langle V, \mathbf{E} \rangle$ *such that*

$$(\mathbf{x}, \mathbf{y}) \in \mathbf{E} \Leftrightarrow \mathbf{x} \leq \mathbf{y} \text{ or } \mathbf{y} \leq \mathbf{x}$$

A second alternative approach is to start with the “global" partial ordering that applies to all entities, and then use the incidence structure to specify “local" partial orderings on the sets of points, lines, and planes. We therefore need to specify how the suborderings on each of these sets is related to the entire partial ordering.

Definition 4. *The upper set for* $\mathbf{x}$ *in a poset* $\mathbb{P}$*, denoted by* $U^{\mathbb{P}}(\mathbf{x})$*, is*

$$U^{\mathbb{P}}(\mathbf{x}) = \{\mathbf{y} \,:\, \mathbf{x} \leq \mathbf{y}\}$$

The lower set for $\mathbf{x}$ *in a poset* $\mathbb{P}$*, denoted by* $L^{\mathbb{P}}(\mathbf{x})$*, is*

$$L^{\mathbb{P}}(\mathbf{x}) = \{\mathbf{y} \,:\, \mathbf{y} \leq \mathbf{x}\}$$

With the aid of this terminology, we can impose the following conditions. The set of points forms a lower set in the partial ordering, so that all parts of points are points. The set of planes forms an upper set in the partial ordering, so that all elements that contain planes as parts are also planes. The set of lines forms an interval within the partial ordering. Alternatively, we can think of the incidence structure as partitioning the elements of the partial ordering into the disjoint classes of points, lines, and planes. Both of these perspectives will be used as the basis for the formalization.

Definition 5. $\mathbb{Q} \oplus \mathbb{I}$ [1]*is a multimereology iff*

[1]The $\oplus$ symbol denotes the amalgamation of structures [10].

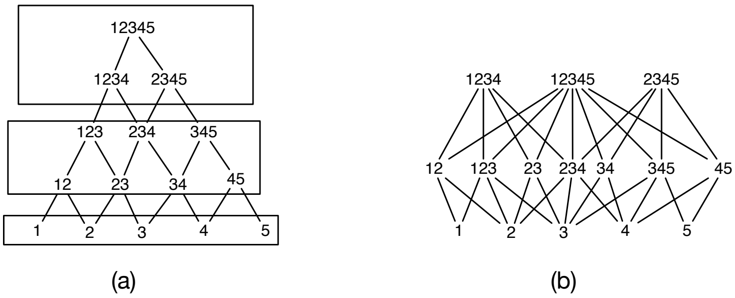

Figure 2: Examples of multimereologies.

1. $\mathbb{Q} = \langle P \cup L \cup Q, \leq \rangle$ *such that* $\mathbb{Q} \in \mathfrak{M}^{partial\_ordering}$;
2. $\mathbb{I} = \langle P, L, Q, \mathbf{I} \rangle$ *such that* $\mathbb{I} \in \mathfrak{M}^{closed\_transitive\_tripartite}$;
3. $\langle P \cup L \cup Q, \mathbf{I} \rangle \subseteq \mathbb{G}^{\mathbb{Q}}$;
4. *if* $\mathbf{x} \in P$, *then* $L^{\mathbb{Q}}(\mathbf{x}) \subseteq P$;
5. *if* $\mathbf{x}, \mathbf{y} \in L$, *then* $U^{\mathbb{Q}}(\mathbf{x}) \cap L^{\mathbb{Q}}[\mathbf{y}] \subseteq L$;
6. *if* $\mathbf{x} \in Q$, *then* $U^{\mathbb{Q}}(\mathbf{x}) \subseteq Q$.

Examples of multimereologies can be seen in Figure 2. Figure 2(a) shows a mereology that has been partitioned into three disjoint sets containing equidimensional elements, while Figure 2(b) is the corresponding tripartite incidence structure. In this example, the element $\mathbf{34}$ is an equidimensional part of the element $\mathbf{345}$, and the element $\mathbf{2345}$ is an equidimensional part of the element $\mathbf{12345}$. On the other hand, since $\mathbf{345}$ is a line and $\mathbf{2345}$ is a plane in the incidence structure, they are related by incidence, even though in the “global" mereology they are related by parthood.

2.1.1. Representation Theorem

Intuitively, each partitioning of a partial ordering corresponds to a unique multimereology, and each multimereology can be used to specify a partitioning of a partial ordering. This intuition forms the basis for the representation theorem for multimereologies. We begin by formalizing the idea of partitioning.

Definition 6. *Suppose* $\mathbb{Q}, \mathbb{P} \in \mathfrak{M}^{partial\_ordering}$ *such that* $\mathbb{Q} = \langle V_1, \leq \rangle, \mathbb{P} = \langle V_2, \preceq \rangle$.
A mapping $\mu : \mathbb{Q} \rightarrow \mathbb{P}$ *is a poset homomorphism iff*

$$\mathbf{x} \leq \mathbf{y} \Rightarrow \mu(\mathbf{x}) \preceq \mu(\mathbf{y})$$

For poset homomorphism, we want to explicitly identify the equivalence class of elements of $\mathbb{Q}$ that all map to the same element of $\mathbb{P}$:

Definition 7. *Suppose the mapping $\mu : \mathbb{Q} \to \mathbb{P}$ is a poset homomorphism.*

$$S^{\mu}(\mathbf{x}) = \{\mathbf{y} \ : \ \mu(\mathbf{x}) = \mu(\mathbf{y})\}$$

For example, in Figure 2(a),

$$S^{\mu}(\mathbf{12}) = \{\mathbf{12}, \mathbf{23}, \mathbf{34}, \mathbf{45}, \mathbf{123}, \mathbf{234}, \mathbf{345}\}$$

The idea is that all elements with the same dimension are in the same equivalence class. One of the semantic requirements for multimereologies is that there be a linear ordering on dimensions; if we consider the poset homomorphism to be the mapping from elements to their dimension, then we are particularly interested in the set of poset homomorphisms between partial orderings and finite linear orderings.

Theorem 1. *Let $Hom(\mathfrak{M}^{partial\_ordering}, \mathbf{3})$ denote the set of all poset homomorphisms between partial orderings and the finite linear ordering with 3 elements.*
There is a bijection: $\varphi : Hom(\mathfrak{M}^{partial\_ordering}, \mathbf{3}) \to \mathfrak{M}^{multimereology}$ such that for any $\mathbb{Q} \in \mathfrak{M}^{partial\_ordering}$, $\varphi(\mu) = \mathbb{Q} \oplus \mathbb{I}$ iff $\mu : \mathbb{Q} \to \mathbf{3}$ and

$$N^{\mathbb{I}}(\mathbf{x}) = ((L^{\mathbb{Q}}[\mathbf{x}] \cup U^{\mathbb{Q}}[\mathbf{x}]) \setminus S^{\mu}(\mathbf{x}))$$

In other words, each multimereology corresponds to a poset homomorphism between a partial ordering and a linear ordering, and any poset homomorphism between a partial ordering and a linear ordering can be used to construct a multimereology.

In Figure 2(a), all elements of the partial ordering that are points in the incidence structure are exactly the elements that are mapped to the minimum element of the linear ordering. Similarly, all elements of the partial ordering that are planes in the incidence structure are exactly the elements that are mapped to the maximum element of the linear ordering.

2.1.2. Axiomatization

$T_{multimereo}$ can be found in Figure 2.1.2. The first three axioms are for partial orderings (condition (1) in Definition 5) axioms (4)-(10) are for closed transitive tripartite incidence structures (condition (2) in Definition 5). Axiom (11) axiomatizes condition (3) in Definition 5. Finally, axioms (12)-(14) axiomatize conditions (4),(5), and (6) respectively in Definition 5.

Theorem 2. *There exists a bijection $\varphi : Mod(T_{multimereo})$[2] $\to \mathfrak{M}$[3]^{multimereo} such that*

1. $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{part}^{\mathcal{M}}$[4] *iff* $\mathbf{x} \in L^{\mathbb{Q}}[\mathbf{y}]$.
2. $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{in}^{\mathcal{M}}$ *iff* $\mathbf{x} \in N^{\mathbb{I}}[\mathbf{y}]$.

[2]Mod(.) denotes the class of models of the given ontology.
[3]$\mathfrak{M}$ denotes a class of mathematical structures.
[4]$\mathcal{M}$ is a specific model

$$\forall x \, part(x,x) \tag{1}$$
$$\forall x,y \, part(x,y) \wedge part(y,x) \supset (x=y) \tag{2}$$
$$\forall x,y,z \, part(x,y) \wedge part(y,z) \supset part(x,z) \tag{3}$$
$$\forall x \, in(x,x) \tag{4}$$
$$\forall x,y \, in(x,y) \supset in(y,x) \tag{5}$$
$$\forall x,y \, point(x) \wedge point(x) \wedge in(x,y) \supset (x=y) \tag{6}$$
$$\forall x,y \, line(x) \wedge line(x) \wedge in(x,y) \supset (x=y) \tag{7}$$
$$\forall x,y \, plane(x) \wedge plane(y) \wedge in(x,y) \supset (x=y) \tag{8}$$
$$\forall x,y,z \, in(x,y) \wedge in(y,z) \supset in(x,z) \tag{9}$$
$$\forall x \, (point(x) \vee line(x) \vee plane(x)) \tag{10}$$
$$\forall x,y \, (in(x,y) \supset (part(x,y) \vee part(y,x))) \tag{11}$$
$$\forall x,y \, ((point(x) \wedge part(y,x)) \supset point(y)) \tag{12}$$
$$\forall x,y,z \, ((line(x) \wedge line(y) \wedge part(x,z) \wedge part(z,y)) \supset line(z)) \tag{13}$$
$$\forall x,y \, ((plane(x) \wedge part(x,y)) \supset plane(y)) \tag{14}$$
$$\forall x,y \, (eqpart(x,y) \equiv \tag{15}$$
$$((point(x) \wedge point(y) \wedge part(x,y)) \vee$$
$$(line(x) \wedge line(y) \wedge part(x,y)) \vee (plane(x) \wedge plane(y) \wedge part(x,y))))$$

Figure 3: $T_{multimereo}$: Axiomatization of multimereology.

3. $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{eqpart}^{\mathcal{M}}$ *iff* $\mathbf{x} \in L^{\langle P, \leq \rangle}[\mathbf{y}] \cup L^{\langle L, \leq \rangle}[\mathbf{y}] \cup L^{\langle Q, \leq \rangle}[\mathbf{y}]$.

Theorem 2 together with Theorem 1 show that we have characterized the models of $T_{multimereo}$ up to isomorphism as the class of multimereologies. Moreover, we can easily see how multimereologies satisfy the semantic requirements. The incidence structure guarantees that elements are assigned a unique dimension and that the dimensions are linearly ordered. By partitioning the mereology into intervals of elements with the same dimension, we have both a parthood relation that is specified on entities with the same dimension., and a multidimensional parthood relation.

Although any partial ordering can be amalgamated with an incidence structure to construct a multimereology, two questions arise:

1. What is the appropriate partial ordering to represent the “global mereology"?
2. What criteria do we need to impose on the partitioning of the partial ordering into “local" equidimensional mereologies?

To answer these questions, we turn to the field of algebraic topology and the notion of topological manifold.

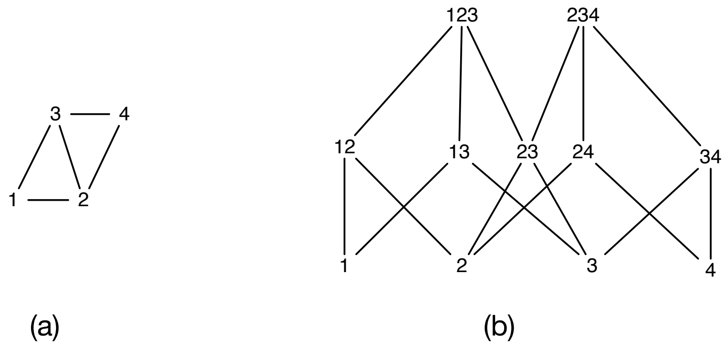

Figure 4: Simplicial complex associated with a graph.

3. Mereology and Topological Manifolds

Mathematicians have long used the notion of manifold to formalize the topology of shape and space. Of key interest for this paper is that topological manifolds provide an explicit notion of dimension – an n-dimensional manifold is a topological space in which every point has a neighbourhood that is homeomorphic to $\mathbb{R}^n$. In this section, we explore how different combinatorial structures for topological manifolds are related to multimereologies, and arrive at a suitable generalization of both that can serve as the basis for the framework that we seek to unify the nondimensional and multidimensional approaches to mereologies.

3.1. Simplicial Complexes

One combinatorial approach to the notion of manifold can be found in the notion of an abstract simplicial complex, which is a collection of finite sets that is closed under taking subsets:

Definition 8. *A family of sets Δ is called an abstract simplicial complex if, for every set X in Δ and every non-empty subset $Y \subseteq X$, we have $Y \in \Delta$.*

An example of an abstract simplicial complex can be found in Figure 4(b). The dimension of an element in an abstract simplicial complex is equal to one less than the cardinality of the set corresponding to the element. Thus, the dimension of the element $\mathbf{23}$ is one and the dimension of the element $\mathbf{123}$ is two. A simplicial n-complex is a simplicial complex in which the maximal dimension of an element in n.

Interestingly, we can associate a mereology with any abstract simplicial complex. Recall that classical extensional mereology $T_{cem\_mereology}$ is the axiomatization of classical mereology together with the Strong Supplementation Principle[5]. Let $T_{2Dcem\_mereology}$ be the extension of $T_{cem\_mereology}$ in which all models have rank 3 (i.e. all maximal chains in the model have cardinality 3).

Theorem 3. *$T_{2Dcem\_mereology}$ is logically synonymous [10] to $T_{simplicial\_2complex}$ [6].*

[5]https://github.com/gruninger/colore/blob/master/ontologies/mereology/cem_mereology.clif
[6]https://github.com/gruninger/colore/blob/master/ontologies/tripartite_incidence/simplicial_2complex.clif

In other words, there is a one-one-correspondence between simplicial 2-complexes and models of $T_{2Dcem\_mereology}$. In this sense, classical extensional mereology is the natural mereology for simplicial complexes. Nevertheless, even though we can associate a mereology with simplicial complexes, this approach falls short because it is trivial on equidimensional elements – no chain in the mereology contains elements with the same dimension. On the other hand, multimereologies allow models in which there is a nontrivial mereology on equidimensional elements.

3.2. CW Complexes

An alternative approach to topological manifolds, from a combinatorial point of view, is the notion of a CW-complex, and we can ask how it can be related to mereologies. To start, we adopt Hatcher's definition [11]:

Definition 9. *CW complex is a space X constructed in the following way:*

1. *Start with a discrete set X^0, the 0-cell of X.*
2. *Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e_n^α via maps $\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$. This means that X^n is the quotient space of $X^{n-1} \bigsqcup_\alpha D_\alpha^n$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \delta D_\alpha^n$. The cell e_α^n is the homeomorphic image of $D_\alpha^n - \delta D_\alpha^n$.*
3. *$X = \bigcup_n X^n$ with the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.*

A CW complex is constructed by induction via a cell attachment process. Simply put, each cell is attached to the existing one at its boundary in increasing order of dimensions. The connected graph in Figure 5(a) is the 1-skeleton of a CW complex. Figure 5(b) is the mereological representation corresponding to the construct of a CW complex. Since the 1-cells in the construction of a CW-complex are edges, the notion of "a path in a graph", like that of $\mathbf{123}$ in Figure 5(a), does not exist in the context of a CW complex. Nonetheless, in multimereology, we indeed want to differentiate between an edge and a path. Even though a path is homeomorphic to an edge, we want to treat paths and edges as two separate entities that are of the same dimension.

In this sense, CW complexes suffer from the same drawback as simplicial complexes on non-trivially representing equidimensional elements. Even though we need a more general structure than CW complexes to formalize mereologies and dimensionality, we still obtain the key insight that each 1-complex is a connected graph. Therefore, the associated mereology should correspond to the connected induced subgraphs of the graph that is the 1-complex.

3.3. Connected Induced Subgraph Containment Orderings

In the mereotopology T_{cisco_mt} [12], the sum of two elements exists iff they are connected. A weaker mereotopology $T_{weak\_cisco\_mt}$ alllows connected elements that do not have sums, although elements for which sums do exist must be connected. In a model of T_{cisco_mt}, there is one-to-one correspondence between elements in the mereology and connected induced

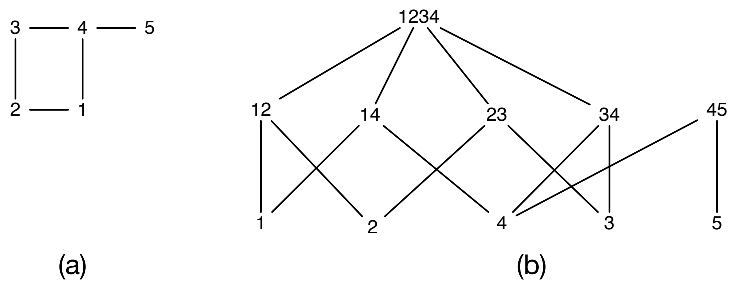

Figure 5: CW complex associated with a graph

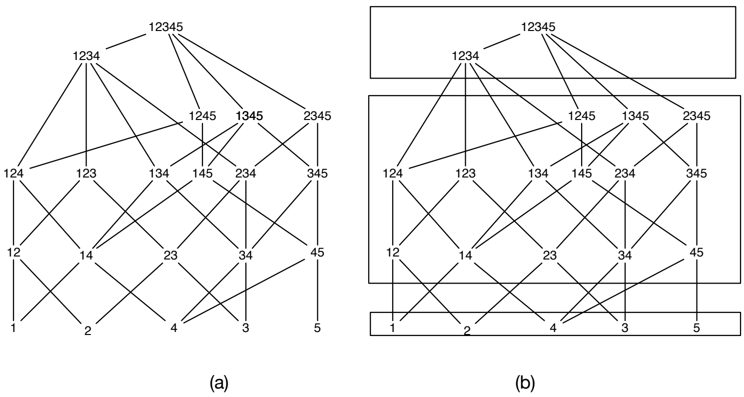

Figure 6: CISCO mereology representation and an associated example of partitioning the mereology

subgraphs of a connected graph, and the parthood relation is isomorphic to the containment ordering on the set of subgraphs.

Furthermore, many graph-theoretic properties are definable within the mereotopology. Paths in the graphs correspond to minimal upper bounds in the mereology and cycles in the graph correspond to triangles within the connection structure of the mereotopology. The identification of paths and cycles within an underlying graph is precisely the way in which we can use T_{cisco_mt} as the mereology within a multimereology that extends the structure of a CW-complex. Even though T_{cisco_mt} axiomatizes a mereology which is nondimensional, it is able to distinguish among elements which in the context of CW-complexes have different dimensions. On the other hand, T_{cisco_mt} also axiomatizes the mereology among elements with the same dimension, such as subpaths of paths and cycles which are subgraphs of 2-complexes.

Figure 6(a) showcases CISCO structure in representation of a 2-connected graph. We can non-trivially partition this given mereology into Figure 6(b). Element $\mathbf{124}$ is incident to $\mathbf{1234}$, and it is an equi-dimensional part of $\mathbf{1245}$.

An application of a multimereology based on T_{cisco} can be seen in the Molecular Structure Ontology (MoSt) [13]. MoSt is an ontology to describe the shape of a molecule; it treats functional groups, that are composed of rings and chains, to be incident to skeletons, while

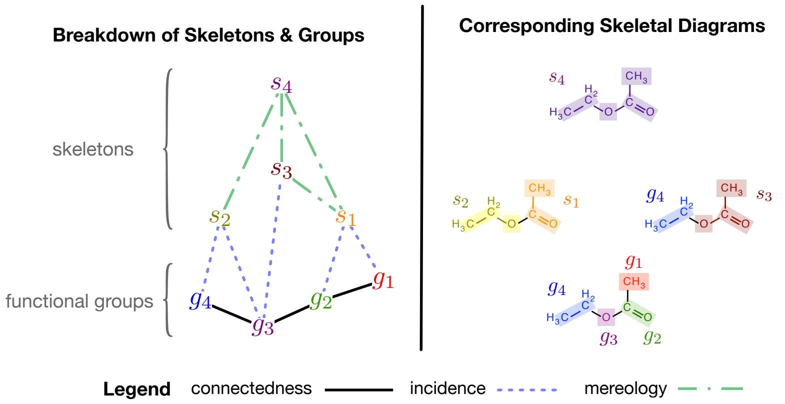

Figure 7: The breakdown of ethyl acetate. $\mathbf{g1}$, $\mathbf{g2}$, $\mathbf{g3}$, $\mathbf{g4}$ embody the primitive functional groups. $\mathbf{s1}$, $\mathbf{s2}$, $\mathbf{s3}$, $\mathbf{s4}$ embody the skeletons. The bolded black lines represent connectivity among functional groups. The dotted blue lines represent parthood relations. The dash-dotted green lines represent incident relations.

allowing a skeleton to be part of another skeleton. In other words, functional groups are one dimension lower than skeletons. Due to the nature of chemistry, not all decomposition of molecules is viable. There is a fixed set of criteria to outline how functional groups can be associated to form a skeleton. As outlined by Figure 7, even though $\mathbf{g3}$ is connected to $\mathbf{g2}$, they do not have a sum. If we were to explain MoSt in the language of multimereology, we would identify an equi-dimensional parthood relation among skeletons, and partitioning criteria across dimensions to accommodate for chemical bonding.

Another application can be seen from the motivating scenario, in which we want to represent hollow object like a gap in the wall. In $T_{multimereo}$, we define a surface to be two-dimensional. Holes and surfaces share the same subgraph in CISCO since a hole forms a cycle, exhibiting the properties of a two-dimensional surface. Nevertheless, intuitively, we know that a solid surface should be a dimension higher than a hollow surface. Since multimereology has the freedom to arbitrarily partition partial orderings according to dimensions, we can modify T_{cisco_mt} to characterize the fact that not every cycle constitutes a plane. In Figure4(a), imagine $\mathbf{123}$ is a hole, while $\mathbf{234}$ is a surface. The representation of this simplicial complex would still be true as shown in Figure4(b). The incidence structure will differ, however, since $\mathbf{123}$ is no longer classified as a two-dimensional plane, but is instead a 1-simplex.

3.4. Summary

We began this section with questions about which mereology should be used in a multimereology and which criteria should be used to partition it. Starting from the combinatorial structures used to represent manifolds in algebraic topology, we claim that T_{cisco_mt} is the right mereology, since it extends the notion of CW complex. Furthermore, the expressiveness of T_{cisco_mt} allows us to specify different partitionings, depending on the classes of subgraphs (e.g. paths, cycles, blocks).

4. Hahmann's Multidimensional Mereotopology

A further application of multimereology is that it can be used as the basis for evaluating other multidimensional mereotopologies. In particular, we can use multimereology to provide a verification of CODI [9]. Recall that the major benefit of CODI, compared to Gotts' and Galton's approaches, is its expressive power insofar as it generalizes relations between spatial entities up to finite dimensions within a single model. T_{codi_down} introduces three primitive relations for specifying a multidimensional mereotopology. The first two relations ($<_{dim} (x, y)$, $EqDim(x, y)$) specify the relative dimension of two elements; in particular, $<_{dim} (x, y)$ is a linear ordering over dimensions. The third relation ($Cont(x, y)$) is a parthood relation that applies to all elements regardless of their dimension such that the dimension of an element has a dimension greater than or equal to its parts.

To compare our axiomatization in $T_{multimereo}$ with T_{codi_down}, we want to find a common ground where the set of minimal and maximal dimensions allowed is the same for both sides. Since $T_{multimereo}$ can only capture entities up to two-dimensional planes, we must restrict T_{codi_down} to two dimensions as well:

Definition 10. *$T_{2d\_codi\_down}$ is the following set of axioms:*

$$\forall x, y\, (Cont(x, y) \supset (<_{dim} (x, y) \lor EqDim(x, y))) \tag{16}$$

$$\exists x\, (MinDim(x)) \tag{17}$$

$$\forall x\, (\neg <_{dim} x, x) \tag{18}$$

$$\forall x, y\, (<_{dim} x, y) \supset (\neg <_{dim} (y, x)) \tag{19}$$

$$\forall x, y, z\, (<_{dim} (x, y) \land (<_{dim} (y, z) \lor EqDim(y, z)) \supset (<_{dim} (x, z))) \tag{20}$$

$$\forall x, y\, ((Cont(x, y) \land Cont(y, x)) \supset (x = y)) \tag{21}$$

$$\forall x, y, z\, ((Cont(x, y) \land Cont(y, z)) \supset (Cont(x, z))) \tag{22}$$

$$\forall x\, (MinDim(x) \lor Curve(x) \lor ArealRegion(x)) \tag{23}$$

Theorem 4. *$T_{2d\_codi\_down}$ is logically synonymous with $T_{multimereo}$.*

By this Theorem, we know that there is a one-to-one correspondence between models of $T_{multimereo}$ and models of $T_{2d\_codi\_down}$. Furthermore, because of Theorem 1, we know that all models of $T_{2d\_codi\_down}$ can be constructed by partitioning a partial ordering into intervals.

5. Unifying boundary definitions

We want the weakest possible mereotopology in which the notions of boundary can be represented and unified. In this section, we demonstrate the capabilities of multimereologies in terms of representing different notions of boundary within the same theory.

5.1. Nondimensional Approach to Boundary

We exploit the dual nature of the models of T_{cisco_mt} – on the one hand, models of T_{cisco} are nondimensional mereologies, but because the elements of these mereologies correspond to connected subgraphs of a graph, they can also represent properties of graphs. The obvious place to start is to note that within graph theory there is a notion of the boundary of a subgraph [14]:

Definition 11. *Let $G = (V, E)$ be a simple graph. The boundary of $\mathbf{v} \in V$ is the set of all vertices of G which are adjacent to $\mathbb{V}$:*

$$B(\mathbf{v}) = \{\mathbf{u} \in V : (\mathbf{u}, \mathbf{v}) \in E\}.$$

The edge boundary of H is the set of edges in G that contain one vertex in H and one vertex not in H.

Based on this definition of graph boundary, we can define the corresponding notion for elements within the mereology in a model of T_{cisco_mt}:

Definition 12.

$$\forall x\, (edge(x) \equiv \exists y, z\, (atom(y) \wedge atom(y) \wedge sum(y, z) \wedge \neg(y = z))) \tag{24}$$

$$\forall x, y\, (graph\_boundary(x, y) \equiv edge(x) \wedge PO(x, y)) \tag{25}$$

The boundary of the cycle $\mathbf{1234}$ in Figure 6 is $\{\mathbf{45}\}$, while the boundary of the edge $\mathbf{12}$ in the same Figure is $\{\mathbf{23}, \mathbf{14}\}$.

5.2. Multidimensional Approach to Boundary

Inspired by algebraic topology [15], our interpretation of multidimensional notion of boundary exhibits the following two characteristics:

1. The boundary is codimension 1 to the space that contains it.
2. x is said to be the boundary of y iff x is a pendant element [7] of y.

which can be axiomatized as

Definition 13.

$$\forall x, y\, (boundary(x, y) \equiv in(x, y) \wedge m\_covers(y, x) \wedge pendant(x, y)) \tag{26}$$

$$\forall x, y\, (pendant(x, y) \equiv \exists z, u\, (C(x, z) \wedge C(x, u) \wedge in(u, y) \wedge in(z, y) \wedge \neg(z = u))) \tag{27}$$

$$\forall x, y\, (m\_covers(x, y) \equiv$$

$$(part(y, x) \wedge x \neq y \wedge \neg\exists z\, (part(y, z) \wedge part(z, x) \wedge in(z, y)) \wedge y \neq z \wedge x \neq z)) \tag{28}$$

According to this multidimensional characterization of boundary, in the same Figure6(b), element $\mathbf{2}$ is not in the boundary of $\mathbf{123}$ because it is incident to more than one incomparable equi-dimensional parts. Contrarily, $\mathbf{1}$ and $\mathbf{3}$ are the boundary of 123 because it is incident to exactly one incomparable equi-dimensional part. In the same example in Figure5(a), the set of boundary that was generated based on graph-theory definition is different from that of a multidimensional based one. Note that even though multimereology successfully captures both non- and multi-dimensional approaches to boundary, the exact relationship that holds between these two approaches remains an open question.

[7] In graph theory, a pendant element is connected to exactly one neighboring element.

6. Summary

This work was motivated by the need to integrate different characterizations of boundary. From the Winograd Schema Challenge to Kandinsky's painting, we saw that boundary is only of a special form to the dimension discrepancy founded by different mathematical theories, and borrowed by various mereotopologies. Instead of favoring one theory over the other, we established a common ground for non- and multi-dimensional approaches.

We started the discussion of unification by specifying ontological commitments that a multi-dimensional mereology must satisfy. We introduced the notion of multimereology, an amalgamation of a partial ordering and incidence structure that formalizes these commitments. A key result is that each partitioning of a partial ordering corresponds to a unique multimereology, and each multimereology specifies a partitioning of a partial ordering. This forms the basis of the representation theorem for the class of multimereologies with respect to the set of poset homomorphisms between partial orderings and finite linear orderings. Finally, we axiomatized the class of multimereologies up to isomorphism.

Coming up with root theories for multimereology left us with the problem of amalgamating an appropriate partial ordering with partitioning criteria. We therefore turned to the field of algebraic topology to find the "right" manifold that can characterize our notion of partitioning a partial ordering. We examined abstract simplicial complexes and regular CW complexes as two possibilities. Due to the sets of construction rules they impose on topological spaces, both CW and simplicial complexes fall short in representing non-trivial mereology within a single dimension. We proceeded to showcase the expressiveness of our multimereology by interpreting and representing other non- and multi-dimensional approaches.

So far, this work provided a middle ground to compare different notions of boundary. In the future, we plan to explore and extend what multimereology is capable of. This can be achieved in two directions. First, it would be useful to compose an exact theorem that captures the relationship between non- and multi-dimensional boundary. Next, if we could use multimereology to validate Smith's and Hahmann's mereotopologies, we would be one step closer to harmonizing this dimension disagreement.

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