Tractability of the Crisp Representations of Tractable Fuzzy Description Logics

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Abstract. An important line of research within the field of fuzzy DLs is the computation of an equivalent crisp representation of a fuzzy ontology. In this short paper, we discuss the relation between tractable fuzzy DLs and tractable crisp representations. This relation heavily depends on the family of fuzzy operators considered.

Introduction. Despite the undisputed success of ontologies, classical ontology languages are not appropriate to deal with vagueness or imprecision in the knowledge, which is inherent to most of the real world application domains. As a solution, several fuzzy extensions of Description Logics (DLs) have been proposed in the literature. For a good survey we refer the reader to [1].

An important line of research within the field of fuzzy DLs is the computation of an equivalent crisp representation of a fuzzy ontology. This way, it is possible to reason with the obtained crisp ontology, making it possible to reuse classical ontology languages (e.g., OWL 2), DL reasoners, and other resources. It is possible to reason with very expressive fuzzy DLs, and with different families of fuzzy operators (also called *fuzzy logics*), namely Zadeh [2], Gödel [3], and Łukasiewicz [4]. To be precise, in Gödel and Łukasiewicz it is necessary to restrict to the finite case, i.e., where the set of degrees of truth is finite and fixed.

In the last years, there is a growing interest in the study of *tractable DLs*. In these logics, the expressive power is compromised for the efficiency of reasoning. In OWL 2, the current standard language for ontology representation, three fragments (called *profiles*) have been identified, namely *OWL 2 EL*, *OWL 2 QL*, and *OWL 2 RL* [5]. Table 1 shows the relation of some OWL 2 constructors and its fragments. In *OWL 2 EL* and *OWL 2 RL*, the basic reasoning tasks can be performed in a time which is polynomial with respect to the size of the ontology. In *OWL 2 QL*, conjunctive query answering can be performed in LogSpace with respect to the size of the assertions.

Sometimes, the crisp representation of a fuzzy KB enjoys the following property: given a fuzzy ontology $\mathcal{O}$ in a fuzzy DL language $\mathcal{X}$, the crisp representation of $\mathcal{O}$ is in the (crisp) DL $\mathcal{X}$. The objective of this paper is to determine in a precise way when this property is verified, focusing on the case of tractable fuzzy DLs, which is a very interesting case in real-world applications.

Definition 1. *A fuzzy DL language $\mathcal{X}$ is* closed under reduction *iff the crisp representation of a fuzzy ontology in $\mathcal{X}$ is in the (crisp) DL language $\mathcal{X}$.*

Table 1. Summary of the relation among OWL 2 and its three profiles.

OWL 2	OWL 2 EL	OWL 2 QL	OWL 2 RL
Class	✓	✓	✓
ObjectIntersectionOf	✓	restricted	✓
ObjectUnionOf			restricted
ObjectComplementOf		restricted	restricted
ObjectAllValuesFrom			restricted
ObjectSomeValuesFrom	✓	restricted	restricted
DataAllValuesFrom			restricted
DataSomeValuesFrom	✓	✓	restricted
...			
ObjectProperty	✓	✓	✓
DatatypeProperty	✓	✓	✓
...			
ClassAssertion	✓	✓	✓
ObjectPropertyAssertion	✓	✓	✓
SubClassOf	✓	✓	✓
SubObjectPropertyOf	✓	✓	✓
SubDataPropertyOf	✓	✓	✓
...			

In the following, we will assume that $\mathcal{X}$ is not more expressive than $\mathcal{SROIQ}(\mathbf{D})$.

Fuzzy DLs. We assume the reader to be familiar with fuzzy DLs [1]. We note that the many existing proposals usually differ in syntax, semantics, and logical properties. In this paper, we consider fuzzy DLs with the following features:

- Concepts and roles are syntactically the same as in the crisp case.
- Axioms are syntactically the same as in the crisp case, with the exception of concept assertions, role assertions, general concept inclusions (GCIs), and role hierarchies, where a crisp axiom τ is extended with a lower bound as $\langle \tau \rhd \alpha \rangle$, with $\rhd \in \{\geq, >\}$, and $\alpha \in [0, 1]$. For instance, $\langle a : C \sqcap D \geq 0.6 \rangle$ means that the concept assertion $a : C \sqcap D$ is true with degree at least 0.6.
- The semantics of classes, properties and axioms depends on some fuzzy logical operators, namely a t-norm, a t-conorm, a negation, and an implication. For instance, the semantics of the conjunction is given by a t-norm. Fuzzy DLs with different fuzzy operators have many different logical properties.

Crisp representations of fuzzy DLs. The basic idea of the crisp representation is to use some basic crisp concepts and roles, representing the α-cuts of the fuzzy concepts and roles. To keep the semantics of the α-cuts, some axioms must be introduced, namely GCIs and role hierarchies. Finally, every axiom of the fuzzy ontology is represented, independently from other axioms, using these basic crisp elements. An important property of these crisp representations is that, although the number of axioms in the TBox and the RBox increase, the number of axioms in the ABox is constant. Let us illustrate this with an example.

Example 1. Assume that a fuzzy ontology $\mathcal{K}$ includes the set of axioms $\{\langle a : \exists R.C \geq 0.6 \rangle, \langle a : \neg \exists R.C > 0.8 \rangle\}$. The crisp representation of the ontology must consider the crisp concepts $C_{\geq 0.6}, C_{\geq 0.8}$, and the crisp roles $R_{\geq 0.6}, R_{\geq 0.8}$, which produce the GCI $C_{\geq 0.8} \sqsubseteq C_{\geq 0.6}$ and the role hierarchy $R_{\geq 0.8} \sqsubseteq R_{\geq 0.6}$. Assuming that the t-norm is the minimum and the negation is the standard (Łukasiewicz), the crisp representation of the axioms is $\{a : \exists R_{\geq 0.6}.C_{\geq 0.6}, a : \forall R_{\geq 0.8}.(\neg C_{\geq 0.8})\}$.

The case of Zadeh fuzzy logic. The full details of the crisp representation in Zadeh $\mathcal{SROIQ}(\mathbf{D})$ can be found in [2]. Zadeh logic makes it possible to obtain smaller crisp representations than with Gödel and Łukasiewicz logics. For instance, in Zadeh logic, from $\langle a : C \sqcap D \geq 0.6\rangle$ we can deduce both $\langle a : C \geq 0.6\rangle$ and $\langle a : D \geq 0.6\rangle$. However, in Łukasiewicz logic, this is not possible, and we have to build a disjunction over all the possibilities. In Gödel implication, we have a similar problem. In the case of Zadeh logic, we have the following property:

Property 1. In Zadeh fuzzy logic, a fuzzy DL language $\mathcal{X}$ is closed under reduction iff it includes GCIs and role hierarchies. □

The proof of this property is trivial from the crisp representation [2]. This result applies, for instance, to logics more expressive than $\mathcal{ALCH}$, such as $\mathcal{SROIQ}(\mathbf{D})$. Furthermore, it also applies to the DLs that are equivalent to the profiles *OWL 2 EL*, *OWL 2 QL*, and *OWL 2 RL* (see Table 1).

Example 2. Consider again the fuzzy ontology $\mathcal{K}$ from Example 1, and assume that the language of $\mathcal{K}$ is $\mathcal{ALC}$. Since $\mathcal{ALC}$ does not contain role hierarchies, the second condition of Property 1 fails, and hence fuzzy $\mathcal{ALC}$ is not closed under reduction. This is intuitive, because the crisp representation contains role hierarchies ($R_{\geq 0.8} \sqsubseteq R_{\geq 0.6}$). □

The case of Gödel fuzzy logic. The full details of the crisp representation in Gödel $\mathcal{SROIQ}(\mathbf{D})$ can be found in [3]. This case is very similar to the previous one. In fact, using a similar reasoning, it can be seen that the following property is verified by the three OWL 2 profiles.

Property 2. In Gödel fuzzy logic, a fuzzy DL language $\mathcal{X}$ is closed under reduction iff it verifies each of the following conditions:

- $\mathcal{X}$ includes GCIs.
- $\mathcal{X}$ includes role hierarchies.
- If $\mathcal{X}$ includes universal (all) restrictions, then it also include conjunction. □

The case of Łukasiewicz fuzzy logic. The full details of the crisp representation in Łukasiewicz $\mathcal{ALCHOI}$ can be found in [4].

Property 3. In Łukasiewicz fuzzy logic, a fuzzy DL language $\mathcal{X}$ is not closed under reduction if it verifies some of the following conditions:

- $\mathcal{X}$ does not include GCIs.
- $\mathcal{X}$ does not include role hierarchies.
- $\mathcal{X}$ includes one and only one of the constructors disjunction and conjunction.

- $\mathcal{X}$ includes existential (some) restrictions, but it does not include disjunction.
- $\mathcal{X}$ includes universal (all) restrictions, but it does not include conjunction.

□

Again, the proof of this property is trivial from the crisp representation [4]. The three OWL 2 profiles verify this property. *OWL 2 EL* and *OWL 2 QL* support conjunction but not disjunction (see Table 1); and *OWL 2 RL* allows intersection as a superclass expression, but does not allow disjunction there [5].

Note that this property is formulated in a different way. The reason is that a crisp representation for a fuzzy DL more expressive than $\mathcal{ALCHOI}$ is still unknown. Hence, rather than a general result, we only have a partial one.

Size of the crisp representations. In Zadeh and Gödel *OWL 2 QL* we obtain a crisp ontology where the ABox has the same number of axioms as the original fuzzy ABox. Hence, tractability is preserved, since the complexity of reasoning depends on the number of assertions.

In Zadeh and Gödel *OWL 2 EL* and *OWL 2 RL*, we obtain a crisp ontology in a tractable language. However, the TBox and the RBox are larger than in the original fuzzy ontology. This increase in the size is an issue to consider when dealing with tractable fuzzy DLs from a practical point of view, as reasoning depends on the size of the ontology.

In Gödel *OWL 2 QL*, a fuzzy universal restriction is mapped into a (crisp) conjunction of universal restrictions. Hence, the resulting ontology is bigger than in the Zadeh case. This does not happen in *OWL 2 EL* nor in *OWL 2 QL*, as they do not allow universal restrictions (see Table 1).

In tractable fuzzy DLs, it is specially important to use optimized crisp representations. For instance, domain and range restrictions can be treated as GCIs, but their crisp representation are more efficient if treated as special cases [2].

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