Proc. 23rd Int. Workshop on Description Logics (DL2010), CEUR-WS 573, Waterloo, Canada, 2010.

Paraconsistent Description Logics Revisited

Norihiro Kamide

Waseda Institute for Advanced Study, Waseda University,
1-6-1 Nishi Waseda, Shinjuku-ku, Tokyo 169-8050, JAPAN.
logician-kamide@aoni.waseda.jp

Abstract. Inconsistency handling is of growing importance in Knowledge Representation since inconsistencies may frequently occur in an open world. Paraconsistent (or inconsistency-tolerant) description logics have been studied by several researchers to cope with such inconsistencies. In this paper, a new paraconsistent description logic, $\mathcal{PALC}$, is obtained from the description logic $\mathcal{ALC}$ by adding a paraconsistent negation. Some theorems for embedding $\mathcal{PALC}$ into $\mathcal{ACL}$ are proved, and $\mathcal{PALC}$ is shown to be decidable. A tableau calculus for $\mathcal{PALC}$ is introduced, and the completeness theorem for this calculus is proved.

1 Introduction

Inconsistency handling is of growing importance in Knowledge Representation since inconsistencies may frequently occur in an open world. *Paraconsistent (or inconsistency-tolerant) description logics* have been studied by several researchers [5–8, 11–13, 16, 18, 19] to cope with such inconsistencies.

However, the existing paraconsistent description logics have no good compatibility with the standard description logics such as $\mathcal{ALC}$ [15] etc. in the following sense:

1. these paraconsistent description logics are not a straightforward extension of the standard ones,
2. some paraconsistent description logics have no translation into a standard description logic.

Such compatibility is important to adopt and re-use the existing applications and algorithms for the standard description logics. A translation or reduction of a paraconsistent description logic into a standard description logic is especially important for such a compatibility issue [5, 6].

The aim of this paper is thus to introduce a compatible paraconsistent description logic which is a straightforward extension of $\mathcal{ALC}$ and is also embeddable into $\mathcal{ALC}$. To construct such a compatible paraconsistent description logic, some merits of some existing paraconsistent description logics are adopted and combined.

Some examples of studies of paraconsistent description logics are presented as follows. An *inconsistency-tolerant four-valued terminological logic* was originally introduced by Patel-Schneider [13], three *inconsistency-tolerant constructive description logics*, which are based on intuitionistic logic, were studied by Odintsov and Wansing [11, 12], some *paraconsistent four-valued description logics* including $\mathcal{ALC}4$ were studied by Ma et al. [5, 6], some *quasi-classical description logics* were developed by Zhang et al. [18, 19], a sequent calculus for reasoning in four-valued description logics was introduced by Straccia [16], and an application of four-valued description logic to information retrieval was studied by Meghini et al. [7, 8].

The logic $\mathcal{ALC}4$ [5] has a good translation into $\mathcal{ALC}$, and using this translation, the satisfiability problem for $\mathcal{ALC}4$ is shown to be decidable. However, $\mathcal{ALC}4$ and its variations have no classical negation (or complement), i.e., these logics are not an extension of the standard description logics. The quasi-classical description logics [18, 19] have the classical negation, i.e., these logics are regarded as extensions of the standard description logics. However, translations of quasi-classical description logics into the corresponding standard description logics have not been proposed yet.

The paraconsistent description logic proposed in this paper supports both the merits of $\mathcal{ALC}4$ and the quasi-classical description logics, i.e., it has the translation and the classical negation. Moreover, a simple dual-interpretation semantics is used in the proposed logic. Such a dual-interpretation semantics is taken over from the dual-consequence Kripke-style semantics for *Nelson's paraconsistent four-valued logic with strong negation* N4 [1, 9].

A description logic (called $\mathcal{ALC}^n_\sim$) with such a dual (or multiple)-interpretation semantics was introduced and studied by Kaneiwa [4] to deal with a negation issue, but not to deal with an issue of inconsistency handling. The logic $\mathcal{ALC}^n_\sim$ is a natural extension of $\mathcal{ALC}$, and $\mathcal{ALC}^n_\sim$ is shown to be decidable (w.r.t. the concept satisfiability problem) and complete (w.r.t. a tableau calculus). But, $\mathcal{ALC}^n_\sim$ is not paraconsistent, and a translation into $\mathcal{ALC}$ has not been proposed yet. The present paper is based on the spirit of $\mathcal{ALC}^n_\sim$ for dual (or multiple)-interpretation semantics.

The contents of this paper are then summarized as follows. A new paraconsistent description logic, $\mathcal{PALC}$, is obtained from $\mathcal{ALC}$ by adding a paraconsistent negation similar to the strong negation in Nelson's N4. A *semantical embedding theorem* of $\mathcal{PALC}$ into $\mathcal{ALC}$ is shown by constructing a standard single-interpretation of $\mathcal{ALC}$ from a paraconsistent dual-interpretation of $\mathcal{PALC}$, and vice versa. By using this embedding theorem, the concept satisfiability problem for $\mathcal{PALC}$ is shown to be decidable. The complexity of the decision procedure for $\mathcal{PALC}$ is also shown to be the same complexity as that of $\mathcal{ALC}$. Next, a tableau calculus, $\mathcal{TPALC}$ (for $\mathcal{PALC}$), is introduced, and a *syntactical embedding theorem* of this calculus into a tableau calculus, $\mathcal{TALC}$ (for $\mathcal{ALC}$), is proved. The completeness theorem for $\mathcal{TPALC}$ is proved by combining both the semantical and syntactical embedding theorems. A comparision of $\mathcal{PALC}$ and other paraconsistent description logics is explained.

2 Paraconsistent Description Logic

In this section, firstly, we present a semantical definition of $\mathcal{ALC}$, and secondly, we introduce $\mathcal{PALC}$ by extending $\mathcal{ALC}$ with a paraconsistent negation.

2.1 $\mathcal{ALC}$

The $\mathcal{ALC}$-language is constructed from atomic concepts, atomic roles, $\sqcap$ (intersection), $\sqcup$ (union), $\neg$ (classical negation or complement), $\forall R$ (universal concept quantification) and $\exists R$ (existential concept quantification). We use the letters A and A_i for atomic concepts, the letter R for atomic roles, and the letters C and D for concepts.

Definition 1 Concepts C *are defined by the following grammar:*

$$C ::= A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \forall R.C \mid \exists R.C$$

Definition 2 *An* interpretation $\mathcal{I}$ *is a pair* $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ *where*

1. $\Delta^{\mathcal{I}}$ *is a non-empty set,*
2. $\cdot^{\mathcal{I}}$ *is an interpretation function which assigns to every atomic concept* A *a set* $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ *and to every atomic role* R *a binary relation* $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The interpretation function is extended to concepts by the following inductive definitions:

1. $(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$,
2. $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$,
3. $(C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}$,
4. $(\forall R.C)^{\mathcal{I}} := \{a \in \Delta^{\mathcal{I}} \mid \forall b\ [(a,b) \in R^{\mathcal{I}} \Rightarrow b \in C^{\mathcal{I}}]\}$,
5. $(\exists R.C)^{\mathcal{I}} := \{a \in \Delta^{\mathcal{I}} \mid \exists b\ [(a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}]\}$.

An interpretation $\mathcal{I}$ *is a model of a concept* C *(denoted as* $\mathcal{I} \models C$*) if* $C^{\mathcal{I}} \neq \emptyset$. *A concept* C *is said to be* satisfiable *in* $\mathcal{ALC}$ *if there exists an interpretation* $\mathcal{I}$ *such that* $\mathcal{I} \models C$.

The syntax of $\mathcal{ALC}$ is extended by a non-empty set N_I of individual names. We denote individual names by o, o_1, o_2, x, y and z.

Definition 3 *An* ABox *is a finite set of expressions of the form:* $C(o)$ *or* $R(o_1, o_2)$ *where* o, o_1 *and* o_2 *are in* N_I, C *is a concept, and* R *is an atomic role. An expression* $C(o)$ *or* $R(o_1, o_2)$ *is called an* ABox statement. *An interpretation* $\mathcal{I}$ *in Definition 2 is extended to apply also to individual names* o *such that* $o^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. *Such an interpretation is a model of an ABox* $\mathcal{A}$ *if for every* $C(o) \in \mathcal{A}$, $o^{\mathcal{I}} \in C^{\mathcal{I}}$ *and for every* $R(o_1, o_2) \in \mathcal{A}$, $(o_1^{\mathcal{I}}, o_2^{\mathcal{I}}) \in R^{\mathcal{I}}$. *An ABox* $\mathcal{A}$ *is called* satisfiable *in* $\mathcal{ALC}$ *if it has a model.*

We adopt the following *unique name assumption*: for any $o_1, o_2 \in N_I$, if $o_1 \neq o_2$, then $o_1^{\mathcal{I}} \neq o_2^{\mathcal{I}}$.

Definition 4 *A* TBox *is a finite set of expressions of the form:* $C \sqsubseteq D$. *The elements of a TBox are called* TBox statments. *An interpretation* $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ *is called a* model *of* $C \sqsubseteq D$ *if* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. *An interpretation* $\mathcal{I}$ *is said to be a model of a TBox* $\mathcal{T}$ *if* $\mathcal{I}$ *is a model of every element of* $\mathcal{T}$. *A TBox* $\mathcal{T}$ *is called* satisfiable *in* $\mathcal{ALC}$ *if it has a model.*

Definition 5 *A knowledge base* Σ *is a pair* $(\mathcal{T}, \mathcal{A})$ *where* $\mathcal{T}$ *is a TBox and* $\mathcal{A}$ *is an ABox. An interpretation* $\mathcal{I}$ *is a model of* Σ *if* $\mathcal{I}$ *is a model of both* $\mathcal{T}$ *and* $\mathcal{A}$. *A knowledge base* Σ *is called* satisfiable *in* $\mathcal{ALC}$ *if it has a model.*

Since the satisfiability for an ABox, a TBox or a knowledge base can be reduced to the satisfiability for a concept [2], we focus on the concept satisfiability in the following discussion.

2.2 $\mathcal{PALC}$

Similar notions and terminologies for $\mathcal{ALC}$ are also used for $\mathcal{PALC}$. The $\mathcal{PALC}$-language is constructed from the $\mathcal{ALC}$-language by adding $\sim$ (paraconsistent negation).

Definition 6 Concepts C *are defined by the following grammar:*

$$C ::= A \mid \neg C \mid {\sim} C \mid C \sqcap C \mid C \sqcup C \mid \forall R.C \mid \exists R.C$$

Definition 7 *A* paraconsistent interpretation $\mathcal{PI}$ *is a structure* $\langle \Delta^{\mathcal{PI}}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-} \rangle$ *where*

1. $\Delta^{\mathcal{PI}}$ *is a non-empty set,*
2. $\cdot^{\mathcal{I}^+}$ *is an interpretation function which assigns to every atomic concept* A *a set* $A^{\mathcal{I}^+} \subseteq \Delta^{\mathcal{PI}}$ *and to every atomic role* R *a binary relation* $R^{\mathcal{I}^+} \subseteq \Delta^{\mathcal{PI}} \times \Delta^{\mathcal{PI}}$,
3. $\cdot^{\mathcal{I}^-}$ *is an interpretation function which assigns to every atomic concept* A *a set* $A^{\mathcal{I}^-} \subseteq \Delta^{\mathcal{PI}}$ *and to every atomic role* R *a binary relation* $R^{\mathcal{I}^-} \subseteq \Delta^{\mathcal{PI}} \times \Delta^{\mathcal{PI}}$,
4. *for any atomic role* R, $R^{\mathcal{I}^+} = R^{\mathcal{I}^-}$.

The interpretation functions are extended to concepts by the following inductive definitions:

1. $({\sim} C)^{\mathcal{I}^+} := C^{\mathcal{I}^-}$,
2. $(\neg C)^{\mathcal{I}^+} := \Delta^{\mathcal{PI}} \setminus C^{\mathcal{I}^+}$,
3. $(C \sqcap D)^{\mathcal{I}^+} := C^{\mathcal{I}^+} \cap D^{\mathcal{I}^+}$,
4. $(C \sqcup D)^{\mathcal{I}^+} := C^{\mathcal{I}^+} \cup D^{\mathcal{I}^+}$,
5. $(\forall R.C)^{\mathcal{I}^+} := \{a \in \Delta^{\mathcal{PI}} \mid \forall b\ [(a, b) \in R^{\mathcal{I}^+} \Rightarrow b \in C^{\mathcal{I}^+}]\}$,
6. $(\exists R.C)^{\mathcal{I}^+} := \{a \in \Delta^{\mathcal{PI}} \mid \exists b\ [(a, b) \in R^{\mathcal{I}^+} \wedge b \in C^{\mathcal{I}^+}]\}$,
7. $({\sim} C)^{\mathcal{I}^-} := C^{\mathcal{I}^+}$,
8. $(\neg C)^{\mathcal{I}^-} := \Delta^{\mathcal{PI}} \setminus C^{\mathcal{I}^-}$,

9. $(C \sqcap D)^{\mathcal{I}^-} := C^{\mathcal{I}^-} \cup D^{\mathcal{I}^-}$,
10. $(C \sqcup D)^{\mathcal{I}^-} := C^{\mathcal{I}^-} \cap D^{\mathcal{I}^-}$,
11. $(\forall R.C)^{\mathcal{I}^-} := \{a \in \Delta^{\mathcal{PI}} \mid \exists b\ [(a,b) \in R^{\mathcal{I}^-} \wedge b \in C^{\mathcal{I}^-}]\}$,
12. $(\exists R.C)^{\mathcal{I}^-} := \{a \in \Delta^{\mathcal{PI}} \mid \forall b\ [(a,b) \in R^{\mathcal{I}^-} \Rightarrow b \in C^{\mathcal{I}^-}]\}$.

An expression $\mathcal{I}^* \models C$ $(* \in \{+,-\})$ *is defined as* $C^{\mathcal{I}^*} \neq \emptyset$. *A paraconsistent interpretation* $\mathcal{PI} := \langle \Delta^{\mathcal{PI}}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-} \rangle$ *is a model of a concept* C *(denoted as* $\mathcal{PI} \models C$*) if* $\mathcal{I}^+ \models C$. *A concept* C *is said to be* satisfiable *in* $\mathcal{PALC}$ *if there exists a paraconsistent interpretation* $\mathcal{PI}$ *such that* $\mathcal{PI} \models C$.

The interpretation functions $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^-}$ are intended to represent "verification" and "falsification", respectively.

Definition 8 *A paraconsistent interpretation* $\mathcal{PI}$ *in Definition 7 is extended to apply also to individual names* o *such that* $o^{\mathcal{I}^+}, o^{\mathcal{I}^-} \in \Delta^{\mathcal{PI}}$ *and* $o^{I^+} = o^{I^-}$. *Such a paraconsistent interpretation is a model of an ABox* $\mathcal{A}$ *if for every* $C(o) \in \mathcal{A}$, $o^{\mathcal{I}^+} \in C^{\mathcal{I}^+}$ *and for every* $R(o_1, o_2) \in \mathcal{A}$, $(o_1^{\mathcal{I}^+}, o_2^{\mathcal{I}^+}) \in R^{\mathcal{I}^+}$. *Such a paraconsistent interpretation is called a* model *of* $C \sqsubseteq D$ *if* $C^{\mathcal{I}^+} \subseteq D^{\mathcal{I}^+}$. *The satisfiability of ABox, a TBox or a knowledge base in* $\mathcal{PALC}$ *is defined in the same way as in* $\mathcal{ALC}$.

3 Semantical Embedding and Decidability

In the following, we introduce a translation of $\mathcal{PALC}$ into $\mathcal{ALC}$, and by using this translation, we show a semantical embedding theorem of $\mathcal{PALC}$ into $\mathcal{ALC}$. The translation introduced is a slight modification of the translation introduced by Ma et al. [5] to embed $\mathcal{ALC}4$ into $\mathcal{ALC}$. A similar translation has been used by Gurevich [3] and Rautenberg [14] to embed Nelson's three-valued constructive logic [1, 9] into intuitionistic logic. The way of showing the semantical and syntactical embedding theorems of $\mathcal{PALC}$ into $\mathcal{ALC}$ is a new technical contribution developed in this paper. The semantical and syntactical embedding theorems are used to show the decidability and completeness theorems for $\mathcal{PALC}$.

Definition 9 *Let* N_C *be a non-empty set of atomic concepts and* N'_C *be the set* $\{A' \mid A \in N_C\}$ *of atomic concepts.* [1] *Let* N_R *be a non-empty set of atomic roles and* N_I *be a non-empty set of individual names. The language* $\mathcal{L}^{\sim}$ *of* $\mathcal{PALC}$ *is defined using* N_C, N_R, N_I, $\sim$, $\neg, \sqcap, \sqcup$, $\forall R$ *and* $\exists R$. *The language* $\mathcal{L}$ *of* $\mathcal{ALC}$ *is obtained from* $\mathcal{L}^{\sim}$ *by adding* N'_C *and deleting* $\sim$.

A mapping f *from* $\mathcal{L}^{\sim}$ *to* $\mathcal{L}$ *is defined inductively by*

1. *for any* $R \in N_R$ *and any* $o \in N_I$, $f(R) := R$ *and* $f(o) := o$,
2. *for any* $A \in N_C$, $f(A) := A$ *and* $f({\sim}A) := A' \in N'_C$,
3. *For any* $A(o) \in N_C$, $f(A(o)) := A(f(o))$ *and* $f({\sim}A(o)) := A'(f(o)) \in N'_C$,
4. $f(\neg C) := \neg f(C)$,

[1] A can include individual names, i.e., A can be $A(o)$ for any $o \in N_I$.

5. $f(C \sharp D) := f(C) \sharp f(D)$ *where* $\sharp \in \{\sqcap, \sqcup\}$,
6. $f(\forall R.C) := \forall f(R).f(C)$,
7. $f(\exists R.C) := \exists f(R).f(C)$,
8. $f(\sim\sim C) := f(C)$,
9. $f(\sim\neg C) := \neg f(\sim C)$,
10. $f(\sim(C \sqcap D)) := f(\sim C) \sqcup f(\sim D)$,
11. $f(\sim(C \sqcup D)) := f(\sim C) \sqcap f(\sim D)$,
12. $f(\sim\forall R.C) := \exists f(R).f(\sim C)$,
13. $f(\sim\exists R.C) := \forall f(R).f(\sim C)$.

Lemma 10 *Let f be the mapping defined in Definition 9. For any paraconsistent interpretation $\mathcal{PI} := \langle \Delta^{\mathcal{PI}}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-} \rangle$ of $\mathcal{PALC}$, we can construct an interpretation $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of $\mathcal{ALC}$ such that for any concept C in $\mathcal{L}^{\sim}$,*

1. $C^{\mathcal{I}^+} = f(C)^{\mathcal{I}}$,
2. $C^{\mathcal{I}^-} = f(\sim C)^{\mathcal{I}}$.

Proof. Let N_C be a non-empty set of atomic concepts and N'_C be the set $\{A' \mid A \in N_C\}$ of atomic concepts. Let N_R and N_I be sets of atomic roles and individual names, respectively.

Suppose that $\mathcal{PI}$ is a paraconsistent interpretation $\langle \Delta^{\mathcal{PI}}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-} \rangle$ where

1. $\Delta^{\mathcal{PI}}$ is a non-empty set,
2. $\cdot^{\mathcal{I}^+}$ is an interpretation function which assigns to every atomic concept $A \in N_C$ a set $A^{\mathcal{I}^+} \subseteq \Delta^{\mathcal{PI}}$, to every atomic role $R \in N_R$ a binary relation $R^{\mathcal{I}^+} \subseteq \Delta^{\mathcal{PI}} \times \Delta^{\mathcal{PI}}$ and to every individual name $o \in N_I$ an element $o^{\mathcal{I}^+} \in \Delta^{\mathcal{PI}}$,
3. $\cdot^{\mathcal{I}^-}$ is an interpretation function which assigns to every atomic concept $A \in N_C$ a set $A^{\mathcal{I}^-} \subseteq \Delta^{\mathcal{PI}}$, to every atomic role $R \in N_R$ a binary relation $R^{\mathcal{I}^-} \subseteq \Delta^{\mathcal{PI}} \times \Delta^{\mathcal{PI}}$ and to every individual name $o \in N_I$ an element $o^{\mathcal{I}^-} \in \Delta^{\mathcal{PI}}$,
4. for any $R \in N_R$ and any $o \in N_I$, $R^{\mathcal{I}^+} = R^{\mathcal{I}^-}$ and $o^{\mathcal{I}^+} = o^{\mathcal{I}^-}$.

Suppose that $\mathcal{I}$ is an interpretation $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where

1. $\Delta^{\mathcal{I}}$ is a non-empty set such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{PI}}$,
2. $\cdot^{\mathcal{I}}$ is an interpretation function which assigns to every atomic concept $A \in N_C \cup N'_C$ a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, to every atomic role $R \in N_R$ a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and to every individual name $o \in N_I$ an element $o^{\mathcal{I}} \in \Delta^{\mathcal{I}}$,
3. for any $R \in N_R$ and any $o \in N_I$, $R^{\mathcal{I}} = R^{\mathcal{I}^+} = R^{\mathcal{I}^-}$ and $o^{\mathcal{I}} = o^{\mathcal{I}^+} = o^{\mathcal{I}^-}$.

Suppose moreover that $\mathcal{PI}$ and $\mathcal{I}$ satisfy the following conditions: for any $A \in N_C$ and any $o \in N_I$,

1. $A^{\mathcal{I}^+} = A^{\mathcal{I}}$ and $(A(o))^{\mathcal{I}^+} = (A(o))^{\mathcal{I}}$,
2. $A^{\mathcal{I}^-} = (A')^{\mathcal{I}}$ and $(A(o))^{\mathcal{I}^-} = (A'(o))^{\mathcal{I}}$.

The lemma is then proved by (simultaneous) induction on the complexity of C. The base step is obvious. We show only some cases on the induction step below.

Case $C \equiv \neg D$: For (1), we obtain: $a \in (\neg D)^{\mathcal{I}^+}$ iff $a \in \Delta^{\mathcal{PI}} \setminus D^{\mathcal{I}^+}$ iff $a \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}^+}$ (by the condition $\Delta^{\mathcal{PI}} = \Delta^{\mathcal{I}}$) iff $a \in \Delta^{\mathcal{I}} \setminus f(D)^{\mathcal{I}}$ (by induction hypothesis for 1) iff $a \in (\neg f(D))^{\mathcal{I}}$ iff $a \in f(\neg D)^{\mathcal{I}}$ (by the definition of f). For (2), we obtain: $a \in (\neg D)^{\mathcal{I}^-}$ iff $a \in \Delta^{\mathcal{PI}} \setminus D^{\mathcal{I}^-}$ iff $a \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}^-}$ (by the condition $\Delta^{\mathcal{PI}} = \Delta^{\mathcal{I}}$) iff $a \in \Delta^{\mathcal{I}} \setminus f(\sim D)^{\mathcal{I}}$ (by induction hypothesis for 2) iff $a \in (\neg f(\sim D))^{\mathcal{I}}$ iff $a \in f(\sim\neg D)^{\mathcal{I}}$ (by the definition of f).

Case $C \equiv \sim D$: For (1), we obtain: $a \in (\sim D)^{\mathcal{I}^+}$ iff $a \in D^{\mathcal{I}^-}$ iff $a \in f(\sim D)^{\mathcal{I}}$ (by induction hypothesis for 2). For (2), we obtain: $a \in (\sim D)^{\mathcal{I}^-}$ iff $a \in D^{\mathcal{I}^+}$ iff $a \in f(D)^{\mathcal{I}}$ (by induction hypothesis for 1) iff $a \in f(\sim\sim D)^{\mathcal{I}}$ (by the definition of f).

Case $C \equiv \forall R.D$: We show only (2) below.

$d \in (\forall R.D)^{\mathcal{I}^-}$
iff $d \in \{a \in \Delta^{\mathcal{PI}} \mid \exists b\, [(a,b) \in R^{\mathcal{I}^-} \wedge b \in D^{\mathcal{I}^-}]\}$
iff $d \in \{a \in \Delta^{\mathcal{I}} \mid \exists b\, [(a,b) \in R^{\mathcal{I}} \wedge b \in D^{\mathcal{I}^-}]\}$ (by the conditions $\Delta^{\mathcal{PI}} = \Delta^{\mathcal{I}}$ and $R^{\mathcal{I}^-} = R^{\mathcal{I}}$)
iff $d \in \{a \in \Delta^{\mathcal{I}} \mid \exists b\, [(a,b) \in R^{\mathcal{I}} \wedge b \in f(\sim D)^{\mathcal{I}}]\}$ (by induction hypothesis for 2)
iff $d \in ((\exists R.f(\sim D))^{\mathcal{I}})$
iff $d \in ((\exists f(R).f(\sim D))^{\mathcal{I}})$ (by the definition of f)
iff $d \in ((f(\sim\forall R.D))^{\mathcal{I}})$ (by the definition of f).

∎

Lemma 11 *Let f be the mapping defined in Definition 9. For any paraconsistent interpretation $\mathcal{PI} := \langle \Delta^{\mathcal{PI}}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-} \rangle$ of $\mathcal{PALC}$, we can construct an interpretation $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of $\mathcal{ALC}$ such that for any concept C in $\mathcal{L}^{\sim}$,*

1. $\mathcal{I}^+ \models C$ *iff* $\mathcal{I} \models f(C)$,
2. $\mathcal{I}^- \models C$ *iff* $\mathcal{I} \models f(\sim C)$.

Proof. By Lemma 10. ∎

Lemma 12 *Let f be the mapping defined in Definition 9. For any interpretation $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of $\mathcal{ALC}$, we can construct a paraconsistent interpretation $\mathcal{PI} := \langle \Delta^{\mathcal{PI}}, \cdot^{\mathcal{I}^+}, \cdot^{\mathcal{I}^-} \rangle$ of $\mathcal{PALC}$ such that for any concept C in $\mathcal{L}^{\sim}$,*

1. $\mathcal{I} \models f(C)$ *iff* $\mathcal{I}^+ \models C$,
2. $\mathcal{I} \models f(\sim C)$ *iff* $\mathcal{I}^- \models C$.

Proof. Similar to the proof of Lemma 11. ∎

Theorem 13 (Semantical embedding) *Let f be the mapping defined in Definition 9. For any concept C,*

C is satisfiable in $\mathcal{PALC}$ iff $f(C)$ is satisfiable in $\mathcal{ALC}$.

Proof. By Lemmas 11 and 12. ∎

Theorem 14 (Decidability) *The concept satisfiability problem for $\mathcal{PALC}$ is decidable.*

Proof. By decidability of the satisfiability problem for $\mathcal{ALC}$, for each concept C of $\mathcal{PALC}$, it is possible to decide if $f(C)$ is satisfiable in $\mathcal{ALC}$. Then, by Theorem 13, the satisfiability problem for $\mathcal{PALC}$ is decidable. ∎

The satisfiability problems of a TBox, an ABox and a knowledge base for $\mathcal{PALC}$ are also shown to be decidable.

Since f is a polynomial-time reduction, the complexities of the satisfiability problems of a TBox, an ABox and a knowledge base for $\mathcal{PALC}$ can be reduced to those for $\mathcal{ALC}$, i.e., the complexities of the problems for $\mathcal{PALC}$ are the same as those for $\mathcal{ALC}$. For example, the satisfiability problems of an acyclic TBox and a general TBox for $\mathcal{PALC}$ are PSPACE-complete and EXPTIME-complete, respectively.

For the concept satisfiability problem for $\mathcal{PALC}$, the existing tableau algorithms for $\mathcal{ALC}$ are applicable by using the translation f with Theorem 13.

4 Syntactical Embedding and Completeness

From a purely theoretical or logical point of view, a sound and complete axiomatization is required for the underlying semantics. In this section, we thus give a sound and complete tableau calculus $\mathcal{TAALC}$ for $\mathcal{PALC}$.

Definition 15 *A concept is called a* negation normal form *(NNF) if the classical negation connective $\neg$ occurs only in front of atomic concepts.*

Let $C(x)$ be a concept in NNF. In order to test satisfiability of $C(x)$, the tableau algorithm starts with the ABox $\mathcal{A} = \{C(x)\}$, and applies the inference rules of a tableau calculus to the ABox until no more rules apply.

Definition 16 ($\mathcal{TALC}$) *Let $\mathcal{A}$ be an ABox that consists only of NNF-concepts.*

The inference rules for the tableau calculus $\mathcal{TALC}$ for $\mathcal{ALC}$ are of the form:

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{C_1(x), C_2(x)\}}\ (\sqcap)$$

where $(C_1 \sqcap C_2)(x) \in \mathcal{A}$, $C_1(x) \notin \mathcal{A}$ or $C_2(x) \notin \mathcal{A}$,

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{C_1(x)\} \quad | \quad \mathcal{A} \cup \{C_2(x)\}}\ (\sqcup)$$

where $(C_1 \sqcup C_2)(x) \in \mathcal{A}$ and $[C_1(x) \notin \mathcal{A}$ and $C_2(x) \notin \mathcal{A}]$,

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{C(y)\}}\ (\forall R)$$

where $(\forall R.C)(x) \in \mathcal{A}$*,* $R(x,y) \in \mathcal{A}$ *and* $C(y) \notin \mathcal{A}$*,*

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{C(y), R(x,y)\}}\ (\exists R)$$

where $(\exists R.C)(x) \in \mathcal{A}$*, there is no individual name* z *such that* $C(z) \in \mathcal{A}$ *and* $R(x,z) \in \mathcal{A}$*, and* y *is an individual name not occurring in* $\mathcal{A}$*.*

Definition 17 *Let* $\mathcal{A}$ *be an ABox that consists only of NNF-concepts. Then,* $\mathcal{A}$ *is called* complete *if there is no more rules apply to* $\mathcal{A}$*.* $\mathcal{A}$ *is called* clash *if* $\{A(x), \neg A(x)\} \subseteq \mathcal{A}$ *for some atomic concept* $A(x)$*. A tree produced by a tableau calculus from* $\mathcal{A}$ *is called* complete *if all the nodes in the tree are complete. A branch of a tree produced by a tableau calculus from* $\mathcal{A}$ *is called* clash-free *if all its nodes are not clash.*

The following theorem is known.

Theorem 18 (Completeness) *For any* $\mathcal{ALC}$*-concept* C *in NNF,* $\mathcal{TALC}$ *produces a complete tree with a clash-free branch from the Abox* $\{C\}$ *iff* C *is satisfiable in* $\mathcal{ALC}$*.*

For $\mathcal{PALC}$-concepts, we use the same definition of NNF as that of $\mathcal{ALC}$-concepts, i.e., "negation" in the term NNF means "classical negation." The way of obtaining NNFs for $\mathcal{PALC}$-concepts is almost the same as that for $\mathcal{ALC}$-concepts, except that we also use the law: $\neg{\sim}C \leftrightarrow {\sim}\neg C$, which is justified by the fact: $(\neg{\sim}C)^{\mathcal{I}^+} = ({\sim}\neg C)^{\mathcal{I}^+}$.

Definition 19 ($\mathcal{TPALC}$**)** *Let* $\mathcal{A}$ *be an ABox that consists only of NNF-concepts.*

The inference rules for the tableau calculus $\mathcal{TPALC}$ *for* $\mathcal{PALC}$ *are obtained from* $\mathcal{TALC}$ *by adding the inference rules of the form:*

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{C(x)\}}\ ({\sim})$$

where ${\sim}{\sim}C(x) \in \mathcal{A}$*,* [2]

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{{\sim}C_1(x)\} \quad | \quad \mathcal{A} \cup \{{\sim}C_2(x)\}}\ ({\sim}\sqcap)$$

where $({\sim}(C_1 \sqcap C_2))(x) \in \mathcal{A}$ *and* $[{\sim}C_1(x) \notin \mathcal{A}$ *and* ${\sim}C_2(x) \notin \mathcal{A}]$*,*

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{{\sim}C_1(x), {\sim}C_2(x)\}}\ ({\sim}\sqcup)$$

where $({\sim}(C_1 \sqcup C_2))(x) \in \mathcal{A}$*,* ${\sim}C_1(x) \notin \mathcal{A}$ *or* ${\sim}C_2(x) \notin \mathcal{A}$*,*

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{{\sim}C(y), R(x,y)\}}\ ({\sim}\forall R)$$

[2] We do not use the condition: $C(x) \notin \mathcal{A}$ in $({\sim})$. This is from a technical reason. See the proof of Theorem 20.

where $(\sim\forall R.C)(x) \in \mathcal{A}$*, there is no individual name* z *such that* $\sim C(z) \in \mathcal{A}$ *and* $R(x, z) \in \mathcal{A}$*, and* y *is an individual name not occurring in* $\mathcal{A}$*,*

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{\sim C(y)\}} \ (\sim\exists R)$$

where $(\sim\exists R.C)(x) \in \mathcal{A}$*,* $R(x, y) \in \mathcal{A}$ *and* $\sim C(y) \notin \mathcal{A}$*.*

An expression $f(\mathcal{A})$ denotes the set $\{f(\alpha) \mid \alpha \in \mathcal{A}\}$.

Theorem 20 (Syntactical embedding) *Let* $\mathcal{A}$ *be an ABox that consists only of NNF-concepts in* $\mathcal{L}^{\sim}$*, and* f *be the mapping defined in Definition 9. Then:*

> $\mathcal{TPALC}$ *produces a complete tree with a clash-free branch from* $\mathcal{A}$ *iff* $\mathcal{TALC}$ *produces a complete tree with a clash-free branch from* $f(\mathcal{A})$

Proof. • $(\Longrightarrow)$: By induction on the complete trees T with a clash-free branch from $\mathcal{A}$ in $\mathcal{TPALC}$. We distinguish the cases according to the first inference of T. The base step is obvious. The induction step is considered below. We show only the following case.

Case $(\sim)$: The first inference of T is of the form:

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{C(x)\}} \ (\sim)$$

where $\sim\sim C(x) \in \mathcal{A}$. By induction hypothesis, $\mathcal{TALC}$ produces a complete tree with a clash-free branch from $f(\mathcal{A}) \cup \{f(C(x))\}$ with $f(\sim\sim C(x)) \in f(\mathcal{A})$. By the definition of f, we have $f(\sim\sim C(x)) = f(C(x))$, and hence $f(\mathcal{A}) \cup \{f(C(x))\} = f(\mathcal{A}) \in f(\mathcal{A})$. Therefore, $\mathcal{TALC}$ provides a complete tree with a clash-free branch from $f(\mathcal{A})$.

• $(\Longleftarrow)$: By induction on the complete trees T' with a clash-free branch from $f(\mathcal{A})$ in $\mathcal{TALC}$. We distinguish the cases according to the first inference of T'. We show only the following case.

Case $(\forall R)$: The first inference of T' is of the form:

$$\frac{f(\mathcal{A})}{f(\mathcal{A}) \cup \{f(\sim C(y))\}} \ (\forall R)$$

where $\forall R.f(\sim C(x)) \in f(\mathcal{A})$, $f(R(x, y)) \in f(\mathcal{A})$ and $f(\sim C(y)) \notin f(\mathcal{A})$. By induction hypothesis, $\mathcal{TPALC}$ provides a complete tree with a clash-free branch from $\mathcal{A} \cup \{\sim C(y)\}$. By the definition of f, we have $\forall R.f(\sim C(x)) = \forall f(R).f(\sim C(x)) = f(\sim\exists R.C(x))$ and $f(R(x, y)) = R(x, y)$. Thus, we obtain:

$$\frac{\mathcal{A}}{\mathcal{A} \cup \{\sim C(y)\}} \ (\sim\exists R).$$

Therefore, $\mathcal{TPALC}$ provides a complete tree with a clash-free branch from $\mathcal{A}$. ∎

Theorem 21 (Completeness) *For any $\mathcal{PALC}$-concept C in NNF, $\mathcal{TPALC}$ produces a complete tree with a clash-free branch from the Abox $\{C\}$ iff C is satisfiable in $\mathcal{PALC}$.*

Proof. Let C be a $\mathcal{PALC}$-concept in NNF. Then, we obtain:

$\mathcal{TPALC}$ produces a complete tree with a clash-free branch from $\{C\}$
iff $\mathcal{TALC}$ produces a complete tree with a clash-free branch from $\{f(C)\}$ (by Theorem 20)
iff $f(C)$ is satisfiable in $\mathcal{ALC}$ (by Theorem 18)
iff C is satisfiable in $\mathcal{PALC}$ (by Theorem 13).

∎

5 Remarks

We now explain about some differences and similarities among $\mathcal{ALC}4$ [5], quasi-classical description logics [18, 19] and $\mathcal{PALC}$. In $\mathcal{ALC}4$, a four-valued interpretation $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is defined using a pair $\langle P, N\rangle$ of subsets of $\Delta^{\mathcal{I}}$ and the projection functions $proj^{+}\langle P, N\rangle := P$ and $proj^{-}\langle P, N\rangle := N$. The interpretations of an atomic concept A and a conjunctive concept $C_1 \sqcap C_2$ are then defined as follows:

1. $A^{\mathcal{I}} := \langle P, N\rangle$ where $P, N \subseteq \Delta^{\mathcal{I}}$,
2. $(C_1 \sqcap C_2)^{\mathcal{I}} := \langle P_1 \cap P_2, N_1 \cup N_2\rangle$ if $C_i^{\mathcal{I}} = \langle P_i, N_i\rangle$ for $i = 1, 2$.

In quasi-classical description logics, a reformulation or simplification of the four-valued interpretations of $\mathcal{ALC}4$ is used: An interpretation is defined using a pair $\langle +C, -C\rangle$ of subsets of $\Delta^{\mathcal{I}}$ without using projection functions. The interpretations of an atomic concept A and a conjunctive concept $C_1 \sqcap C_2$ are then defined as follows:

1. $A^{\mathcal{I}} := \langle +A, -A\rangle$ where $+A, -A \subseteq \Delta^{\mathcal{I}}$,
2. $(C_1 \sqcap C_2)^{\mathcal{I}} := \langle +C_1 \cap +C_2, -C_1 \cup -C_2\rangle$.

The pairing functions used in the four-valued and quasi-classical semantics have been used in some algebraic semantics for Nelson's logics (see e.g. [10] and the references therein). On the other hand, the semantics of $\mathcal{PALC}$ is defined using two interpretation functions $\cdot^{\mathcal{I}^+}$ and $\cdot^{\mathcal{I}^+}$ instead of the pairing functions. These interpretation functions have been used in some Kripke-type semantics for Nelson's logics (see e.g. [17] and the references therein). The "horizontal" semantics using paring functions and the "vertical" semantics using two kinds of interpretation functions have thus essentially the same meaning.

Acknowledgments. I would like to thank Dr. Ken Kaneiwa and the referees for their valuable comments. This research was partially supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Young Scientists (B) 20700015.

References

1. A. Almukdad and D. Nelson, Constructible falsity and inexact predicates, *Journal of Symbolic Logic* 49, pp. 231–233, 1984.
2. F. Baader, D. Calvanese, D. McGuinness, D. Nardi and Peter F. Patel-Schneider (Eds.), The description logic handbook: Theory, implementation and applications, Cambridge University Press, 2003.
3. Y. Gurevich, Intuitionistic logic with strong negation, *Studia Logica* 36, pp. 49–59, 1977.
4. K. Kaneiwa, Description logics with contraries, contradictories, and subcontraries, *New Generation Computing* 25 (4), pp. 443–468, 2007.
5. Y. Ma, P. Hitzler and Z. Lin, Algorithms for paraconsistent reasoning with OWL, Proceedings of the 4th European Semantic Web Conference (ESWC 2007), LNCS 4519, pp. 399-413, 2007.
6. Y. Ma, P. Hitzler and Z. Lin, Paraconsistent reasoning for expressive and tractable description logics, Proceedings of the 21st International Workshop on Description Logic (DL 2008), CEUR Workshop Proceedings 353.
7. C. Meghini and U. Straccia, A relevance terminological logic for information retrieval, Proceedings of the 19th Annual International ACM SIGIR Conference on Research and Development in Information Retrieval, pp. 197–205, 1996.
8. C. Meghini, F. Sebastiani and U. Straccia, Mirlog: A logic for multimedia information retrieval, In: Uncertainty and Logics: Advanced Models for the Representation and Retrieval of Information, pp. 151–185, Kluwer Academic Publishing, 1998.
9. D. Nelson, Constructible falsity, *Journal of Symbolic Logic* 14, pp. 16–26, 1949.
10. S.P. Odintsov, Algebraic semantics for paraconsistent Nelson's logic, *Journal of Logic and Computation* 13 (4), pp. 453–468, 2003.
11. S.P. Odintsov and H. Wansing, Inconsistency-tolerant description logic: Motivation and basic systems, in: V.F. Hendricks and J. Malinowski, Editors, Trends in Logic: 50 Years of Studia Logica, Kluwer Academic Publishers, Dordrecht, pp. 301–335, 2003.
12. S.P. Odintsov and H. Wansing, Inconsistency-tolerant Description Logic. Part II: Tableau Algorithms, *Journal of Applied Logic* 6, pp. 343–360, 2008.
13. Peter F. Patel-Schneider, A four-valued semantics for terminological logics, *Artificial Intelligence* 38, pp. 319–351, 1989.
14. W. Rautenberg, Klassische und nicht-klassische Aussagenlogik, Vieweg, Braunschweig, 1979.
15. M. Schmidt-Schauss and G. Smolka, Attributive concept descriptions with complements, *Artificial Intelligence* 48, pp. 1–26, 1991.
16. U. Straccia, A sequent calculus for reasoning in four-valued description logics, Proceedings of International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX 1997), LNCS 1227, pp. 343–357, 1997.
17. H. Wansing, The logic of information structures, LNAI 681, 163 pages, 1993.
18. X. Zhang and Z. Lin, Paraconsistent reasoning with quasi-classical semantics in $\mathcal{ALC}$, Proceedings of the 2nd International Conference on Web Reasoning and Rule Systems (RR 2008), LNCS 5341, pp. 222–229, 2008.
19. X. Zhang, G. Qi, Y. Ma, Z. Lin, Quasi-classical semantics for expressive description logics, Proceedings of the 22nd International Workshop on Description Logic (DL 2009), CEUR Workshop Proceedings 477.