Computing #2-SAT of Grids, Grid-Cylinders and Grid-Tori Boolean Formulas

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Abstract

We present an adaptation of transfer matrix method for signed grids, grid-cylinders and grid-tori. We use this adaptation to count the number of satisfying assignments of Boolean Formulas in 2-CNF whose corresponding associated graph has such grid topologies.

1 Introduction

The transfer matrix method is a general technique which has been used to find exact solutions for a great variety of problems. In particular, have been developed techniques, based on this method, to count structures in a grid graph $G_{n,m}$, e.g., spanning trees, Hamiltonian cycles, independent sets, acyclic orientations, k-coloring, and so on [1, 2, 7, 9]. In the case of others grid topologies, as grid-cylinders and grid-tori, there exists little work done on counting structures. In [9] the transfer matrix technique is used, with some modifications, to count structures in fixed height grid-cylinders and tori. In the case of counting satisfying assignments of Boolean formulas with this type of grid topologies, the work is null as far as we know.

In almost all cases of counting structures in grid graphs, the technique used follows a transfer matrix formulation. For example, Calkin and Wilf [2] used this method for computing the number $I(G_{n,m})$ of independent sets of a grid graph $G_{n,m}$ and Golin in [9] count the same number (and others structures) but in grid-cylinders and grid-tori.

The number of independent sets in a grid graph, problem denoted as $I(G_{m,n})$, is closely related to the "hard-square model" used in statistical physics and, of particular interest is the so-called "hard-square entropy constant" defined as $\lim_{m,n\to\infty} I(G_{m,n})^{1/m\cdot n}$ [1]. Applications also include for instance tiling and efficient coding schemes in data storage [12].

It is well known that the number of satisfying assignments (models) of a monotone formula F in two conjunctive normal form 2-CNF, which

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is a propositional formula formed by a conjunction of disjunctions of two nonnegative literals, is related with the number of independent sets of the constrained undirected graph of the formula [11, 9]. The number of models of a Boolean formula F is denoted as $\#\mathrm{SAT}(F)$ and the computation of $\#\mathrm{SAT}(F)$ for formulas in 2-CNF, denoted as #2-SAT, is a classic #P-complete problem.

There is a significant amount of works on the design of algorithms for solving #SAT, #2-SAT and #3-SAT [6, 16, 10, 3, 4, 8, 5, 15]. Most of them are based on branch-and-bound techniques, for example, applying the recursive decomposition of the input formula based on the classical Davis and Putnam division rule [8, 4].

Regarding to #2-SAT problem, considering formulas with n variables, the better time bounds than the trivial $O(2^n)$ have been achieved in the works of Dahllöf et al. [4], F$\ddot{u}$rer [8] and Wahlstr$\ddot{o}$n [15]. Wahlstr$\ddot{o}$n uses a refinement of the method of analysis, where is extended the concept of compound measures to multivariate measures in which a leading running time of $O(1.2377^n)$ has obtained, for weighted formulas in 2-CNF.

An important line of research is related to the determination of the constraints on the 2-CF formulas which allow us to compute #2-SAT in polynomial time. In this address, there are few general results, one of them is due to Vadhan [14] who showed that #2-SAT is solved in polynomial time for monotone 2-CNF where all variables appear twice at the most. Roth [11] generalizes the previous results for non only the monotone case, but continuing to consider two ocurrence per variable at the most. In this paper, we extend the class of formulas in 2-CNF in which, counting the number of satisfying assignments can be done in polynomial time.

On the other hand, Bubbley has shown that #2μ-SAT (conjunction of clauses without bound in its length and where each variable may appear at most twice) is a #P-Complete problem [13].

In order to extend the transfer matrix method for considering any kind of 2-CNF's we have to deal with grid graphs with signed edges. In the case of counting models of Boolean formulas with this type of grid topologies, the work is null as far as we know. In this article, we adapt the transfer matrix method considering three classes of grid topologies: grid graphs, grid-cylinders and grid-tori obtained from 2-CNF's not restricted to the monotone case, and we show how to compute the number of models for these classes of formulas. The complexity of our method when counting models in structures of fixed height is polynomial.

2 Preliminaries

For k and l integers such that $k < l$, we denote the set $\{k, k+1, ..., l\}$ by $[k, l]$. The Euclidean distance between points u and v in Euclidean 2-space

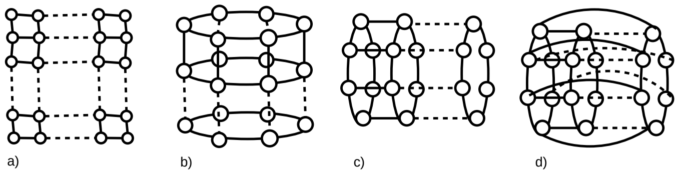

Figure 1: a) Grid, b), c) grid-cylinders and d) grid-tori.

is denoted by $d(u, v)$.

A *grid graph* of size $m \times n$ is a graph $G_{n,m}$ with vertex set $V(n, m) = [0, n] \times [0, m]$ and edge set $E(n, m) = \{(u, v) \in V^2(n, m) : d(u, v) = 1\}$. Let $E_1(n, m) = (\{0\} \times [0, m]) \times (\{n\} \times [0, m])$ and $E_2(n, m) = ([0, n] \times \{0\}) \times ([0, n] \times \{m\})$ be two sets of edges.

A *grid-cylinder* of size $m \times n$ is a graph $C(n, m)$ with vertex set $V(n, m)$ and edge set $EC(n, m) = E(n, m) \cup E'(n, m)$, where $E'(n, m) \in \{E_1(n, m), E_2(n, m)\}$ (see figures 1b and 1c). A *grid-tori* of size $m \times n$ is a graph $T(n, m)$ with the same vertex set $V(n, m)$ but its edge set is $ET(n, m) = E(n, m) \cup E_1(n, m) \cup E_2(n, m)$ (see figure 1d).

A set $I \subseteq V$ is called an independent set if no two of its elements are joined by an edge. We describe the method used by Calkin as follows.

Let $I(G_{n,m})$ be the number of independent sets of $G_{n,m}$, and let $\mathcal{C}_m$ be the set of all $(m + 1)$-vectors $\mathbf{v}$ of $0's$ and $1's$ without two consecutive $1's$ (the number of these vectors is Fib_{m+2}, the $(m + 2)$-th Fibonacci number). Let T_m be an $Fib_{m+2} \times Fib_{m+2}$ symmetric matrix of $0's$ and $1's$ whose rows and columns are indexed by the vectors of $\mathcal{C}_m$. The entry of T_m in position $(\mathbf{u}, \mathbf{v})$ is 1 if the vectors $\mathbf{u}, \mathbf{v}$ are orthogonal, and is 0 otherwise, T_m is called the transfer matrix for $G_{n,m}$. Then, $I(G_{n,m})$ is the sum of all entries of the n-th power matrix T_m^n, i.e., $I(G_{n,m}) = \mathbf{1}^t T_m^n \mathbf{1}$, where $\mathbf{1}$ is the (Fib_{m+2})-vector whose entries are all $1's$. For example, if $m = 2$ and $n = 3$ we have that $\mathcal{C}_2 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1)\}$,

$$T_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad T_2^3 = \begin{pmatrix} 17 & 12 & 13 & 12 & 9 \\ 12 & 7 & 10 & 8 & 5 \\ 13 & 10 & 9 & 10 & 8 \\ 12 & 8 & 10 & 7 & 5 \\ 9 & 5 & 8 & 5 & 3 \end{pmatrix},$$

therefore $I(G(2, 3)) = \mathbf{1} T_2^3 \mathbf{1} = 227$.

A Boolean formula F is called *monotone* when each literal appearing in F occurs with just one of its signs. Given a monotone Boolean formula F in 2-conjunctive normal form (2-CNF), we can associate an undirected graph $G_F = (V, E)$, called its constrained graph, where V is the set of variables of F and two vertices of V are connected by an edge in E if they belong to the same clause of F. Conversely, given an undirected graph $G = (V, E)$, we can associate a monotone 2-CNF formula F_G with variables V, and where $F_G = \bigwedge_{(u,v)\in E}(u \vee v)$. We say that a 2-$CNF$ F is a *cycle*, *path*, *tree*, *grid*, *grid-cylinder* or a *grid-tori* formula if its constrained graph is a cycle, path, tree, grid, grid-cylinder or a grid-tori, respectively.

3 Extending the Transfer Matrix Method

In order to extend the transfer matrix method for considering any kind of 2-CNF's we have to deal with grid graphs with signed edges. In this case, the associated graph of a formula F is a graph $G_F = (V, E)$ with labels on the edges, where V is the set of variables appearing in F, and a clause $(l \vee l')$ of F determines an ordered pair (s_1, s_2) of signs assigned as the labels of the edge connecting the variables appearing in l and l'. The signs s_1 and s_2 are related to the signs of the literals l and l' respectively. For example, the clause $(\neg x \vee y)$ determines the labelled edge: "$x \overset{-\ +}{\text{——}} y$" which is equivalent to the edge "$y \overset{+\ -}{\text{——}} x$".

Some authors had considered the signs of the literals in the clauses of a 2-CNF F by using orientation of the edge corresponding to the clause [12, 13], and then the problem of counting models of F is seen as counting the number of orientations in its respective constrained graph, which has no sink.

A graph with labelled edges on a set A is a triplet $G = (V, E, \psi)$, where (V, E) is a graph, and ψ is a function with domain E and range A. The valuation $\psi(e)$ is called the label of the edge $e \in E$.

We denote $S = \{+, -\}$, $\bar{S} = \{\pm, \mp\}$ and $\hat{S} = S \cup \bar{S}$. Let $G_{n,m} = (V, E, \psi)$ be a grid graph with labelled edges on S^2. Let x and y be nodes in V. If $e = \{x, y\}$ is an edge and $\psi(e) = (s, s')$, then s (s') is called the *adjacent sign* to x (y), see figure 2.

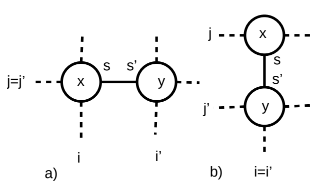

Fig. 2: Adjacent sign to $x(y)$

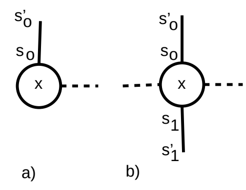

Fig. 3: Incident edges on the node x

Let $e = ((i, j), (i', j'))$ be an edge of a grid graph $G_{n,m}$, if $i = i'$ and $j \neq j'$, e is called a column-edge (see figure 2b), and if $i \neq i'$ and $j = j'$, e is called a row-edge (see figure 2a).

If x is a node of $G_{n,m}$, then either x has one incident column-edge, or x has two incident column-edges. If x has one incident column-edge e_0 whose label is (s_0, s'_0), then we define $sgn_c(x) = s_0$, where s_0 is the adjacent sign to x (see figure 3a).

If x has two incident column-edges e_0 and e_1 with labels (s_0, s_0') and (s_1, s_1') respectively (see figure 3b), we define $sgn_{cc} : V \to \hat{S}$ as follows

$$sgn_{cc}(x) = \begin{cases} + & if \quad (s_0, s_1) = (+,+), \\ - & if \quad (s_0, s_1) = (-,-), \\ \pm & if \quad (s_0, s_1) = (+,-), \\ \mp & if \quad (s_0, s_1) = (-,+). \end{cases}$$

In general, we can consider the function $sgn : V \to \hat{S}$ as $sgn(x) = sgn_c(x)$ if x has one incident column-edge or $sgn(x) = sgn_{cc}(x)$ if x has two incident column-edges.

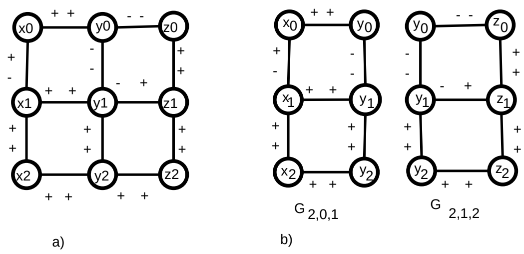

Fig. 4: a) Grid $G_{2,2}$, b) Subgrids $G_{2,0,1}$, $G_{2,1,2}$

Given $G_{n,m} = (V, E, \psi)$ a grid graph with labelled edges on S^2, we consider for $k = 0, ..., n-1$ the sub-grid graph with labelled edges $G_{m,k,k+1} = (V_k, E_k, \psi_k)$, where $V_k = V \cap ([k, k+1] \times [0, m])$, $E_k = \{(u, v) \in V_k^2 : d(u, v) = 1\}$ and $\psi_k = \psi \mid_{E_k}$ *the restriction of* ψ *to* E_k.

Notice that $G_{m,k,k+1}$ specifies a grid of two columns and $m+1$ rows. If x is a node in $G_{m,k,k+1}$, then x has only one incident row-edge e. For $k = 0, \ldots, n-1$ we define $sgn_k : V_k \to S$ as $sgn_k(x) = s$, where s is the adjacent sign of x on the incident row-edge e.

For example, let $G_{2,2}$ be the grid graph illustrated in figure 4a. Then $sgn(x) = +$, for $x \in \{x_0, x_2, y_2, z_0, z_1, z_2\}$, $sgn(y_0) = -$ and $sgn(y_1) = sgn(x_1) = \mp$. In $G_{2,0,1}$, we have $sgn_0(x) = +$ for all $x \in V_0$. In $G_{2,1,2}$, $sgn_1(x) = +$ for $x \in \{y_2, z_1, z_2\}$ and $sgn_1(x) = -$ for $x \in \{y_0, y_1, z_0\}$ (see figure 4b).

Given a vector $\mathbf{v} = (v_0, v_1, \ldots, v_m) \in \{0,1\}^{m+1}$ and a string $\mathbf{s} = s_0 \ldots s_m$ of signs in $\hat{S}$, for $m \geq 0$, we define the family of operators $\varphi_{\mathbf{s}} : \{0,1\}^{m+1} \to \{0,1\}^p$, $(m+1 \leq p \leq 2m+2)$ as $\varphi_{\mathbf{s}}(\mathbf{v}) = (s_0 v_0, \ldots, s_m v_m)$, where

$$s_j v_j = \begin{cases} v_j & if \quad s_j = +, \\ \bar{v_j} & if \quad s_j = -, \\ (v_j, \bar{v_j}) & if \quad s_j = \pm, \\ (\bar{v_j}, v_j) & if \quad s_j = \mp. \end{cases} \tag{1}$$

for $j = 0, \ldots, m$. For $v \in \{0,1\}$, $\bar{v}$ denotes $1 - v$ and $\bar{\mathbf{v}}$ denotes $(\bar{v}_0, ..., \bar{v}_m)$. For instance,

$$\varphi_{+,-,\pm,\mp,-}(1,0,1,1,0) = (+1, -0, \pm 1, \mp 1, -0) = (1, 1, (1,0), (0,1), 1).$$

In general, we can omit the internal parenthesis given the associative property of the cartesian product. In particular, the vector $(1,1,(1,0),(0,1),1)$ can be seen as $(1,1,1,0,0,1,1)$.

Let $\mathcal{F}_m$ be the set of all $(m+1)$-vectors $\mathbf{v}$ of $0's$ and $1's$, and let $\mathcal{C}_m \subset \mathcal{F}_m$ be the set of all $(m+1)$-vectors $\mathbf{v}$ of $0's$ and $1's$, such that $\mathbf{v}$ does not have two consecutive $1's$. The cardinality of $\mathcal{C}_m$ (denoted by $|\mathcal{C}_m|$) is Fib_{m+2} (the $(m+2)$-th Fibonacci number), while $|\mathcal{F}_m| = 2^{m+1}$. Given $\mathbf{s} = s_0 s_1 \cdots s_m$ a string of signs in $\hat{S}$, we define $\mathcal{F}_m^{\mathbf{s}} = \{e \in \mathcal{F}_m : \varphi_{\mathbf{s}}(e) \in \mathcal{C}_{m+\ell}\}$, where $\ell = |\{s \in \{s_0, ..., s_m\} : s \in \bar{S}\}|$.

Remark 1. Notice that $\mathcal{C}_m \subseteq \mathcal{F}_m$ and that the equality holds when $s_i = +$ for all $i = 0, ..., m$. Furthermore, if there exists $i \in \{0, ..., m\}$ such that $s_i \in \hat{S}$, then $|\mathcal{F}_m^{\mathbf{s}}| < |\mathcal{C}_m|$.

Let $G_{n,m}$ be a grid graph of size $m \times n$ with labelled edges on the set S^2, we assume that $x_0^k, \ldots, x_m^k$ and $x_0^{k+1}, \ldots, x_m^{k+1}$ are the nodes of the $k-th$ and $(k+1)-th$ columns respectively of $G_{n,m}$, $0 \le k < n$ (or columns 0 and 1 of $G_{m,k,k+1}$ respectively).

For $j = k, k+1$, let $\mathbf{s}^j = s_0^j s_1^j \cdots s_m^j$ and $\tau^j = \tau_0^j \tau_1^j \cdots \tau_m^j$ be two string of signs, such that $s_i^j = sgn(x_i^j)$ and $\tau_i^j = sgn_k(x_i^j)$ for $i = 0, \cdots, m$. Following the idea proposed in [2], we define a matrix $T_k = T_{m,k}$, the transfer matrix from column k to the column $k+1$ of $G_{n,m}$ as follows. T_k is an $\mid \mathcal{F}_m^{\mathbf{s}^{k+1}} \mid \times \mid \mathcal{F}_m^{\mathbf{s}^k} \mid$ matrix of $0's$ and $1's$ whose rows and columns are indexed by vectors $(\mathbf{v}, \mathbf{u})$ of $\mathcal{F}_m^{\mathbf{s}^{k+1}} \times \mathcal{F}_m^{\mathbf{s}^k}$. The entry of T_k in position $(\mathbf{v}, \mathbf{u})$ is 1 if the vectors $\varphi_{\tau^k}(\mathbf{u})$ and $\varphi_{\tau^{k+1}}(\mathbf{v})$ are orthogonal, and is 0 otherwise.

Notice that if s_i^j and τ_i^j are positive signs for $i = 0, \cdots, m$, $j = k, k+1$, then T_k is the transfer matrix used in the classic transfer method [2].

For example, if $G_{2,2}$ is the grid graph with labelled edges as it is illustrated in figure 4. For $G_{2,0,1}$, we have that $\mathbf{s}^0 = + \mp +$, $\mathbf{s}^1 = - \mp +$ and $\tau^0 = \tau^1 = + + +$, then $\mathcal{F}_2^{+\mp+} = \{\mathbf{u}_1, \cdots, \mathbf{u}_4\}$ and $\mathcal{F}_2^{-\mp+} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where $\mathbf{u}_1 = (0,0,0)$, $\mathbf{u}_2 = (0,1,0)$, $\mathbf{u}_3 = (0,0,1)$, $\mathbf{u}_4 = (1,1,0)$, $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (0,1,0)$, $\mathbf{v}_3 = (1,0,1)$ and $\mathbf{v}_4 = (1,1,0)$. The transfer matrix $T_0 = (a_{ij})_{4\times4}$, is a 4×4 matrix determined, for $1 \le i, j \le 4$, as $a_{ij} = 1$, if $\varphi_{\tau^1}(\mathbf{v_i}) \cdot \varphi_{\tau^0}(\mathbf{u_j}) = 0$ and $a_{ij} = 0$ otherwise. Since $\tau^0 = \tau^1 = + + +$, we have $\varphi_{\tau^1}(\mathbf{v_i}) = \mathbf{v_i}$ and $\varphi_{\tau^0}(\mathbf{u_j}) = \mathbf{u_j}$. Then

$$T_0 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

For $G_{2,1,2}$ that is also depicted in figure 4, we have $\mathbf{s}^1 = - \mp +$, $\mathbf{s}^2 = + + +$, $\tau^1 = - - +$ and $\tau^2 = - + +$, then $\mathcal{F}_2^{-\mp+} = \{\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_4\}$ and $\mathcal{F}_2^{+++} = \{\boldsymbol{\nu}_1, ..., \boldsymbol{\nu}_5\}$, where $\boldsymbol{\mu}_1$=(1,0,0), $\boldsymbol{\mu}_2$=(0,1,0), $\boldsymbol{\mu}_3$=(1,0,1), $\boldsymbol{\mu}_4$=(1,1,0),

$\boldsymbol{\nu}_1$=(0,0,0), $\boldsymbol{\nu}_2$=(1,0,0), $\boldsymbol{\nu}_3$=(0,1,0), $\boldsymbol{\nu}_4$=(0,0,1) and $\boldsymbol{\nu}_5$=(1,0,1). Then,

$$\varphi_{--+}(\mathcal{F}_2^{-\mp+}) = \{(0,1,0),(1,0,0),(0,1,1),(0,0,0)\}$$

and $\varphi_{-++}(\mathcal{F}_2^{-\mp+}) = \{(1,0,0),(0,0,0),(1,1,0),(1,0,1),(0,0,1)\}$.

The transfer matrix $T_1 = (b_{ij})_{5\times 4}$, is such that, for $1 \leq i \leq 5$ and $1 \leq j \leq 4$, $b_{ij} = 1$, if $\varphi_{-++}(\boldsymbol{\nu}_i) \cdot \varphi_{--+}(\boldsymbol{\mu}_j) = 0$ and $b_{ij} = 0$ otherwise. Then

$$T_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{3}$$

In the case, not necessarily monotone, of a formula F having a constrained grid graph $G_{n,m}$ with labelled edges on S^2 and transfer matrices $T_0, \ldots, T_{n-1}$, we conclude that the sum of all entries of the product matrix $T_{n-1} \cdots T_0$ is the number of satisfying assignment of F. This fact is expressed in the following theorem.

Theorem 1. *Let F be a grid formula such that its constrained graph is $G_{n,m}$ $(1 \leq n)$ with labelled edges on S^2, then $\#SAT(F)$ is given by the sum of all entries of the product matrix $T_{n-1} \cdots T_0$, where T_k is the transfer matrix of the two consecutive columns: k and $k+1$ of $G_{n,m}$, $k = 0, ..., n-1$.*

Before detailing the proof, we consider the following example and observations.

Example 1. *Let $F = (x_0 \vee y_0) \wedge (\neg y_0 \vee \neg z_0) \wedge (z_0 \vee z_1) \wedge (z_1 \vee z_2) \wedge (z_2 \vee y_2) \wedge (y_2 \vee x_2) \wedge (x_2 \vee x_1) \wedge (\neg x_1 \vee x_0) \wedge (x_1 \vee y_1) \wedge (\neg y_1 \vee z_1) \wedge (\neg y_1 \vee \neg y_0) \wedge (y_1 \vee y_2)$. The constrained graph of F is the grid graph $G_{2,2}$ with labelled edges depicted in Figure 3. Then, from last example, T_0 and T_1 are the transfer matrices given in (2) and (3) respectively. Now, we have that the product matrix $T_1 T_0$ is the following*

$$T_1 T_0 = \begin{pmatrix} 3 & 2 & 2 & 0 \\ 4 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 3 & 0 \end{pmatrix}$$

therefore, $\#SAT(F) = 30$.

If $F_{n,m}$ denotes a grid formula having as constrained graph a grid $G_{n,m}$, for $n > 0$, we can write

$$F_{n,m} = (\bigwedge_{i=0}^{n} C_i) \wedge (\bigwedge_{\ell=0}^{n-1} R_\ell) \tag{4}$$

where

$$C_i = \bigwedge_{k=0}^{m-1} (\eta^i_{2k} x^i_k \vee \eta^i_{2k+1} x^i_{k+1}) \tag{5}$$

$\eta^i_q \in S$ for $q = 0, ..., 2m-1$,

$$R_\ell = \bigwedge_{j=0}^{m} (\tau^{2\ell}_j x^\ell_j \vee \tau^{2\ell+1}_j x^{\ell+1}_j) \tag{6}$$

$\tau^r_j \in S$ for $j = 0, ..., m$, $r \in \{2\ell, 2\ell+1\}$. Here, the formulas C_i and R_ℓ are called $column\text{-}formula$ and $row\text{-}formula$ respectively.

Notice that for $n, m > 0$

$$F_{n,m} = F_{n,m-1} \wedge C_n \wedge R_{n-1}, \; F_{m,0} = C_0, \; F_{0,n} = R_0. \tag{7}$$

For $i = 0, ..., n-1$, we define

$$F_{m,i,i+1} = C_i \wedge C_{i+1} \wedge R_i \tag{8}$$

Note that

$$F_{n,m} = \bigwedge_{i=0}^{n-1} F_{m,i,i+1} \tag{9}$$

If $\phi : \{x^i_0, \ldots, x^i_m\} \to \{0,1\}$ is an assignment of values for the variables of C_i (partial assignments of the variables of $F_{n,m}$), this is denoted by the $(m+1)$-vector $(\phi(x^i_0), ..., \phi(x^i_m))$. That is, an assignment for the variables of C_i can be seen as a vector in $\{0,1\}^{m+1}$. Observe that, the assignments of the variables of $F_{n,m}$ can be considered as a matrix of n columns formed by the assignments for the variables of $C_0, ..., C_n$.

For $i = 0, ..., n$, let $\xi^i_0 = \eta^i_0$, $\xi^i_m = \eta^i_{2m-1}$ and $\xi^i_q = sgn(x^i_q)$ for $q = 1, ..., m-1$. Also, notice that for $v \in \{0,1\}$

$$\xi^i_q v = \begin{cases} \eta^i_{2q-1} v = \eta^i_{2q} v & if \quad \eta^i_{2q-1} = \eta^i_{2q}, \\ (\eta^i_{2q-1} v, \eta^i_{2q} v) & \quad otherwise. \end{cases} \tag{10}$$

To prove the theorem 1, first, we characterize the partial assignments of the variables of $F_{n,m}$ such that satisfies each column-formula C_i (lemma 1). Second, we characterize the pairs of assignments that satisfies the formula (8), i.e. satisfies two consecutive column-formulas C_i, C_{i+1} and the respective row-formula R_i (lemma 2). Finally, we prove that all matrix of partial assignments derived from the lemmas 1 and 2, satisfies the formula $F_{n,m}$.

Next, for simplicity we omit the superindex i of $v^i_j, x^i_j, \eta^i_j, \tau^i_j$ and ξ^i_j.

Lemma 1. *The vector* $\mathbf{u} \in \{0,1\}^{m+1}$ *satisfies the formula (5) iff* $\overline{\mathbf{u}} \in \mathcal{F}^{\xi_0 \cdots \xi_m}_m$.

Proof. By definition, it is clear that $\varphi_{\xi_0,...,\xi_m}(\overline{\mathbf{u}}) \in \mathcal{F}_{m+k}$. Now, if $\mathbf{u} = (u_0, ..., u_m)$ satisfies the formula (5), then $(\eta_{2\ell} u_\ell \vee \eta_{2\ell+1} u_{\ell+1}) = 1$ for all $\ell \in \{0, ..., m-1\}$, that is equivalent to $(\eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1}) \neq (1,1)$. From (10) we obtain

$$(\xi_\ell \overline{u}_\ell, \xi_{\ell+1}\overline{u}_{\ell+1}) = \begin{cases} (\eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1}) \\ if\ \eta_{2\ell-1} = \eta_{2\ell}\ and\ \eta_{2\ell+1} = \eta_{2\ell+2}, \\ \\ (\eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1}, \eta_{2\ell+2}\overline{u}_{\ell+1}) \\ if\ \eta_{2\ell-1} = \eta_{2\ell}\ and\ \eta_{2\ell+1} \neq \eta_{2\ell+2}, \\ \\ (\eta_{2\ell-1}\overline{u}_\ell, \eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1}) \\ if\ \eta_{2\ell-1} \neq \eta_{2\ell}\ and\ \eta_{2\ell+1} = \eta_{2\ell+2}, \\ \\ (\eta_{2\ell-1}\overline{u}_\ell, \eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1}, \eta_{2\ell+2}\overline{u}_{\ell+1}) \\ if\ \eta_{2\ell-1} \neq \eta_{2\ell}\ and\ \eta_{2\ell+1} \neq \eta_{2\ell+2}. \end{cases}$$

for all $\ell \in \{0, ..., m-1\}$. It is straightforward to verify that $(\xi_\ell \overline{u}_\ell, \xi_{\ell+1}\overline{u}_{\ell+1})$ does have no two consecutive 1's, for example, in the third case, the conditions $(\eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1}) \neq (1,1)$ and $\eta_{2\ell-1} \neq \eta_{2\ell}$ imply that $(\eta_{2\ell-1}\overline{u}_\ell, \eta_{2\ell}\overline{u}_\ell, \eta_{2\ell+1}\overline{u}_{\ell+1})$ does not have two consecutive $1's$. Therefore, $(\xi_0\overline{u}_0, ..., \xi_m\overline{u}_m) = \varphi_{\xi_0,...,\xi_m}(\overline{\mathbf{u}})$ does not have two consecutive $1's$, i.e. $\varphi_{\xi_0,...,\xi_m}(\overline{\mathbf{u}}) \in \mathcal{C}_{m+k}$.

Suppose that $\varphi_{\xi_0,...,\xi_m}(\overline{\mathbf{u}}) \in \mathcal{C}_{m+k}$, for $\ell = 0, ..., m$ then $(\xi_\ell \overline{u}_\ell, \xi_{\ell+1}\overline{u}_{\ell+1})$ does not have two consecutive $1's$. The vector $\mathbf{u}$ satisfies the column-formula C_i (equation (5)), otherwise, there is $\ell \in \{0, ..., m-1\}$ such that $\eta_{2\ell} u_\ell \vee \eta_{2\ell+1} u_{\ell+1} = 0$, then $\eta_{2\ell}\overline{u}_\ell = 1$ and $\eta_{2\ell+1}\overline{u}_{\ell+1} = 1$, from (10) we have $\xi_\ell\overline{u}_\ell \in \{1, (\eta_{2\ell-1}\overline{u}_\ell, 1)\}$ and $\xi_{\ell+1}\overline{u}_{\ell+1} \in \{1, (1, \eta_{2\ell+2}\overline{u}_{\ell+1})\}$. Then $(\xi_\ell \overline{u}_\ell, \xi_{\ell+1}\overline{u}_{\ell+1})$ has two consecutive $1's$. $\square$

For all $i = 0, ..., m$, we denote the strings $\xi_0^i, ..., \xi_m^i$ and $\tau_0^i, ..., \tau_m^i$ by ξ^i and τ^i respectively.

Lemma 2. *The pair* $(\mathbf{u}, \mathbf{v}) \in \{0,1\}^{2m+2}$ *satisfies* $F_{m,i,i+1}$ *iff* $(\overline{\mathbf{u}}, \overline{\mathbf{v}}) \in \mathcal{F}_m^{\xi^i} \times \mathcal{F}_m^{\xi^{i+1}}$ *and* $\varphi_{\tau^{2i}}(\overline{\mathbf{u}}) \cdot \varphi_{\tau^{2i+1}}(\overline{\mathbf{v}}) = 0$.

Proof. Suppose that $\mathbf{u} = (u_0, ..., u_m)$ and $\mathbf{v} = (v_0, ..., v_m)$ are such that $(\mathbf{u}, \mathbf{v})$ satisfies $F_{m,i,i+1}$. From lemma 1, $\overline{\mathbf{u}} \in \mathcal{F}_m^{\xi^i}$ and $\overline{\mathbf{v}} \in \mathcal{F}_m^{\xi^{i+1}}$, we must prove that $\varphi_{\tau^{2i}}(\overline{\mathbf{u}}) \cdot \varphi_{\tau^{2i+1}}(\overline{\mathbf{v}}) = 0$. By hypothesis $\tau_j^i u_j \vee \tau_j^{i+1} v_j = 1$ for all $j = 0, ..., m$, then $\tau_j^i \overline{u}_j \wedge \tau_j^{i+1}\overline{v}_j = 0$ for all $j = 0, ..., m$, therefore $\varphi_{\tau^{2i}}(\overline{\mathbf{u}}) \cdot \varphi_{\tau^{2i+1}}(\overline{\mathbf{v}}) = 0$.

If $\overline{\mathbf{u}} \in \mathcal{F}_m^{\xi^i}$ and $\overline{\mathbf{v}} \in \mathcal{F}_m^{\xi^{i+1}}$, from lemma 1, $\mathbf{u}$ satisfies C_i and $\mathbf{v}$ satisfies C_{i+1}. Now, if $\varphi_{\tau^{2i}}(\overline{\mathbf{u}}) \cdot \varphi_{\tau^{2i+1}}(\overline{\mathbf{v}}) = 0$, then $\tau_j^i \overline{u}_j \cdot \tau_j^{i+1}\overline{v}_j = 0$ for all $j = 0, ..., m$, hence $\tau_j^i u_i \vee \tau_j^{i+1} v_j = 1$ for all $j = 0, ..., m$. Therefore $(\mathbf{u}, \mathbf{v})$

satisfies the row-formula R_j (equation (6)) for $j = 0, ..., m$. $\square$

Remark 2. From previous lemma we have $\mathbf{1}^t T_i \mathbf{1} = \#SAT(F_{m,i,i+1})$, where T_i is the transfer matrix of the column i to the column $i+1$ of $G_{n,m}$ (the constrained graph of $F_{n,m}$).

Finally, we prove the theorem 1.

Proof (Theorem 1). From equation (9), it is clear that the vector $(\mathbf{u}_0, ..., \mathbf{u}_n) \in \{0,1\}^{(n+1)(m+1)}$ satisfies the formula $F_{n,m}$ iff $(\mathbf{u}_i, \mathbf{u}_{i+1})$ satisfies $F_{m,i,i+1}$ for $i = 0, ..., n-1$. By lemma 2, $(\bar{\mathbf{u}}_i, \bar{\mathbf{u}}_{i+1}) \in \mathcal{F}_m^{\xi^i} \times \mathcal{F}_m^{\xi^{i+1}}$ and $\varphi_{\tau^{2i}}(\bar{\mathbf{u}}) \cdot \varphi_{\tau^{2i+1}}(\bar{\mathbf{v}}) = 0$ for $i = 0, ..., n-1$. Let $a^i_{l_{i+1}l_i}$ be the entry of the transfer matrix T_i in the position $(\bar{\mathbf{u}}_{i+1}, \bar{\mathbf{u}}_i) \in \mathcal{F}_m^{\xi^{i+1}} \times \mathcal{F}_m^{\xi^i}$. Then, by definition of T_i and previous analysis, $(\mathbf{u}_0, ..., \mathbf{u}_n) \in \{0,1\}^{(n+1)(m+1)}$ satisfies the formula $F_{n,m}$ iff $(\bar{\mathbf{u}}_0, ..., \bar{\mathbf{u}}_n) \in \mathcal{F}_m^{\xi^0} \times \cdots \times \mathcal{F}_m^{\xi^n}$ and $a^{n-1}_{l_n l_{n-1}} \cdots a^0_{l_1 l_0} = 1$. Therefore $\#SAT(F_{n,m})$ is the cardinality of the set $\{(\bar{\mathbf{u}}_0, \cdots, \bar{\mathbf{u}}_n) \in \mathcal{F}_m^{\xi^0} \times \cdots \times \mathcal{F}_m^{\xi^n} : a^{n-1}_{l_n l_{n-1}} \cdots a^0_{l_1 l_0} = 1\}$.
Taking into account all the terms $a^{n-1}_{l_n l_{n-1}} \cdots a^0_{l_1 l_0} = 0$, we obtain

$\#SAT(F_{n,m}) = \sum_{(l_0,...,l_n) \in I_0 \times \cdots \times I_n} a^{n-1}_{l_n l_{n-1}} \cdots a^1_{l_2 l_1} \cdot a^0_{l_1 l_0} = \mathbf{1}^t T_{n-1} \cdots T_0 \mathbf{1}$,

where $I_k = \{0, ..., r_k\}$, $r_k = \mid \mathcal{F}_m^{\xi^k} \mid$ for $k = 0, ..., n$.$\square$

Remark 3. Note that $T = (T_{n-1} T_{n-2} \dots T_0) = (\alpha_{i,j})_{r_n \times r_0}$ is a $r_n \times r_0$-matrix, where $\alpha_{i,j}$ is the number of models of $F_{n,m}$ with $\bar{u}_i \in \mathcal{F}_m^{\xi_0}$ and $\bar{u}_j \in \mathcal{F}_m^{\xi_n}$ fixed.

4 Counting Models on Grid-Cylinders and Grid-Tori

In this section, we consider grid-cylinder or a grid-tori formulas. We are interested in counting models for formulas with these classes of grid topologies. For this objective, we introduce the Hadamard product "$\diamond$", which is defined for $k \times l$ matrices as follows. Let $A = (a_{i,j})_{k \times l}$ and $B = (b_{i,j})_{k \times l}$ be $k \times l$ matrices. The $k \times l$ matrix $A \diamond B = (a_{i,j} b_{i,j})$ is the Hadamard product.

Notice that a grid-cylinder $C(n,m)$ can be seen as a grid $G_{n,m} = (V(n,m), E(n,m))$ with edges from the column 0 to the column n (row 0 to the row m) of $G_{n,m}$. Then the transfer matrix T_n of the column 0 to the column n (row 0 to the row m) has sense.

Theorem 2. *Let F be a grid-cylinder formula of size $m \times n$ with graph $C(n,m) = (V(n,m), EC(n,m))$, $EC = E \cup E_1$. Then $\#SAT(F) = \mathbf{1}^t T_n \diamond$*

$(T_{n-1}T_{n-2}\dots T_0)\mathbf{1}$, *where* T_k *is the transfer matrix of the two consecutive columns:* k *and* $k+1$ *of* $G_{n,m}$, $k = 0, ..., n-1$ *and* T_n *is the transfer matrix of the columns* 0 *and* n *of* $G_{n,m}$.

Clearly the previous theorem, also is true for $EC = E \cup E_2$ (interchanging n by m and m by n). In the following example is illustrated.

Example 2. *Let* $F = (x_0 \vee y_0) \wedge (\neg y_0 \vee \neg z_0) \wedge (z_0 \vee z_1) \wedge (z_1 \vee z_2) \wedge (z_2 \vee y_2) \wedge (y_2 \vee x_2) \wedge (x_2 \vee x_1) \wedge (\neg x_1 \vee x_0) \wedge (x_1 \vee y_1) \wedge (\neg y_1 \vee z_1) \wedge (\neg y_1 \vee \neg y_0) \wedge (y_1 \vee y_2), (x_0, z_0), (\neg x_1, z_1), (\neg x_2, \neg z_2))$ *(see figure 5).*

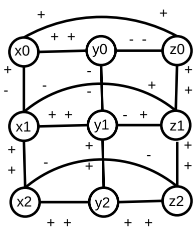

Fig. 5: Grid-Cylinder $C(2, 2)$

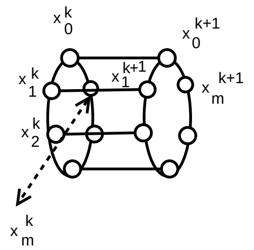

Fig. 6: Consecutive Cycles

We have that the matrix T_1T_0 *is given in example 1. The transfer matrix* T_2 *of columns* 0 *and* 2 *is computed as follows.*

The strings of signs for edges from the column 0 *to column* 2 *are given by:* $s_0^0 s_1^0 s_2^0 = +\mp+$, $s_0^2 s_1^2 s_2^2 = +++$, $\tau_0^0\tau_1^0\tau_2^0 = +--$ *and* $\tau_0^2\tau_1^2\tau_2^2 = ++-$, *then* $\mathcal{F}_2^{+\mp+} = \{\mathbf{u}_1, \cdots, \mathbf{u}_4\}$ *and* $\mathcal{F}_2^{+++} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, *where* $\mathbf{u}_1 = (0, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$, $\mathbf{u}_4 = (1, 1, 0)$, $\mathbf{v}_1 = (0, 0, 0)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (0, 1, 0)$, $\mathbf{v}_4 = (0, 0, 1)$ *and* $\mathbf{v}_5 = (1, 0, 1)$. *The transfer matrix* $T_2 = (a_{ij})_{5\times 4}$, *is a* 5×4 *matrix given by* $a_{ij} = 1$ *if* $\varphi_{++-}(\mathbf{v_i}) \cdot \varphi_{+--}(\mathbf{u_j}) = 0$ *and* $a_{ij} = 0$ *otherwise* $(1 \leq i \leq 5$ *and* $1 \leq j \leq 4)$. *Then*

$$T_2 \diamond (T_1T_0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \diamond \begin{pmatrix} 3 & 2 & 2 & 0 \\ 4 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 1 & 3 & 0 \end{pmatrix}$$

Therefore $\#SAT(F) = 17$.

Proof (Theorem 2). Let F be a grid-cylinder formula of size $m \times n$. We have that F can be expressed as $F = F_{n,m} \wedge R_n$, where $F_{n,m}$ is given by equation (4) and $R_n = \bigwedge_{j=0}^{m} (\tau_j^{2n} x_j^0 \vee \tau_j^{2n+1} x_j^n)$, that is, the graph of F is the graph o $G_{n,m}$ (the constrained graph of $F_{n,m}$) adding new labelled edges (with signs τ_j^{2n} and τ_j^{2n+1}) from the column 0 to column n of $G_{n,m}$. Let $T_n = (\beta_{ij})_{r_n \times r_0}$ be the transfer matrix of the column 0 to column n of $G_{n,m}$ following the arcs given by R_n. From remark 3, $T = (T_{n-1}T_{n-2}\dots T_0) = (\alpha_{ij})_{r_n \times r_0}$, where $\alpha_{i,j}$

is the number of satisfying assignments of $F_{n,m}$ with $\bar{u}_i \in \mathcal{F}_m^{\xi_0}$ and $\bar{u}_j \in \mathcal{F}_m^{\xi_n}$ fixed. Also, the formula R_n is satisfied by u_i and u_j iff $\beta_{ij} = 1$. Therefore, there are $\beta_{ij}\alpha_{ij}$ satisfying assignments of F with $\bar{u}_i \in \mathcal{F}_m^{\xi_0}$ and $\bar{u}_j \in \mathcal{F}_m^{\xi_n}$ fixed. We observe that, the product $\beta_{ij}\alpha_{ij}$ is the entry $a_{i,j}$ of the Hadamard product $T_n \diamond T$.$\Box$

4.1 Transfer Matrix for Cycles

We can adapt our extension for computing the transfer matrix between two consecutive simple cycles instead of two consecutive columns as follows.

Let $\mathcal{F}_m$ be the set of all $(m+1)$-vectors $\mathbf{v}$ of $0's$ and $1's$ (as in section 3), and let $\mathcal{C}_m \subset \mathcal{F}_m$ be the set of all $(m+1)$-vectors $\mathbf{v}$ of $0's$ and $1's$, such that $\mathbf{v}$ does not have two consecutive $1's$ and does not have $1's$ in the first and last positions. Given $\mathbf{s} = s_0 s_1 \dots s_m$ a string in $\hat{S}$, we define $\mathcal{F}_m^{\mathbf{s}} = \{e \in \mathcal{F}_m : \varphi_{\mathbf{s}}(e) \in \mathcal{C}_{m+\ell}\}$, $\ell = \mid \{s \in \{s_0, s_1, \dots, s_m\} : s \in \bar{S}\} \mid$. Assume that $x_0^k, \dots, x_m^k$ and $x_0^{k+1}, \dots, x_m^{k+1}$ are the nodes of the $k-th$ and $(k+1)-th$ cycles respectively of $C_{n,m}$, $0 \le k < n$.

For $j = k, k+1$, let $\mathbf{s}^j = s_0^j s_1^j \cdots s_m^j$ and $\tau^j = \tau_0^j \tau_1^j \cdots \tau_m^j$, where $s_i^j = sgn(x_i^j)$ and $\tau_i^j = sgn_k(x_i^j)$. We define a matrix $T_k = T_{m,k}$, the transfer matrix from cycle k to the cycle $k+1$ as follows. T_k is an $\mid \mathcal{F}_m^{\mathbf{s}^{k+1}} \mid \times \mid \mathcal{F}_m^{\mathbf{s}^k} \mid$ matrix of $0's$ and $1's$ whose rows and columns are indexed by vectors of $\mathcal{F}_m^{\mathbf{s}^{k+1}} \times \mathcal{F}_m^{\mathbf{s}^k}$. The entry of T_k in position $(\mathbf{u}, \mathbf{v})$ is 1 if the vectors $\varphi_{\tau^k}(\mathbf{u})$ and $\varphi_{\tau^{k+1}}(\mathbf{v})$ are orthogonal, and is 0 otherwise (see figure 6).

Example 3. *We compute the transfer matrices: T_0 from cycle $x_0y_0z_0$ to $x_1y_1z_1$ and T_1 from cycle $x_1y_1z_1$ to $x_2y_2z_2$ for F as in example 2 (see figure 5). We have $s_0^0 s_1^0 s_2^0 = + \pm \mp$, $s_0^1 s_1^1 s_2^1 = \mp \pm +$ and $s_0^2 s_1^2 s_2^2 = \mp + \pm$. On the other hand, $\tau^0 = \tau_0^0 \tau_1^0 \tau_2^0 = + - +$, $\tau^1 = \tau_0^1 \tau_1^1 \tau_2^1 = - - +$, and $\tau^2 = \tau_0^2 \tau_1^2 \tau_2^2 = \tau^3 = \tau_0^3 \tau_1^3 \tau_2^3 = + + +$. Then $\mathcal{F}_2^{+\pm\mp} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $\mathcal{F}_2^{\mp\pm+} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\mathcal{F}_2^{\mp\pm+} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$, where $\boldsymbol{u}_1 = (0,1,0)$, $\boldsymbol{u}_2 = (0,0,1)$, $\boldsymbol{u}_3 = (0,1,1)$, $\boldsymbol{v}_1 = (0,0,0)$, $\boldsymbol{v}_2 = (1,0,0)$, $\boldsymbol{v}_3 = (0,1,0)$, $\boldsymbol{w}_1 = (1,0,0)$, $\boldsymbol{w}_2 = (0,0,1)$ and $\boldsymbol{w}_3 = (1,0,1)$. Computing $\varphi_{\tau^{2k}}(\mathbf{u}) \cdot \varphi_{\tau^{2k+1}}(\mathbf{v})$ for $k = 0,1$ and following the definition of transfer matrix, we have that T_0 and T_1 are 3×3 matrices given by*

$$T_0 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Remark 4. For F from example 2,

$$\mathbf{1} T_1 T_0 \mathbf{1} = \mathbf{1} \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix} \mathbf{1} = 17 = \#SAT(F).$$

The following theorem can also be used for computing $\#SAT(F)$ for F, a grid-cylinder.

Theorem 3. *Let F be a grid-cylinder of size $m \times n$ with graph $C(n,m)$, then $\#SAT(F) = \mathbf{1}^t T_{n-1} \ldots T_0 \mathbf{1}$, where T_k is the transfer matrix of two consecutive cycles: k and $k+1$ of $C(n,m)$, for $k = 0, ..., n-1$.*

Proof. The proof is similar to the proof of theorem 1, taking $\mathcal{F}_m$, $\mathcal{C}_m$, $\mathcal{F}_m^{\mathbf{s}}$ and the transfer matrix for cycles as in section 4.1. We observe that, in this case the column formulas C_i given by equation (5) are simple cycles.$\square$

Using theorem 3 and Hadamard product we can compute $\#SAT(F)$ for F, a grid-tori. The following theorem shows us how to proceed.

Theorem 4. *Let F be a grid-tori of size $m \times n$ with graph $T(n,m) = (V(n,m), E'(n,m))$, $E' = E_1 \cup E_2$. Then $\#SAT(F) = T_n \diamond (T_{n-1}T_{n-2} \ldots T_0)$, where T_k is the transfer matrix of the two consecutive cycles of $T(n,m)$: k and $k+1$ of $G_{n,m}$, $k = 0, ..., n-1$ and T_n is the transfer matrix of the cycle 0 and n.*

Example 4. *Let $F_1 = F \cup \{(x_0 \vee x_2), (\neg y_0, y_2), (\neg z_0, \neg z_2)\}$, where F is like in example 2 (see figure 7).*

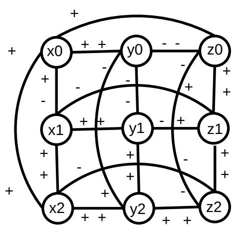

Fig. 7: Grid-Tori of example 4

We compute the transfer matrix T_2 from the cycle $x_0y_0z_0$ to the cycle $x_2y_2z_2$ as follows. We have $\mathcal{F}_2^{+\pm\mp} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $\mathcal{F}_2^{\mp\pm+} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$, where the vectors $\boldsymbol{u}_i' s$ and $\boldsymbol{w}_j' s$ are as the example 3, only that now $\tau^0 = \tau_0^0\tau_1^0\tau_2^0 = + - -$ and $\tau^3 = \tau_0^3\tau_1^3\tau_2^3 = + + -$. The transfer matrix T_2 is obtained by the evaluation of $\varphi_{\tau^3}(\mathbf{w_i}) \cdot \varphi_{\tau^0}(\mathbf{u_j})$ for $1 \le i \le 3$ and $1 \le j \le 3$. Then

$$T_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

In example 2, T_0, T_1 and T_1T_0 are computed, therefore

$$T_2 \diamond (T_1T_0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \diamond \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

and $\#SAT(F_1) = \mathbf{1} T_2 \diamond (T_1T_0) \mathbf{1} = 15$.

Proof (Theorem 4). Using the theorem 3, the proof is similar to the proof of theorem 2 taking $F_{n,m}$ as a grid cylinder formula and $R_n = C_0 \wedge C_n \wedge E$, where C_0 and C_n corresponding to the first cycle and n-th cycle of $C(n, m)$ respectively ($C(n, m)$ is the grid cylinder associated to $F_{n,m}$). The formula E is formed by new clauses corresponding to edges from the vertices of the first cycle to the vertices of n-th cycle of $C(n, m)$. $\square$

5 Conclusion

We have presented an extension of the transfer matrix method that allows to consider signed edges on grid graphs, grid-cylinders and grid-tori. We argued about the advantage of this extension in the problem of counting assignments of Boolean formulas in 2-CNF.

We have designed a procedure for computing #2SAT(F) where F is a grid, grid-cylinder or grid-tori Boolean formula, based on the sum of all entries of the product matrix of the transfer matrix of each two consecutive columns for the case of a grid. In a grid cylinder we have two result for computing #2SAT(F) one uses the sum of the entries of the Hadamard product between the transfer matrix of the first column (row) and the top column (row) with the product matrix of the transfer matrix of each two consecutive columns (row). The second result uses the sum of all entries of the product matrix of the transfer matrix of each two consecutive cycles. Finally, if F is a grid tori, we use the sum of the entries of the Hadamard product between the transfer matrix of the first cycle and the top cycle of tori with the product matrix of the transfer matrix of each two consecutive cycles of the tori.

A work in progress is the detailed determination of the complexity of the proposed extension. However, based on previous works in the transfer matrix method and our preliminary experiments, the complexity remains polynomial as long as the starting grid graphs are of fixed height, we consider the complexity with a fixed-parameter.

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