The geometry of N-body orbits and the DFT —Extended Abstract for work in progress —

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1. Extended Abstract

The context is the Newtonian equal-mass three-body problem [Newton1687]. It's been a couple of decades since the discovery by Cris Moore [Moore1993] of a new periodic choreographic orbit, the first since Euler [Euler1767] and Lagrange [Lagrange1772]. Choreographic means that the all the particles follow the same orbital path. This figure-eight orbit was a numerical solution done on a Mac SE when looking for braids in orbits. The proof of its mechanical existence by Richard Montgomery [Montgomery1998] and Alain Chenciner [ChencinerMontgomery2000]was seen as important [see also [Chen2001] and [Nauenberg2007]] . Poincaré [Poincare1890] had discussed the necessarily complex, even chaotic, nature of 3-body orbits [Poincare1890] . This led to additional hundreds of new periodic choreographic orbits found numerically by Carles Simó [Simo2002] and later others [SuvakovDmitrasinovic2013]. The required proofs that these were also more than numerical objects still remain to be provided, with a few exceptions.

At about the same time, there was a renewal of interest in the use of the discrete Fourier transform (DFT) in Euclidean geometry. This subject goes back to Jesse Douglas [Douglas1940a] and Isaac Schoenberg [Schoenberg1950]. The second simplest consideration of this type is based on the harmonic analysis of the cyclic group of order 3 (second because order 2 is even simpler than 3). The basic assertion is then the classical construction of Napoleon's Theorem. Any triangle, seen as a triple of points in the complex plane, may be written as a complex linear combination of the totally degenerate triangle consisting of three coincident points located at 1, and the two standard equilateral triangles drawn in the unit disk with a vertex at 1, one for each possible orientation.

Returning to mechanics, one remarks that a solution of a three-body problem means giving the evolution in space of the three coordinates of the point masses involved. If the masses are all equal we're looking at the evolution of a simple triangle in the plane, thanks to the conservation laws of mechanics. Viewing the triangle in terms of harmonic coordinates as mentioned above, the first coordinate is the constant center of gravity of the three masses, so unmoving. Thus, to a 3-body solution correspond two more plane curves which are the tracks of the two non-degenerate harmonic coordinates.

In the case of the new figure-eight choreography, the DFT leads to two symmetrical 'triangular platelet boundaries'. It is known that the figure-eight orbit is not a lemniscate, or indeed parametrizable in terms of well-known special functions. So it might seem there may be some collection of special functions associated with Newtonian mechanics and good for parameterizing such curves.

It is appealing to see what the apparently very complicated higher-order Simó choreographies may lead to. One takes the conventional orbits and performs a DFT as above, then plots the resulting curves. These display visually a high degree of symmetry and regularity not apparent in the original orbits.

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Actually Simó's published discussions of how he found and calculated his 343 new periodic 3-body choreographies, and a number of choreographies for more bodies (some simple ones being just equally spaced rings of more than 3 particles) do not provide full sets of initial conditions that allow reproducing his results in, say, Octave (an open source analogue of Matlab). He remarks in his work that published initial values are often not precise enough to allow numerical following of orbits that are claimed, or indeed illustrated. So to produce the required DFT images I had to reverse engineer the (to me) rather odd plot format made public by Simó. The results I put up on a personal website at the University of Michigan [IonWeb]. Then I redid another version, creating SVG images using Mathematica 4, and added those for viewing.

Early on, there was much interest in recreating the original figure-eight orbit; many people did so. There were contributions from numerical analysis experts — such as Broucke [Broucke1975], Hadjidemetriou [Hadjidemetriou1975], Kapela et al. [Kapela2005]– and celestial orbit people — such as Marchal [Marchal2002], Hénon [Henon1976], Aarseth [Aarseth2003], Alexander D.Bruno, Montaldi and Steckles [MontaldiSteckles2013], Gerver [Gerver2003a], Moeckel [Moeckel2012], Terracini [Terracini2006], Ferrario [Ferrario2024], Zhifu Xie [Xie2022] — and also by others — such as Jenkins [JenkinsWeb], Vanderbei [VanderbeiWeb]; Jenkins, a self-proclaimed amateur, like others, also created a notable web site allowing orbit viewing using Java. The methods ranged from Runge-Kutta numerics of various types to action minimization and other variational routines, or used built-in solvers like those of [Mathematica], [Maple], or [Matlab] and [Octave]. At one point I counted about 40 different approaches. Of course, a number of the web presences of these efforts have by now disappeared. Notable to me was that though there were lots of figure eights, say, there was no clarity that they were all describing the same orbit—the results are given as a finite sequence of computed coordinate values of widely varying precisions. Phil Sharp [Sharp2006] (and I) produced a Matlab routine that showed the choreographic eight, but a change of 1 part in 10^{12} in initial conditions splits the result into three parallel orbits that were, of course, visually indistinguishable, if plotted ordinarily, from the true choreography's single repeated orbit.

More recently, in 2019, Li and Liao [LiLao2019] announced discovery of 313 more periodic collisionless orbits. Then in 2023 Hristov, Hristova, Dmitrašinović and Tanikawa [HristovEtAl2024] announced more than 12,000 distinct 3-body orbits, derived using newer computing hardware and a refined assignment of symbol sequences to trajectories that made search for suitable orbits easier. They also pointed out some edge problems with Li and Liao's listing. It is now time to examine the new orbits from the DFT point of view. This involves reviving some older constructions which ran fine under earlier versions of scripting languages (e.g. Python, Javascript), graphics technology (e.g. SVG), numerical technology (e.g., [Octave], [Numpy] etc., Java) and symbolic computation platforms (e.g. [Mathematica] and [Maple]).

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A. Online Resources

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