(t, r)-Broadcast Domination in Networks*

Gennaro Cordasco[1,*,†], Luisa Gargano[2,*,†] and Adele A. Rescigno[2,*,†]

[1]Department of Psychology, University of Campania "L.Vanvitelli", Italy.
[2]Department of Computer Science, University of Salerno, Italy.

Abstract

We study a recently introduced generalization of distance domination in graphs known as (t, r)-Broadcast Domination: A set S of broadcasting vertices transmit a signal of initial strength t; the strength of the signal decays linearly along edges according to distance, that is, a vertex at a distance $d < t$ from a broadcasting vertex $v \in S$ receives a signal of strength $t - d$, for each $v \in S$. The goal is to determine a set S of broadcasting vertices of minimum size, which ensures that every vertex in the network receives a cumulative signal strength of at least r.

In this paper, we initiate the study of the (t, r)-Broadcast Domination problem in general graphs. Our results include a general approximation algorithm and optimal polynomial time algorithms for cographs. Moreover, we consider graphs of bounded *Neighborhood diversity* (`nd`), and graphs of bounded *Iterated type partition number* (`itp`) and give: (i) a *fixed parameter tractable (FPT)* algorithm for (t, r)-Broadcast Domination parameterized by `nd`; (ii) a FPT algorithm for (t, r)-Broadcast Domination parameterized by `itp` plus the solution size $\beta = |S|$; (iii) a FPT algorithm for (t, r)-Broadcast Domination parameterized by `itp` plus the demand r.

Keywords

Broadcast domination, Complexity, Approximation, Parameterized complexity

1. Introduction

The concept of graph domination was introduced by Claude Berge in [3] and formally defined by Oystein Ore in [31]. Since then it has been extensively studied and generalized in quite many interesting variants [20, 21]. These include versions of graph domination with distance parameters that can be applied in multi-agent security and pursuit, city planning (such as the placement of hospitals, radio stations, and nuclear reactors), routing in communication networks, and sensor placement in power networks. One significant generalization of the graph domination problem is distance domination, first studied in [23]. In this variant, a vertex in the dominating set can "dominate" its directly adjacent vertices and all vertices within a certain distance. Specifically, a distance-k dominating set of a graph $G = (V, E)$ is a set $S \subseteq V$ such that every vertex $v \in V$ is either in S or it is within distance k from a vertex in S. This means

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*Corresponding author.
†These authors contributed equally.
gennaro.cordasco@unicampania.it (G. Cordasco); lgargano@unisa.it (L. Gargano); arescigno@unisa.it (A. A. Rescigno)
https://www.psicologia.unicampania.it/dipartimento/docenti (G. Cordasco); https://docenti.unisa.it/000779/home (L. Gargano); https://docenti.unisa.it/001727/home (A. A. Rescigno)
0000-0001-9148-9769 (G. Cordasco); 0000-0003-3459-1075 (L. Gargano); 0000-0001-9124-610X (A. A. Rescigno)

that S is a distance-k dominating set of G if any vertex v in G can be reached by a path of length at most k starting from a vertex in S. The distance-k domination number of a graph G is the smallest cardinality of such a set. Notably, distance-k domination generalizes standard domination, where distance-1 domination is equivalent to classical domination.

A recent variation of distance domination is the (t, r)-Broadcast Domination, first defined by Blessing *et al.* in [4]. As an application consider the following scenario in a wireless network G: A set of broadcast towers on a subset of a graph's vertices, each has a signal strength t. Each tower provides itself a signal strength of t, its neighbors $t - 1$, and so on, decreasing by 1 as it traverses each edge until the signal dies out. The goal of the (t, r)-Broadcast Domination problem is to determine the minimal number of towers needed to ensure every vertex in G receives a cumulative signal strength of at least r.
Another application addresses the issue of effectively crafting accurate and evidence-based information to combat misinformation. In epidemiology, graph-based information diffusion algorithms can be used to find the smallest set of individuals who can cooperate in immunizing the vertices in order to prevent the spreading of negative narratives [12, 17, 26]. While these algorithms are not designed for intercepting fakes, they can be used as a component in a broader strategy for identifying and mitigating the spread of fake information. (t, r)-Broadcast Domination assumes that the strength of the message decreases by passing from one individual to another.

Since its introduction, the (t, r)-Broadcast Domination problem has been extensively studied in various special classes of graphs: Two-dimensional grids, paths, triangular grids, matchstick graphs, and n-dimensional grids [1, 2, 4, 6, 7, 13, 15, 16, 22, 25, 32].

Our results. In this paper, we initiate the study of the (t, r)-Broadcast Domination problem in general graphs.
Our results include: (i) a general approximation algorithm and (ii) an optimal polynomial time algorithm for cographs. Moreover, we consider graphs of bounded *Neighborhood diversity (*`nd`*)*, and graphs of bounded *Iterated type partition number (*`itp`*)* and give: (iii) a *fixed parameter tractable (FPT)* algorithm for (t, r)-Broadcast Domination parameterized by `nd`; (iv) a FPT algorithm for (t, r)-Broadcast Domination parameterized by `itp` plus the solution size $\beta = |S|$; (v) a FPT algorithm for (t, r)-Broadcast Domination parameterized by `itp` plus r.
We recall that a problem with input size n and parameter p is called *fixed-parameter tractable (FPT)* if it can be solved in time $f(p) \cdot n^c$, where f is a computable function only depending on p and c is a constant [14, 30].

2. The problem

Let $G = (V(G), E(G))$ be an undirected graph and two vertices $u, v \in V(G)$, we denote by $n = |V(G)|$ the number of vertices in G and by $d(u, v)$ the distance between u and v in G. Moreover, for a vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) \mid (u, v) \in E(G)\}$ the neighborhood of v and $N_G[v] = N_G(v) \cup \{v\}$. Furthermore, we denote by $N_{G,t}(v) = \{u \in V(G) \mid d_G(u, v) \leq t\}$ the *neighborhood of radius t around v*. Clearly, $N_{G,1}(v) = N_G[v]$. In the above notations, we will omit the subscript G whenever the graph is clear from the context.

A *Dominating set* in a graph $G = (V, E)$ is a subset of V such that every vertex not in the set has at least one neighbor in the set.

Definition 1. *Given an undirected graph $G = (V, E)$ and integers $t \geq 2$, $r \geq 1$, a subset of vertices $S \subseteq V$ is a (t, r)-* Broadcast Dominating set *of G if for each $v \in V$, it holds*

$$\sum_{u \in S \cap N_t(v)} (t - d(u, v)) \geq r, \tag{1}$$

In this paper, we will consider the following problem.

> (t, r)-Broadcast Domination:
> **Input:** An undirected graph $G = (V, E)$ and two integers $t \geq 2$, $r \geq 1$
> **Output:** Find a (t, r)-Broadcast Dominating set of minimum size.

We assume that for each $v \in V$ we have $r \leq \sum_{u \in N_t(v)} (t - d(u, v))$, otherwise, the problem does not admit any solution. This assumption is harmless, in the sense that it is a polynomially verifiable condition that does not impact the results.

It is worth observing that when $t = 2$ and $r = 1$ the (t, r)-Broadcast Domination problem corresponds to the classical Dominating set problem.

Finally, since the problem can be solved independently in each connected component of the input graph, from now on, we assume that the input graph is connected.

3. Approximation algorithm

Knowing that (t, r)-Broadcast Domination generalizes the Dominating set problem, by the hardness results in [8, 18], we immediately have the following theorem.

Theorem 1. (t, r)-Broadcast Domination *cannot be approximated to within a factor of $(1 - \epsilon) \ln n$ in polynomial time for any constant $\epsilon > 0$ unless $NP \subseteq DTIME(n^{O(\log \log n)})$.*

Moreover, using the same arguments of the proof of Theorem 1 in [9], one can easily get a logarithmic approximation algorithm. We show that the (t, r)-Broadcast Domination problem can be recast as a submodular cover problem, and apply a classical results due to Wosley [33].

For a graph $G = (V, E)$ and integers $t \geq 2$, $r \geq 1$, we define a function $f : 2^V \to \mathbb{N}$, as follows: for all $S \subseteq V$, let

$$f(S) = \sum_{v \in V} \pi_v(S), \quad \text{where} \quad \pi_v(S) = \min\left(r, \sum_{s \in S \cap N_t(v)} (t - d(s, v))\right). \tag{2}$$

Lemma 1. *The function $f : 2^V \to \mathbb{N}$ given in (2), satisfies the following properties:*
(i) f is integer valued; (ii) $f(\emptyset) = 0$; (iii) f is non-decreasing; (iv) A set $S \subseteq V$ satisfies $f(S) = f(V)$ if and only if S is (t, r)-Broadcast Dominating set; (v) f is submodular.

Hence, one can apply the natural greedy algorithm, call it $\mathcal{A}$, which starts with $S = \emptyset$ and iteratively adds to S the element $v \in V \setminus S$ s.t. $f(S \cup \{v\}) - f(S)$ is maximum, until $f(S) = f(V)$ is achieved.

Theorem 2. (t, r)-Broadcast Domination *problem can be approximated in polynomial time by a factor* $\ln n + \ln(\min(r, t)) + 1$.

Proof. By a classical result of Wolsey [33], it follows that algorithm $\mathcal{A}$ is a $(\ln(\max_{w\in V} f(\{w\}))+1)$-approximation algorithm for (t,r)-Broadcast Domination. For each $w \in V$, we have

$$f(\{w\}) = \sum_{v\in V} \min\left(r, \sum_{s\in\{w\}\cap N_t(v)} (t-d(s,v))\right) = \sum_{v\in N_t(w)} \min\big(r, t-d(w,v)\big) \le n\min(r,t).$$

□

It is worth observing that we can always assume that $t \le n-1$, since the distances between two nodes are at most equal to $n-1$. Hence, the approximation guaranteed by the Theorem is at most $2\ln n + 1$.

4. A polynomial time algorithm for cographs

Cographs have been discovered independently many times since the 1970s, with different equivalent definitions [5]. We are going to adopt the following definition.

Definition 2. *A* cograph *is a graph that can be constructed using the following recursive rules:*

- *Any single-vertex graph is a cograph;*
- *The disjoint union $G_1 \oplus G_2$ of two cographs is a cograph.*
  $G_1 \oplus G_2$ is the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$.
- *The join $G_1 \otimes G_2$ of two cographs is a cograph.*
  $G_1 \otimes G_2$ has vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup\{(u,w) \mid u \in V(G_1),\ w\in V(G_2)\}$.

Definition 3. *A cotree $T(G)$ is a binary parse tree defining a cograph G, in which the leaves are the vertices of G and each internal vertex labeled $\oplus$ (resp. $\otimes$) represents the disjoint union (resp. join) operation.*

Observation 1. *If a cograph G is connected, then the root of the cotree is labeled $\otimes$ and the diameter is at most 2.*

Consider a connected cograph $G = (V,E)$, the cotree T associated to G, and integers $t \ge 2$, $r \ge 1$. We recursively compute a solution of the (t,r)-Broadcast Domination problem. The algorithm uses the dynamic-programming design pattern and traverses the cotree T in a breadth-first fashion.

Fix a node x in T, we denote by $G(x) = (V(x), E(x))$ the subgraph of G associated to x. Clearly $G = G(y)$ where y is the root of T. To reconstruct the solution recursively, we calculate optimal solutions, for each internal node of $x \in T$ considering all the possible contributions, of vertices in $V \setminus V(x)$. Specifically, we compute bottom-up the solutions associated with each internal node $x \in T$ for each demand $r' = 1,\dots,r$. A solution with demand $r' = r-a$ will be used when we assume that each node in $V(x)$ receives a cumulative signal strength a from vertices in $V\setminus V(x)$.

The following definition introduces the values that will be computed by the algorithm in order to be able to compute the solution of the (t,r)-Broadcast Domination problem.

Definition 4. *Given a cograph $G=(V,E)$ and two integers $t\ge 2$, $r\ge 1$ we denote by $\gamma_t(G,r)$ the size of a smallest (t,r)-Broadcast Dominating set for G, where the distance function is redefined as $d'(u,v) = \min(2, d_G(u,v))$, for each $u,v\in V$.*

For $r=0$, we also define this value for any cograph G as $\gamma_t(G,0)=0$.

Formally, we are going to compute the value $\gamma_t(G(x),r')$, for each $x\in T$ and $r' = 0,1,\dots,r$.

Observation 2. *The reason for using the redefined distance function is as follows. For each internal node $x \in T$ labeled with $\otimes$ (join) the redefined function matches the original one, while when x is labeled with $\oplus$ (disjoint union), we have that x is associated with two disconnected components. In this case, since the graph G is connected, and the vertices in $V(x)$, by construction, will share the same neighborhood in $V \setminus V(x)$, we know that the distance between vertices belonging to different components of $G(x)$ is 2 in G.*

The solution for the instance $\langle G, t, r\rangle$ of our original (t, r)-Broadcast Domination problem is $\gamma_t(G(y), r)$ where y is the root of T.

Lemma 2. *For each $x \in T$, the computation of $\gamma_t(G(x), r')$ for each $r' = 1, \ldots, r$ can be done recursively in time $O(r^2)$.*

Proof. Consider a node $x \in T$, we show how to use a bottom-up strategy to compute all the values of $\gamma_t(G(x), r')$, for each $r' = 1, \ldots, r$.

For each leaf $x \in T$ we have that $G(x)$ is a single vertex. Then,

$$\gamma_t(G(x), r') = \begin{cases} 1 & \text{if } t \geq r', \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

For any internal node x, assuming that we have already computed the solution for its children nodes, we show how to calculate each value $\gamma_t(G(x), r')$, for each $r' = 1, \ldots, r$, in time $O(r')$.

We have two cases to consider according to the label of x (cf. Definition 3):

1) Node x is labeled $\oplus$ (i.e., represents the disjoint union operation).
In this case we have that $G(x) = G_1 \oplus G_2$ where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the graphs associated with the children of x in T. By Observation 2, we assume that the distance between vertices belonging to the two components G_1 and G_2 is 2.

For each $r' = 1, \ldots, r$, we can fix the demand r_1 (r_2) for G_1 (G_2) and compute the solution for the other graph G_2 (G_1) ensuring that both solutions satisfy the demand r'.

Fixed the value of r', we denote by m_1 the minimum value of r_1, which represents the portion of the demand r' satisfied by vertices in V_1. Since the distance between vertices belonging to the two components is 2, we have that each vertex in V_2 provides a signal strength of $t-2$ to vertices in V_1 and overall they should cover the residual demand $r' - m_1$. Hence, the value of m_1 corresponds to the smallest non negative integer such that $|V_2| \geq \left\lceil \frac{r'-m_1}{t-2} \right\rceil$. Indeed, by choosing a value of r_1 smaller than m_1, we have that the contribution of vertices in V_2 is not enough to reach r' and hence such values are not compatible with a solution for $G(x)$. Then, for each $m_1 \leq r_1 \leq r'$, we compute the value $\overline{r_2} = \max(0, r' - (t-2)\gamma_t(G_1, r_1))$, which corresponds to the residual demand on G_2. Similarly, let m_2 the smallest non negative integer such that $|V_1| \geq \left\lceil \frac{r'-m_2}{t-2} \right\rceil$. For each $m_2 \leq r_2 \leq r'$, we compute the value $\overline{r_1} = \max(0, r' - (t-2)\gamma_t(G_2, r_2))$, which corresponds to the residual demand on G_1. We have,

$$\begin{aligned} \gamma_t(G(x), r') \quad = \quad \min\Bigg(& \min_{m_1 \leq r_1 \leq r'} \left(\gamma_t(G_1, r_1) + \max\left\{ \left\lceil \frac{r'-r_1}{t-2} \right\rceil, \gamma_t(G_2, \overline{r_2}) \right) \right), \\ & \min_{m_2 \leq r_2 \leq r'} \left(\gamma_t(G_2, r_2) + \max\left(\left\lceil \frac{r'-r_2}{t-2} \right\rceil, \gamma_t(G_1, \overline{r_1}) \right) \right) \Bigg). \end{aligned} \tag{4}$$

2) Node x is labeled $\otimes$ (i.e., represents the join operation).
Let $G(x) = G_1 \otimes G_2$. We repeat the reasoning above by considering that, in this case,the distance between vertices belonging to the two components G_1 and G_2 is 1.

Let m_1 be the smallest non negative integer such that $|V_2| \geq \left\lceil \frac{r'-m_1}{t-1} \right\rceil$. For each $m_1 \leq r_1 \leq r'$, we compute the value $\overline{r_2} = \max(0, r' - (t-1)\gamma_t(G_1, r_1))$.

Similarly, let m_2 the smallest non negative integer such that $|V_1| \geq \left\lceil \frac{r'-m_2}{t-1} \right\rceil$. For each $m_2 \leq r_2 \leq r'$, we compute the value $\overline{r_1} = \max(0, r' - (t-1)\gamma_t(G_2, r_2))$. We have,

$$\begin{aligned}
\gamma_t(G(x), r') \quad &= \quad \min\Bigg(\min_{m_1 \leq r_1 \leq r'} \left(\gamma_t(G_1, r_1) + \max\left(\left\lceil \frac{r'-r_1}{t-1} \right\rceil, \gamma_t(G_2, \overline{r_2}) \right) \right), \\
&\qquad \min_{m_2 \leq r_2 \leq r'} \left(\gamma_t(G_2, r_2) + \max\left(\left\lceil \frac{r'-r_2}{t-1} \right\rceil, \gamma_t(G_1, \overline{r_1}) \right) \right) \Bigg). \qquad (5)
\end{aligned}$$

By induction on the tree, we can prove that the recursive formula presented in (4)-(5) coincides with the definition of $\gamma_t(\cdot, \cdot)$ and both the values are computed in time $O(r')$; hence, the algorithm is correct and the overall computation associated to a node $x \in T$ is $O(r^2)$. $\square$

Theorem 3. *When G is a cograph, the (t, r)-*Broadcast Domination *problem is solvable in time $O(nr^2 + m)$.*

Proof. We recall that: (i) Building the cotree T associated to a given cograph G can be done in linear time ($O(n + m)$); (ii) the cotree contains $2n - 1$ vertices (it has n leaves).
Hence, exploiting Lemma 2, we can build the desired solution $\gamma_t(G, r)$ in time $O(nr^2 + m)$. The optimal set S can be computed in the same time by standard backtracking techniques. $\square$

5. Graphs of bounded neighborhood diversity or itp number

In this section, we recall the definitions of Neighborhood diversity and Iterated type partition number of a graph and give FPT algorithms for graphs in which such parameters are bounded.

Definition 5. *The following operations can be used to construct any graph:*

(O1) The creation of an isolated vertex.

(O2) The substitution of the vertices $1, \ldots, \ell$ of an outline graph H by the graphs $G_1, \ldots, G_\ell$, denoted by $H(G_1, \ldots, G_\ell)$, is the graph $G = (V, E)$ with
$V = \bigcup_{1 \leq i \leq \ell} V(G_i)$
$$E = \bigcup_{1 \leq i \leq \ell} E(G_i) \cup \{(u, w) \mid u \in G_i,\ w \in G_j,\ (i, j) \in E(H),\ 1 \leq i < j \leq \ell\}.$$

Notice that the disjoint union and join operations of Definition 2 are special cases of (O2) with $\ell = 2$.

Let $G = H(G_1, \ldots, G_\ell)$ be a connected graph. According to operation (O2): H is a connected outline graph with ℓ vertices and G_i is a subgraph of G such that for all $u, v \in V(G_i)$, $N(u) \setminus V(G_i) = N(v) \setminus V(G_i)$, for each $i = 1, \ldots, \ell$.

Let S be a (t, r)-Broadcast Dominating set of G. Denote by S_i the subset of S including the vertices in G_i, that is, $S = \bigcup_{1 \leq i \leq \ell} S_i$ where $S_i \subseteq V(G_i)$.

In the following, we give a reformulation of the condition in (1) of Definition 1 that takes into account the construction of G in terms of the operation (O1)–(O2) described in Definition 5 and that will be useful to present our algorithms.

Since G is a connected graph then also H is a connected graph. Hence, we have that each vertex $v \in V(G_i)$, with $1 \leq i \leq \ell$, is at distance at most 2 from any $u \in V(G_i)$, and $d_G(u,v) = d_H(i,j)$ for each $u \in V(G_j)$ for $j \neq i$. Hence, knowing that $t \geq 2$, for each $v \in V(G_i)$, we have

$$|S \cap N_{G,t}(v)| = |S \cap V(G_i)| + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S \cap V(G_j)| = |S_i| + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|$$

and

$$\sum_{u \in S \cap N_{G,t}(v)} d_G(u,v) = \sum_{u \in S_i} d_G(u,v) + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|\, d_H(i,j)$$

$$= |S_i \cap N_{G_i}(v)| + 2\,|S_i \setminus N_{G_i}[v]| + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|\, d_H(i,j).$$

Summarizing,

$$\begin{aligned}
&\sum_{u \in S \cap N_{G,t}(v)} (t - d_G(u,v)) = t\,|S \cap N_{G,t}(v)| - \sum_{u \in S \cap N_{G,t}(v)} d_G(u,v) = \\
&= t\left(|S_i| + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|\right) - \left(|S_i \cap N_{G_i}(v)| + 2\,|S_i \setminus N_{G_i}[v]| + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|\, d_H(i,j)\right) \\
&= t\,|S_i| - |S_i \cap N_{G_i}(v)| - 2\,|S_i \setminus N_{G_i}[v]| + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|\,(t - d_H(i,j)). \qquad (6)
\end{aligned}$$

Now, we consider that $(S_i \cap N_{G_i}(v)) \cap (S_i \setminus N_{G_i}[v]) = \emptyset$ and

– for $v \notin S_i$ it holds $(S_i \cap N_{G_i}(v)) \cup (S_i \setminus N_{G_i}[v]) = S_i$, and

– for $v \in S_i$ it holds $(S_i \cap N_{G_i}(v)) \cup (S_i \setminus N_{G_i}[v]) = S_i \setminus \{v\}$. Then,

$$|S_i \cap N_{G_i}(v)| + 2\,|S_i \setminus N_{G_i}[v]\,| = \begin{cases} 2|S_i| - |S_i \cap N_{G_i}[v]| & \text{if } v \notin S_i, \\ 2|S_i| - 2 - |S_i \cap N_{G_i}[v]| & \text{if } v \in S_i. \end{cases}$$

Hence, by (6), to verify (1), i.e., $\sum_{u \in S \cap N_{G,t}(v)} (t - d_G(u,v)) \geq r$, for each vertex $v \in V(G_i)$, is equivalent to verify

$$((t-2)\,|S_i| + |S_i \cap N_{G_i}[v]\,|) + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|(t - d_H(i,j)) \geq r \qquad \text{if } v \notin S_i, \qquad (7)$$

$$((t-2)\,|S_i| + 2 + |S_i \cap N_{G_i}[v]\,|) + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} |S_j|(t - d_H(i,j)) \geq r \qquad \text{if } v \in S_i. \qquad (8)$$

Notice that the first part in each of the two sums of (7) and (8) refers to the relation between $v \in V(G_i)$ and the vertices of the solution set in G_i (i.e., S_i) while the second part refers to the relation between vertex i in $V(H)$ and all the other vertices j in $V(H)$.

5.1. Neighborhood Diversity

The neighborhood diversity of a graph was introduced by Lampis in [28]. Given a graph $G = (V, E)$, two vertices $u, v \in V$ have the same *type* iff $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. The *neighborhood diversity* of a graph G, $\mathtt{nd}(G)$, is the minimum number ℓ of sets in a partition $V_1, V_2, \dots, V_\ell$, of the vertex set V, such that all the vertices in V_i have the same type, for $i = 1, \dots, \ell$. By definition, each V_i induces either a *clique* or an *independent set* in G. In this paper, we will use the following equivalent definition based on the operations (O1)-(O2).

Definition 6. *A graph G has neighborhood diversity $\ell \geq 1$ if ℓ is the minimum integer such that $G = H(G_1, \dots, G_\ell)$ where G_i is either a clique or an independent set, for each $i = 1, \dots, \ell$.*

The following theorem states that the (t, r)-Broadcast Domination problem is FPT with respect to the neighborhood diversity of the input graph.

We denote by $\mathtt{nd}$ the neighborhood diversity of the input graph $G = H(G_1, \dots, G_{\mathtt{nd}})$.

Theorem 4. *The (t, r)-*Broadcast Domination *problem is solvable in time $O(\mathtt{nd}^{5\mathtt{nd}+o(\mathtt{nd})} \log D)$ where $D = \max\{t, r, |V(G)|\}$.*

Proof. Denote by V_i the vertex set of G_i. Let S be a (t, r)-Broadcast Dominating set of G. For each $i = 1, \dots, \mathtt{nd}$, define $S_i = S \cap V_i$.
In order to give an algorithm solving the (t, r)-Broadcast Domination problem for G, we characterize (7) and (8) for G. Let $v \in V_i$ with $1 \leq i \leq \mathtt{nd}$. If G_i is a clique then

$$|S_i \cap N_{G_i}(v)| = \begin{cases} |S_i| & \text{if } v \notin S_i, \\ |S_i| - 1 & \text{otherwise.} \end{cases}$$

By (7) and (8) we have

$$(t-1)\ |S_i| + \sum_{\substack{j | j \neq i \\ d_H(i,j) \leq t}} |S_j|(t - d_H(i,j)) \geq r \qquad \text{if } v \notin S_i,$$

$$(t-1)\ |S_i| + 1 + \sum_{\substack{j | j \neq i \\ d_H(i,j) \leq t}} |S_j|(t - d_H(i,j)) \geq r \qquad \text{if } v \in S_i.$$

If G_i is an independent set then $|S_i \cap N_{G_i}(v)| = 0$ and by (7) and (8) we have

$$(t-2)\ |S_i| + \sum_{\substack{j | j \neq i \\ d_H(i,j) \leq t}} |S_j|(t - d_H(i,j)) \geq r \qquad \text{if } v \notin S_i$$

$$(t-2)\ |S_i| + 2 + \sum_{\substack{j | j \neq i \\ d_H(i,j) \leq t}} |S_j|(t - d_H(i,j)) \geq r \qquad \text{if } v \in S_i.$$

By the above inequalities, it is easy to see that an instance $\langle G, t, r \rangle$ of (t, r)-Broadcast Domination has a solution $S = \bigcup_{1 \leq i \leq \mathtt{nd}} S_i$ with $S_i \subseteq V_i$ if and only if the following Integer Linear Programming has a solution $\mathbf{x} = (x_1, \dots, x_{\mathtt{nd}})$ such that S_i consists of any $x_i = |S_i|$ vertices of V_i. The binary variable y_i, indicates whether $S_i = V_i$ or not; in particular, $y_i \in \{0,1\}$ and by constraints (3)-(4) below, we have that if $y_i = 1$ then $x_i = |V_i|$.

$$\min \sum_{i=1,\dots,\texttt{nd}} x_i \qquad \textbf{subject to:}$$

$$(1) \quad (t-1)x_i + y_i + \sum_{\substack{j|j\neq i\\ d_H(i,j)\leq t}} x_j(t - d_H(i,j)) \geq r \quad \forall i \text{ such that } 1 \leq i \leq \texttt{nd} \text{ and } G_i \text{ is a clique}$$

$$(2) \quad (t-2)x_i + 2y_i + \sum_{\substack{j|j\neq i\\ d_H(i,j)\leq t}} x_j(t - d_H(i,j)) \geq r \quad \forall i \text{ such that } 1 \leq i \leq \texttt{nd} \text{ and } G_i \text{ is an ind. set}$$

$$(3) \quad x_i \geq |V_i| y_i \qquad \forall i = 1,\dots,\texttt{nd}$$

$$(4) \quad x_i \leq |V_i| \qquad \forall i = 1,\dots,\texttt{nd}$$

$$(5) \quad y_i \in \{0,1\} \qquad \forall i = 1,\dots,\texttt{nd}$$

To evaluate the time to solve the above ILP, we use a well-known result, stated in [19, 29]: Any ℓ-Variable Integer Linear Programming Feasibility can be solved in time $O(\ell^{2.5\ell+o(\ell)} \cdot L)$ where L is the number of bits in the input. [19, 29], where

> **ℓ-Variable Integer Linear Programming Feasibility**
> **Instance:** A matrix $A \in \mathbb{Z}^{r\times\ell}$ and a vector $b \in \mathbb{Z}^r$.
> **Question:** Is there a vector $x \in \mathbb{Z}^\ell$ such that $Ax \geq b$?

Hence, observing that the considered ILP uses at most 2 nd variables and that the coefficients are upper bounded by $D = \max\{t, r, |V(G)|\}$, we have that it can be solved within time $O(\texttt{nd}^{5\texttt{nd}+o(\texttt{nd})} \log D)$. □

5.2. Iterated type partition number

Given a graph G, the *iterated type partition* number of G, introduced in [10], is defined by iteratively contracting *clique* and *independent set* subgraphs having the same neighborhood until a *prime* graph is obtained; a graph is called *prime* if no more contractions are possible. The iterated type partition number, denoted $\texttt{itp}(G)$, is the number of vertices of the obtained prime graph. It can be shown that the vertices of the obtained *prime graph* represent subgraphs that are *cographs*. An example of a graph G with $\texttt{itp}(G) = 5$ and its iterative identification is given in Figure 1.

Trivially, for each graph G we have $\texttt{itp}(G) \leq \texttt{nd}(G)$. See also [10] for a discussion on the relations with other graph parameters.

In this paper, we will use the following equivalent definition based on the operations (O1)-(O2).

Definition 7. *A graph G has* iterated type partition *number $\ell \geq 1$ if ℓ is the minimum integer such that $G = H(G_1, \dots, G_\ell)$ for cographs $G_1, G_2, \dots, G_\ell$ and an outline graph H.*

We denote by itp the iterated type partition number of the input graph G.

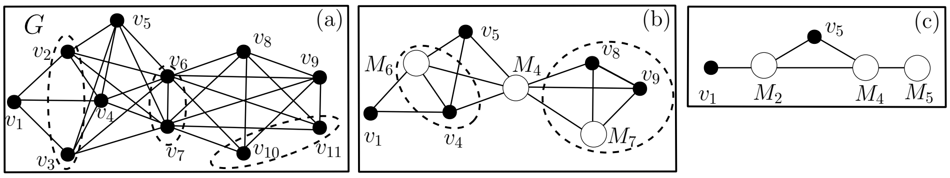

Figure 1: (a)-(c) A graph G with *iterated type partition* number 5 and its iterative identification. Dashed circles describe the identified *clique* or *independent set* subgraphs sharing the same neighborhood.

Observation 3. *We notice that if $G = H(G_1, \ldots, G_{\texttt{itp}})$ is connected then for each $u, v \in V(G_i)$ the distance between u and v in G_i is at most 2, for any $i = 1, \ldots, \texttt{itp}$. Indeed, if G_i is connected, then it is a connected cograph and has a diameter at most 2. If G_i is not connected, there exists j such that (i, j) is an edge in H. Hence, u and v have common neighbors in $V(G_j)$. Hence, as in Observation 2, the use of the values in Definition 4 is correct.*

Algorithm for bounded `itp` and solution size β

Using the strategy adopted in Section 4, we are able to compute in time $O(nr^2 + m)$ the values $\gamma_t(G_i, r')$ as well as the corresponding solutions denoted $\Gamma_{G_i,r'}$, for each cograph G_i with $i = 1, \ldots, \texttt{itp}$ and $r' = 1, \ldots, r$

Definition 8. *Given a cograph $G = (V, E)$ and two integers $t \geq 2$, $r \geq 1$ we denote by $\beta_t(G, b)$ the value of the largest demand r' such that there exists a set S of size at most b ($|S| \leq b$) and S is a (t, r')-Broadcast Dominating set of G, where the distance function is redefined as $d'(u, v) = \min(2, d_G(u, v))$, for each $u, v \in V$.*

Using the values $\gamma_t(G_i, r')$ and the solutions $\Gamma_{G_i,r'}$, for each cograph G_i with $i = 1, \ldots, \texttt{itp}$ and each $r' = 1, \ldots, r$ we are able to compute in time $O(nr)$ the values $\beta_t(G_i, b)$ for $i = 1, \ldots, \texttt{itp}$ and $b = 1, 2, \ldots, |V(G_i)|$, as well as the corresponding solutions denoted $\mathrm{B}_{G_i,b}$.

Theorem 5. *(t, r)-Broadcast Domination is solvable in time $O(\texttt{itp}(\beta + 1)^{\texttt{itp}+1} + nr^2 + m)$.*

Proof. Algorithm ITP-β starting from $b = 1$ increases the budget until a solution is identified. We recall that we are assuming that for each $v \in V$ we have $r \leq \sum_{u \in N_t(v)} (t - d(u, v))$, hence the problem does admit a solution.

Fixed a budget b, the algorithm considers all the possible vectors $\mathbf{s} = (s_1, \ldots, s_{\texttt{itp}})$ where $b = \sum_{i=1}^{\texttt{itp}} s_i$ and $0 \leq s_i \leq \min(b, |V(G_i)|)$. For each vector, the algorithm evaluates whether there exists a solution S such that $|S_i| = s_i$ where $S_i = S \cap V(G_i)$, for each $1 \leq i \leq \texttt{itp}$. Specifically, it computes the values r'_i, corresponding to the residual demand on G_i considering the contribution of nodes in $S \setminus S_i$. Such contribution depends only on the sizes s_j of each S_j with $j \neq i$. Then the values $\beta_t(\cdot, \cdot)$ are exploited to check whether each component G_i is able to reach the residual demand r'_i, using the assigned budget s_i. If this is the case, the solution set is determined in line 4 and returned by the algorithm.

For each $i = 1, \ldots, \texttt{itp}$ and $v \in V(G_i)$ we have

$$\beta_t(G_i, s_i) = \sum_{u \in \mathrm{B}_{G_i,s_i} \cap N_{G,t}(v)} (t - d(u, v)) \geq r'_i = r - \sum_{\substack{j | j \neq i \\ d_H(i,j) \leq t}} s_j(t - d_H(i, j)),$$

Algorithm 1: ITP-$\beta(G = H(G_1, \ldots, G_{\mathtt{itp}}), t, r, \beta_t(\cdot,\cdot), \mathrm{B}_{\cdot,\cdot})$

Input: A graph $G = H(G_1, \ldots, G_{\mathtt{itp}})$, a radius t, a demand r, and values $\beta_t(G_i, b)$ for each $i = 1, \ldots, \mathtt{itp}$ and $b = 1, \ldots, |V(G_i)|$, with their associated solutions $\mathrm{B}_{G_i,b}$.

Output: S a solution for the (t, r)-Broadcast Domination problem for G.

```
for b = 1 to β do
    for each s = (s_1, ..., s_itp) | Σ_{i=1}^{itp} s_i = b and 0 ≤ s_i ≤ min(b, |V(G_i)|) do
        for i = 1, ..., itp do  r'_i = r − Σ_{j|j≠i, d_H(i,j)≤t} s_j (t − d_H(i,j))
        if ⋀_{i=1}^{itp} (β_t(G_i, s_i) ≥ r'_i) then return S = ⋃_{i=1}^{itp} B_{G_i,s_i}
```

and consequently

$$\begin{aligned}\sum_{u \in S \cap N_{G,t}(v)} (t - d(u,v)) &= \sum_{u \in S_i \cap N_{G,t}(v)} (t - d(u,v)) + \sum_{j \neq i} \left(\sum_{u \in S_j \cap N_{G,t}(v)} (t - d(u,v)) \right) \\ &= \sum_{u \in \mathrm{B}_{G_i,s_i} \cap N_{G,t}(v)} (t - d(u,v)) + \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} s_j (t - d_H(i,j)) \geq r.\end{aligned}$$

Finally, we notice that Algorithm ITP-β requires time $O(\mathtt{itp}\ (\beta+1)^{\mathtt{itp}})$. Moreover, the time to obtain all the values $\beta_t(G_i, b)$ and the corresponding solutions $\mathrm{B}_{G_i,b}$ for $i = 1, \ldots, \mathtt{itp}$ and $b = 1, 2, \ldots, |V(G_i)|$, is $O(nr^2 + m)$. □

Algorithm for bounded `itp` and demand r

Let $G = H(G_1, \ldots, G_{\mathtt{itp}})$. In this section, we design an FPT algorithm to solve the (t, r)-Broadcast Domination problem for G parameterized by `itp` and the demand r. The algorithm exploits the values $\gamma_t(G_i, r')$ that can be obtained using the strategy adopted in Section 4, for each cograph G_i with $1 \leq i \leq \mathtt{itp}$ and $r' = 1, \ldots, r$. We recall that $\gamma_t(G, 0) = 0$, for each cograph G.

Theorem 6. *(t, r)-Broadcast Domination is solvable in time $O(\mathtt{itp}\ (r+1)^{\mathtt{itp}} + nr^2 + m)$.*

Proof. Algorithm ITP-r considers all the possible vectors $\mathbf{r} = (r_1, \ldots, r_{\mathtt{itp}})$, where $r_i = 0, \ldots, r$ and $i = 1, \ldots, \mathtt{itp}$, and for each of them verifies if each value r_i satisfies

$$r_i \geq r - \sum_{\substack{j|j\neq i \\ d_H(i,j)\leq t}} \gamma_t(G_j, r_j)(t - d_H(i,j)). \tag{9}$$

Assuming that (9) holds for each r_i in $\mathbf{r}$, select the vertex set $S(\mathbf{r}) = \bigcup_{1\leq i\leq \mathtt{itp}} \Gamma_{G_i,r_i}$, where $\Gamma_{G_i,r_i} \subseteq V(G_i)$ is obtained by using the strategy in Section 4 with $|\Gamma_{G_i,r_i}| = \gamma_t(G_i, r_i)$. For $v \in V(G_i)$ and $i = 1, \ldots, \mathtt{itp}$, we have

$$\sum_{u \in \Gamma_{G_i,r_i} \cap N_{G,t}(v)} (t - d(u,v)) \geq r_i. \tag{10}$$

Algorithm 2: ITP-$r(G = H(G_1, \dots, G_{\mathtt{itp}}), t, r, \gamma_t(\cdot,\cdot))$

Input: A graph $G = H(G_1, \dots, G_{\mathtt{itp}})$, a radius t, a demand r, and values $\gamma_t(G_i, r')$ for each $i = 1, \dots, \mathtt{itp}$ and $r' = 1, \dots, r$.

Output: The vector of demands $\mathbf{r}_s$.

1 $s = \infty$ and $\mathbf{r}_s = (0, \dots, 0)$

2 **for *each*** $\mathbf{r} = (r_1, \dots, r_{\mathtt{itp}})$ such that $0 \leq r_i \leq r,\ 1 \leq, i \leq \mathtt{itp}$ **do**

3 **if** $\bigwedge_{i=1}^{\mathtt{itp}} \left(r_i \geq r - \sum_{\substack{j \mid j \neq i \\ d_H(i,j) \leq t}} \gamma_t(G_j, r_j)(t - d_H(i,j)) \right)$ **then**

4 **if** $s > \sum_{1 \leq i \leq \mathtt{itp}} \gamma_t(G_i, r_i)$ **then** $s = \sum_{1 \leq i \leq \mathtt{itp}} \gamma_t(G_i, r_i)$ and $\mathbf{r}_s = \mathbf{r}$

5 **return** $\mathbf{r}_s$;

By (9) and (10), it holds

$$\sum_{u \in S(\mathbf{r}) \cap N_{G,t}(v)} (t - d(u,v)) = \sum_{u \in \Gamma_{G_i, r_i} \cap N_{G,t}(v)} (t - d(u,v)) + \sum_{j \neq i} \left(\sum_{u \in \Gamma_{G_j, r_j} \cap N_{G,t}(v)} (t - d(u,v)) \right)$$

$$\geq r_i + \sum_{j \neq i} \left(\sum_{u \in \Gamma_{G_j, r_j} \cap N_{G,t}(v)} (t - d(u,v)) \right) = r_i + \sum_{\substack{j \mid j \neq i \\ d_H(i,j) \leq t}} \gamma_t(G_j, r_j)(t - d_H(i,j)) \geq r.$$

Hence, the set $S(\mathbf{r}) = \bigcup_{1 \leq i \leq \mathtt{itp}} \Gamma_{G_i, r_i}$ is a (t,r)-Broadcast Dominating set of G.

Furthermore, since Algorithm ITP-r returns, among all the vectors $\mathbf{r}$ whose components r_i satisfy (9) and (10) for each $i = 1, \dots, \mathtt{itp}$, the vector $\mathbf{r}_s$ (see lines 4-5) such that

$$\mathbf{r}_s = \arg\min_{\mathbf{r}} |S(\mathbf{r})|,$$

we can reconstruct the set $S(\mathbf{r}_s)$ that is a solution for the instance $\langle G, t, r \rangle$ of the (t,r)-Broadcast Domination problem.

Finally, we notice that Algorithm ITP-r requires time $O(\mathtt{itp}\ (r+1)^{\mathtt{itp}})$. Moreover the time to obtain all the values $\gamma_t(G_i, r')$ and the solutions $\Gamma_{G_i, r'}$, for $i = 1, \dots, \mathtt{itp}$ and $r' = 1, \dots, r$, is $O(nr^2 + m)$. □

6. Discussion

The (t,r)-Broadcast Domination problem has been recently introduced and studied in some special classes of graphs (mainly grid graphs and lattices). We have initiated the study of the (t,r)-Broadcast Domination problem in general graphs. We have designed an approximation algorithm for general graphs and optimal polynomial time algorithms for cographs and graphs of bounded *Neighborhood diversity (*`nd`*)*. Moreover, we have presented FPT algorithms parameterized by *Iterated type partition number (*`itp`*)* plus the solution size $\beta = |S|$ and by `itp` plus the demand r. It eluded us the design of an FPT algorithm for the problem parameterized by `itp` only, which we leave as an open problem.

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