Completeness Proof Strategies for Euler Diagram Logics

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Abstract. Visual logics based on Euler diagrams have recently been developed, including generalized constraint diagrams and concept diagrams. Establishing the metatheories of these logics includes providing completeness proofs where possible. Completeness has been established for such logics, including Euler diagrams, spider diagrams and a fragment of the constraint diagram logic. In this paper, we identify commonality in their completeness proof strategies, showing how, as expressiveness increases, the strategy readily extends. We identify a fragment of concept diagrams and demonstrate that the completeness proof strategy does not extend to this fragment. Thus, we have established that the existing completeness proof strategies are limited. Consequently, we examine the challenge of devising new approaches to proving completeness in more expressive logics.

1 Introduction

There has been a lot of recent interest in logics that, in various ways, extend Euler diagrams. This interest was sparked by pioneering work in the mid 1990s, by Hammer [3] and Shin [9]. Hammer developed a very simple sound and complete Euler diagram logic, whereas Shin devised a logic, called Venn-II, that was more expressive than Euler diagrams and which she also proved to be sound and complete. Since these early days we have seen the development of diagrammatic logics with ever-increasing levels of expressiveness. Amongst these logics, perhaps the most studied is that of spider diagrams, introduced by Gil et al. [2], which arose from Kent's constraint diagram logic [6], formalised in [1]. Building on from the complete systems of Hammer and Shin, spider diagrams have been shown to be complete [4], as has a fragment of the constraint diagram logic [11]. Other related logics include the Euler/Venn system of Swoboda and Allwein [13] and the Euler system of Mineshima et al. [7].

One reason that significant emphasis has been placed on deriving completeness results for logics is that completeness means that the logic is capable of proving all theorems expressible within the logic. Formally, a theorem is a statement that semantically follows from a set of statements formulated in the logic, called axioms. In the case of diagrammatic logics, the set of axioms is a set of diagrams and a theorem is a diagram whose informational content is derivable from the axioms. For a theorem to be provable from the axioms, we need to be

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able to apply so-called *inference rules*, which are (informally) transformations that alter the syntax of the axioms, until we obtain the theorem.

This paper has two key parts. First, in section 2, we will demonstrate that there are substantial similarities in existing completeness proof strategies for Euler-based diagrammatic logics, with the result that we can consider the strategies to be variations on a single approach. In section 3 we describe the task of extending the proof strategy to a fragment of concept diagrams and show that the strategy breaks down. We examine the factors whose interaction prevents the ready extension of the strategy and show that completeness proofs for more expressive notations will require a different approach. We conclude in section 4 by describing some of the approaches that may be taken to finding suitable new strategies.

2 Completeness Strategies for Euler Diagram Logics

There have been a number of sound and complete logics based on Euler diagrams developed to date. All of the proofs of completeness have used constructive strategies, providing a proof that the theorem follows from the axioms (in fact, those strategies we demonstrate are restricted to a single axiom). Moreover, they all adopt a similar framework, converting the diagrams involved into normal forms that are easily comparable. As we shall demonstrate in this section, the completeness proof for each considered logic is an extension of the completeness proofs for its fragments. We show this by detailing the strategies used for a hierarchy of increasingly expressive logics: Euler diagrams [3], spider diagrams [4] and, briefly, constraint diagrams as considered in [11].

2.1 Euler Diagrams

Euler diagrams, as investigated by Hammer [3], are the simplest logic that we will consider. They comprise closed curves, each with a label. In any given diagram, no two distinct curves have the same label. Examples can be seen in figure 1, where d expresses that (the sets) A and C are disjoint, B is a subset of A, and D is a subset of C. The diagram d' expresses that D is a subset of C. Whilst the curves (labelled) A and E are present in d', no information is given about the relationship of the sets they represent to C and D.

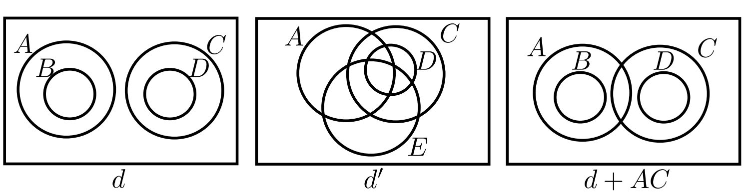

Fig. 1. Three Euler diagrams.

Hammer's logic contains just three inference rules: *Erasure* (of a curve), *Introduction of a New Curve*, and *Weakening* which allows new regions to be added; Weakening is illustrated in figure 1, where $d + AC$ is obtained from d by adding a region inside both A and C. To prove completeness of this logic, Hammer proceeds by constructing a proof-writing algorithm: given an axiom d and a theorem d', carry out the following steps to prove d' follows from d:

1. *Apply the Introduction of a New Curve rule*, adding one curve labelled L for each curve label, L, in d' that is not in d, to give a diagram d_c.
2. *Apply the Erasure rule*, erasing all curves from d_c that have labels not appearing in d', to give a diagram d_e.
3. *Apply the Weakening rule*, adding minimal regions to d_e until it is the same as d'.

The proof of completeness involves showing that it is possible to apply this algorithm whenever $d \vDash d'$ (i.e. d semantically entails d'), thus establishing that $d \vdash d'$ (i.e. there is a proof that d' follows from d). We observe that the first step of this proof can be considered as kind of *maximising* step: syntax is added to the axiom that is used in the theorem. The last step of the proof also adds syntax. In fact, we can interchange the last two steps without significantly impacting the details of the completeness proof. Thus, if we add minimal regions before erasing curves we would genuinely have *maximised* the syntax in the axiom diagram so that only inference steps that erase syntax are required in order to obtain d'. In what follows, we denote the maximised version of d by d_{max}, and we have, instead:

1. *Apply the Introduction of a New Curve rule*, adding one curve labelled L for each curve label, L, in d' that is not in d, to give a diagram d_c; we can similarly obtain d'_c, which we will use to determine inference rule applications at the next step.
2. *Apply the Weakening rule*, adding minimal regions to d_w until it has the same the same minimal regions as d'_c, to obtain $d_w = d_{max}$, the maximised version of d.
3. *Apply the Erasure rule*, erasing all curves from d_{max} that have labels not appearing in d', to give a diagram d_e. Then $d_e = d'$.

That is, we have:

$$d \vdash d_c \vdash d_w = d_{max} \vdash d_e = d'.$$

To determine which minimal regions to add to obtain d_{max}, we constructed a diagram, d'_c from d' by adding the curves with labels that occur in d but not in d'. Then d_c and d'_c have the same curve labels, so the minimal regions are immediately comparable; we add regions that are in d'_c but are not in d_c, thus maximising the syntax to get d_{max}. As we shall see, this concept of maximising the syntax in the axiom diagram is a recurring theme in subsequently developed completeness proofs. The completeness proof strategy is illustrated in figure 2, where we show $d \vdash d'$.

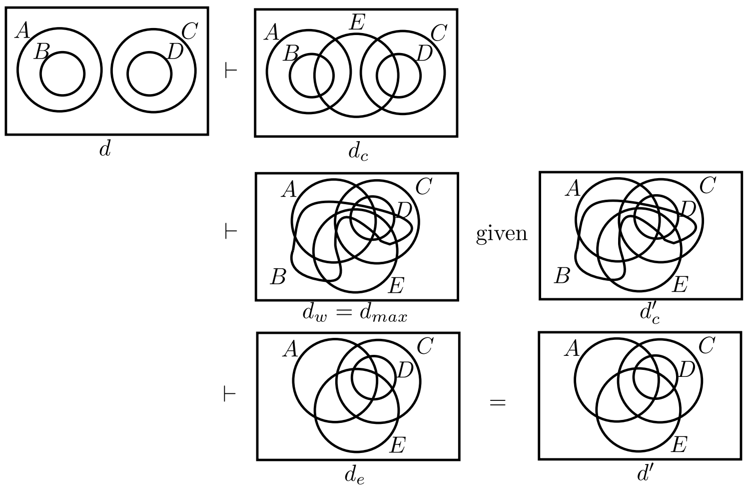

Fig. 2. Proving $d \vdash d'$.

In order to illustrate the extension of this strategy to more expressive systems, including spider diagrams in the next subsection, we formalize the notion of maximal forms. First, we define an (abstract) Euler diagram:

Definition 1. *An **Euler diagram** is a pair, $d = (L, R)$, where L is a finite set of curve labels and $R \subseteq \{(in, L - in) : in \subseteq L\}$ is a finite set of regions.*

So, d in figure 2 is, formally, $d = (L, R)$ where $L = \{A, B, C, D\}$ and

$$R = \{(\emptyset, \{A, B, C, D\}), (\{A\}, \{B, C, D\}), (\{A, B\}, \{C, D\}), (\{C\}, \{A, B, D\}), (\{C, D\}, \{A, B\})\}.$$

For example, $(\{A\}, \{B, C, D\})$ corresponds to the region inside the curve labelled A but outside the curves labelled B, C, and D[1].

Now, going back to the completeness proof strategy, we have seen that d_{max} is created by constructing the diagram d'_c which does not formally comprise part of the proof that $d \vdash d'$; in figure 2 we have $d_{max} = d'_c$ when $d \vdash d'$. We define the maximal form as follows:

Definition 2. *Let $d_c = (L, R)$ and $d'_c = (L', R')$ be Euler diagrams such that $L = L'$. The diagram d_c is **maximal** with respect to d'_c provided $R' \subseteq R$.*

It can be shown, given that d_c is maximal with respect to d'_c, $d \vDash d'_c$ if and only if $R = R'$. In terms of the completeness proof strategy, this means that $d_{max} = d'_c$. Our re-ordering of the steps in Hammer's completeness proof can now be informally justified. Firstly, to d_c we add precisely the minimal regions in d'_c that are not in d_c to give $d_w = d_{max}$. Since d'_c is semantically equivalent to d' and $d'_c = d_{max}$, it should be easy to see that we can then merely delete curves from d_{max} to give d', establishing completeness.

[1] The elements of R are often called *zones* but in this paper we call them regions for consistency with Hammer's work.

2.2 Spider Diagrams

Spider diagrams extend the Euler diagram logic of Hammer in two distinct ways: they are augmented with (a) trees, called *spiders*, and shading, both of which are used within diagrams to place constraints on set cardinality, and (b) logical connectives which are used to allow more complex expressions to be formed. Whilst Euler diagrams form a very simple monadic first-order logic, spider diagrams take the level of expressiveness to monadic first-order logic with equality [12].

Examples of spider diagrams can be seen in figure 3 where, in addition to the information provided by the underlying Euler diagram, d_1 expresses – using spiders – that there are at least two elements, one of which is in B and the other of which is in $B \cup D$. Diagram d_1 also expresses – using shading – that no further elements are in B. Here, each of the spiders (one of which comprises a single node) represents the existence of an element. The shading in a region, r, expresses that all elements in the set represented by r must be represented by spiders. The spider diagram $d_2 \vee d_3$ is semantically equivalent to d_1.

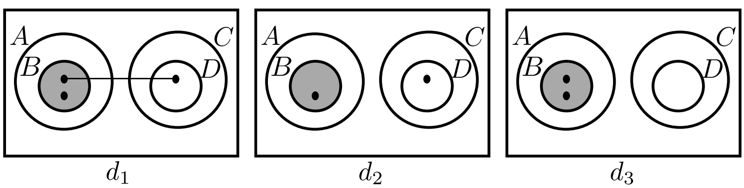

Fig. 3. Three spider diagrams.

The completeness proof strategy for spider diagrams, from [4], starts with axiom d and theorem d', so $d \vDash d'$, and, as with Hammer's approach, constructs a proof to show that $d \vdash d'$. In brief, the process starts off by converting d to a normal form where the only logical connective used is $\vee$ and the spiders each comprise just a single node, giving a diagram we will denote by d_{NF} (*NF* for Normal Form). Of note is that the construction of d_{NF} includes some of the steps we need to maximise syntax in the axiom: all of the so-called *unitary* diagrams contain all of the curve labels that occur somewhere in either the axiom or theorem. Unitary diagrams are spider diagrams which do not involve any logical connectives. In addition, the unitary diagrams in this normal form contain the same sets of regions. For Euler diagrams we constructed d'_c to direct which regions we needed to add. The same approach is used for spider diagrams: we convert diagram d' to, in this case, d'_{NF} in order to allow us to identify which inference rules to apply to d_{NF} to give d'_{NF} and, subsequently, to obtain d' (which is both syntactically and semantically equivalent to d'_{NF}).

Since the rules applied to convert to normal forms are equivalences, we see that if it can be shown that $d_{NF} \vdash d'_{NF}$, then we have established

$$d \vdash d_{NF} \vdash d'_{NF} \vdash d'.$$

We focus on the part of the completeness proof that establishes $d_{NF} \vdash d'_{NF}$. Since d_{NF} is in normal form, by definition this means that

$$d_{NF} = \bigvee_{1 \le i \le n} d_i,$$

where each d_i is a unitary spider diagram containing only spiders that are single nodes. Similarly,

$$d'_{NF} = \bigvee_{1 \le i \le m} d'_i.$$

Returning to our consideration of regions, these normal forms ensure that, for each d_i and d'_j in d_{NF} and d'_{NF} respectively, the sets of regions are the same. That is, if we consider the underlying Euler diagrams, $L_i = L'_j$ and $R_i = R'_j$. So, the 'Euler part' of d_i is maximal with regard to the 'Euler part' of d_j and we have the right 'Euler conditions' for semantic entailment (i.e. if there were no spiders or shading then $d_i \vdash d'_j$). What remains is to consider the effects of spiders and shading. By comparing d_{NF} and d'_{NF}, an inference rule can be applied to d_{NF} in order to add spiders and shading to its components, increasing the number of diagrams in the disjunction, until it can be established that each unitary diagram, d_i, in the axiom logically entails a unitary diagram, d'_j, in the theorem. In this sense, the completeness proof strategy for spider diagrams maximizes the syntax in the axiom by adding curves (to get the 'right' curve label set), adding regions, and finally adding spiders and shading. Similar to the Euler diagram case, once this maximal form is achieved it is merely a matter of erasing syntax from d_i to obtain d'_j. We have $d_i \vdash d'_{NF}$ (by using an inference rule analogous to $P \vdash P \vee Q$ in propositional logic). Subsequently, it can be trivially shown that $d_{NF} \vdash d'_{NF}$, as required. We refer to [4] for full details.

For our purposes, it is sufficient for us to now define unitary spider diagrams where spiders comprise only single nodes, and to extend the definition of maximal to this case.

Definition 3. *A **spider diagram** is a tuple, $d = (L, R, R^*, S, \eta)$, where (L, R) is an Euler diagram, $R^* \subseteq R$ is a set of shaded regions, S is a finite set whose elements are called spiders and $\eta \colon S \to R$ is a function that identifies the region in which each spider is placed.*

In figure 3, the spider diagram d_2 has d of figure 2 as its underlying Euler diagram for which we previously specified L and R. In addition, there is one shaded region, and we have $R^* = \{(\{A, B\}, \{C, D\})\}$, two spiders, so $S = \{s_1, s_2\}$, and these spiders are placed in regions as given by $\eta(s_1) = (\{A, B\}, \{C, D\})$ and $\eta(s_2) = (\{C, D\}, \{A, B\})$. For diagrams with spiders comprising single nodes, the definition of maximal is as follows:

Definition 4. *Let $d = (L, R, R^*, S, \eta)$ and $d' = (L', R', R^{*\prime}, S', \eta')$ be spider diagrams such that $L = L'$. The diagram d is **maximal** with respect to d' provided $R' = R$, $R^{*\prime} \subseteq R^*$ and there exists an injection, $f \colon S' \to S$ such that for each $s' \in S'$, $\eta'(s') = \eta(f(s'))$.*

The definition of maximal given for spider diagrams generalizes that for Euler diagrams[2]. Intuitively, our definition of maximal is saying that *everything that occurs in d' also occurs in d*. The next lemma follows from a similar result in [4] (essentially restated here using our terminology):

Lemma 1. *Let $d = (L, R, R^*, S, \eta)$ and $d' = (L', R', R^{*\prime}, S', \eta')$ be spider diagrams such that $L = L'$ and $R = R'$. Suppose d is maximal with respect to d'. Then $d \vDash d'$ if and only if for each shaded region, r', in $R^{*\prime}$, the number of spiders in r' in both diagrams is the same.*

Theorem 1. *Let $d = (L, R, R^*, S, \eta)$ and $d' = (L', R', R^{*\prime}, S', \eta')$ be spider diagrams such that $L = L'$ and $R = R'$. If d is maximal with respect to d' and $d \vDash d'$ then $d \vdash d'$.*

Proof (Sketch). By lemma 1, each region that is shaded in d' contains the same number of spiders in d. Thus we can erase shading from d until $R^* = R^{*\prime}$, obtaining d_i, then remove spiders from d_i, that are not mapped to by the injective function $f: S(d') \to S(d)$. Finally, rename the spiders to obtain d'.

Theorem 2. *Let $d = (L, R, R^*, S, \eta)$ and $d' = (L', R', R^{*\prime}, S', \eta')$ be spider diagrams such that $L = L'$ and $R = R'$. If $d \vDash d'$ then d is maximal with respect to d'.*

Proof (Sketch). The proof is by contradiction. Suppose $d \vDash d'$ but d is not maximal with respect to d'. Then either $R^{*\prime} \nsubseteq R^*$ or there is no suitable injection, f, from the spiders of d' to those of d. In the first case, d' contains a shaded region, r, which is non-shaded in d. Then d' asserts that the set represented by r contains exactly n elements, where n is the number of spiders in r in d'. However, d allows the set represented by r to contain $n+1$ elements, so $d \nvDash d'$. In the second case, where there is no suitable injection, f, there is a region, r', in d' that contains more spiders than in d. Here, d asserts that the set represented by r' contains at least n elements, where n is the number of spiders in r' in d. However, d' asserts that the set represented by r' contains at least $n+j$ elements, where j is the number of 'extra' spiders in r' in d'. Again, $d \nvDash d'$. Thus, since $d \vDash d'$, the conditions for maximality must be satisfied.

We now have a completeness result concerning the fragment of the spider diagram logic that we have defined:

Theorem 3 (Completeness). *Let $d = (L, R, R^*, S, \eta)$ and $d' = (L', R', R^{*\prime}, S', \eta')$ be spider diagrams such that $L = L'$ and $R = R'$. If $d \vDash d'$ then $d \vdash d'$.*

Proof. If $d \vDash d'$ then, by theorem 2, d is maximal with respect to d'. By theorem 1, $d \vdash d'$.

To summarize, whilst the overall strategy is more complex, we still have a process of adding syntax to the axiom in order to maximize it with respect to the theorem we are aiming to prove.

[2] Note that for Euler diagrams we stipulated $R' \subseteq R$, but for spider diagrams we have $R = R'$. This difference is not significant; it merely makes the details of our argument more straightforward.

2.3 Constraint Diagrams

Constraint diagrams build on spider diagrams by adding further syntax, in particular arrows, to place constraints on binary relations. The system developed in [11] was shown to be sound and complete, with the completeness proof strategy directly extending that for spider diagrams. We omit formal definitions of these diagrams and maximal forms, and just illustrate the concepts by example.

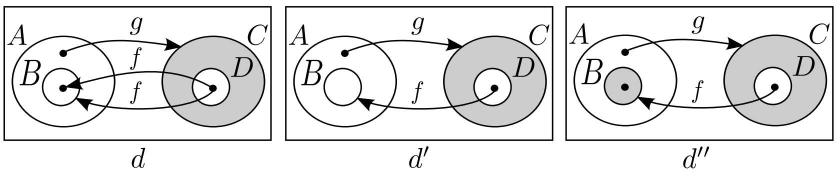

Fig. 4. Constraint diagrams: maximal forms and completeness.

First, to give a brief introduction to the meaning of arrows, consider d in figure 4. The arrow labelled g tells us that (the element represented by) the spider at its source is related to precisely the elements in C, the target, under (the relation represented by) g. Similarly, the spider in D is related to the unique element in B under f, and no other elements. In this example, d is maximal with respect to both d' and d''. Here, $d \vDash d'$ but $d \nvDash d''$. In the case of d and d', we can injectively map the spiders from d' to d in such a manner that the regions in which they are placed match. That is, there is an injective function $f \colon S' \to S$ where $\eta(s') = \eta(f(s'))$ and f ensures that the induced function $g \colon A' \to A$ is also an injection, where A' and A are the sets of arrows for d' and d respectively. Arrows are of the form $(\mathit{label}, \mathit{source}, \mathit{target})$ and $g(l, s, t) = (l, f(s), f(t))$ in the case where s and t are both spiders. We then delete shading, along with spiders and arrows that are not mapped to by f and g, from d to obtain d'.

Similar functions exist for d and d'', but this time we cannot apply inference rules to erase syntax from d to give d''. We would need to erase a spider from a shaded region, which is not sound; as with spider diagrams, the numbers of spiders in the shaded regions of the theorem, d', must match those in the axiom, d. Similarly to the spider diagram case, the number of spiders in the shaded regions of the theorem must be the same as in the axiom. These examples give the idea of the maximal forms used in constraint diagrams. We refer the reader to [11] for the full details which are too complex to illustrate in full here.

3 Concept Diagrams: The End of the Strategy?

We now consider extending the proof strategy described in the previous section to concept diagrams, which are intended to be used to model ontologies; see [5] for a practical example. They were introduced by Oliver et al. in 2009 [8] and extend constraint diagrams. Concept diagrams may include unlabelled curves, which we call *anonymous curves* and which represent anonymous subsets of the universal set. These provide an increase in expressiveness over notations such as constraint diagrams. As with our consideration of spider diagrams, we only permit spiders to comprise single nodes. Thus, taking a concept diagram from this fragment and removing its arrows and anonymous curves yields a spider diagram from the fragment defined in section 2.2.

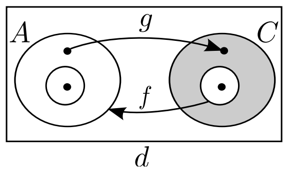

Fig. 5. A concept diagram.

The diagram in figure 5 is a concept diagram. The part of the diagram made up of labelled curves, shading and spiders is a spider diagram. The arrows provide information about binary relations. The diagram d expresses the following, in addition to the information given in the underlying spider diagram:

1. there are two sets, x and y, the former is a subset of A and the latter is a subset of C,
2. the image of the relation f, when its domain is restricted to y, is A,
3. there is an element, a, in $A - x$ such that the image of the relation g, when its domain is restricted to a, is the element in $C - y$.

We now present the syntax of the fragment of concept diagrams under consideration, adapted from [10].

Definition 5. *A* ***unitary concept diagram*** *is a tuple* $d = (L, C, R, R^*, S, \eta, A)$*, where*

1. *$L = L(d)$ is a finite set whose elements are called labelled curves,*
2. *$C = C(d)$ is a finite set whose elements are called anonymous curves,*
3. *$R = R(d)$ is a set of regions such that*
$$R \subseteq \{(in, (L \cup C) - in) : in \subseteq L \cup C\}.$$
4. *$R^* = R^*(d) \subseteq R$ is a set of shaded regions.*
5. *$S = S(d)$ is a finite set whose elements are called spiders,*
6. *$\eta = \eta_d \colon S \to R$ is a function that returns the location of each spider.*
7. *$A = A(d)$ is a finite set of arrows, each of the form (l, s, t), where l is the label, $s \in L \cup C$ is the source and $t \in S \cup L \cup C$ is the target.*

If $d = (L, C, R, R^*, S, \eta, A)$ is a concept diagram and $C = \emptyset$ then $d_s = (L, R, R^*, S, \eta)$ is a spider diagram. Semantics are assigned to concept diagrams similarly to the previous notations discussed: in addition to the usual interpretation of the underlying spider diagram, the arrows of a concept diagram place restrictions on binary relations and the anonymous curves represent the existence of sets as illustrated in our examples[3].

In order to extend the strategy discussed in the previous section to concept diagrams, we first extend the definition of maximality:

Definition 6. *Let $d = (L, C, R, R^*, S, \eta, A)$ and $d' = (L', C', R', R^{*\prime}, S', \eta', A)$ be concept diagrams such that $L = L'$. The diagram d is* ***maximal*** *with respect to d' provided:*

1. *there exists a bijection $g\colon L' \cup C' \to L \cup C$ such that*
   (a) *g is the identify map when its domain is restricted to L',*
   (b) *g induces a bijection $h\colon R' \to R$, defined by $h(in, out) = (in', out')$, where*
      i. *$in' = \{g(c') : c \in in\}$, and*
      ii. *$out' = \{g(c') : c' \in out\}$,*
      which ensures for each $(in, out) \in R^{*\prime}$, $h(in, out) \in R^*$,
2. *there exists an injection, $f\colon S' \to S$ such that for each $s' \in S'$, $\eta'(s') = \eta(f(s'))$, and*
3. *g and f induce an injection $p\colon A' \to A$ defined by*

$$p(l, s, t) = \begin{cases} (l, g(s), g(t)) & \text{if } s, t \in L' \cup C' \\ (l, g(s), f(t)) & \text{if } s \in L' \cup C' \wedge t \in S' \\ (l, f(s), g(t)) & \text{if } s \in S' \wedge t \in L' \cup C' \\ (l, f(s), f(t)) & \text{if } s, t \in S'. \end{cases}$$

Equipped with the definition of maximality, we can examine how to generalize the lemma and theorems from section 2.2 in order to extend the strategy. We start by considering the equivalent of lemma 1:

Conjecture 1. Let $d = (L, C, R, R^*, S, \eta, A)$ and $d' = (L', C', R', R^{*\prime}, S', \eta', A')$ be concept diagrams such that $L = L'$. Suppose d is maximal with respect to d'. Then $d \vDash d'$ if and only if for each shaded region, r', in $R^{*\prime}$, the number of spiders in r' is the same as the number of spiders in $h(r')$ in d.

Figure 6 shows a counterexample to conjecture 1. First, it is obvious that d_1 is maximal with respect to d_2. Here, d_1 tells us that there exists a set containing exactly two elements. From this we can deduce that there exists a set containing exactly one element. That is, d_2 follows logically from d_1. The shaded region in d_2 contains *fewer* spiders than in d_1. In both these diagrams, we see that the shading is actually redundant: removing the shading does not alter the informational content of the diagrams.

[3] Here, we note that our representation of concept diagrams assumes that spiders represent the existence of elements; strictly, in concept diagrams, spiders act as free variables. Formally, unitary diagrams as we have defined them would need to be prefixed by existential quantifies (one for each spider) to get our interpretation. However, to avoid diagram clutter, we simply omit the existential quantifiers since no ambiguity arises.

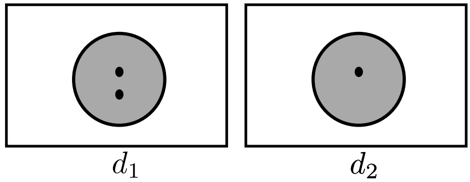

Fig. 6. The spiders in the shaded regions do not match.

Clearly, such problems concerning spiders and shading impact our ability to obtain a completeness result for the fragment under consideration. In order to obtain completeness, we need inference rules that allow us to identify when (a) shading is redundant, (b) we can delete spiders from shaded regions, and (c) when anonymous curves are redundant. Worthy of note is that the diagrams in figure 6 are semantically equivalent to spider diagrams (on removing the anonymous curves, the informational content is unaltered). Thus, the problems here arise from the syntactic richness of the notation and are not merely because of an increase in expressive power.

To proceed with our exposition of problems that arise when attempting to extend the previously used proof strategies to concept diagrams, we extend the definition of maximal to incorporate the condition on shading given in conjecture 1:

Definition 7. *A concept diagram d is **strongly maximal** with respect to d' provided*

1. *d is maximal with respect to d', and*
2. *for each shaded region, r', in $R^{*\prime}$, the number of spiders r' is the same as the number of spiders in $h(r')$ in d.*

By doing this, we are following the standard mathematical process of applying further constraints to a conjecture for which we have found a counterexample. In fact, conjecture 1 is trivially true in the strongly maximal case. As a consequence, our attempts to extend the completeness strategy apply to a smaller fragment of concept diagrams.

Next, we consider extending theorem 1 to concept diagrams:

Conjecture 2. Let $d = (L, C, R, R^*, S, \eta, A)$ and $d' = (L', C', R', R^{*\prime}, S', \eta', A')$ be concept diagrams such that $L = L'$. If d is strongly maximal with respect to d' and $d \vDash d'$ then $d \vdash d'$.

To establish the truth of conjecture 2, we start by defining the inference rules which are needed to establish $d \vdash d'$, if d is strongly maximal with respect to d' and $d \vDash d'$.

Inference rule 1: Remove arrow. Let $d = (L, C, R, R^*, S, \eta, A)$ be a concept diagram and let $a \in A$ be an arrow in d. Let $d' = (L, C, R, R^*, S, \eta, A - \{a\})$ be the diagram obtained by removing a from d. Then d logically entails d'.

Inference rule 2: Remove shading. Let $d = (L, C, R, R^*, S, \eta, A)$ be a concept diagram and let $r^* \in R^*$ be a shaded region in d. Let $d' = (L, C, R, R^* - \{r^*\}, S, \eta, A)$ be the diagram obtained by removing the shading from r^* in d. Then d logically entails d'.

Inference rule 3: Remove spider. Let $d = (L, C, R, R^*, S, \eta, A)$ be a concept diagram and let $x \in S$ be a spider with an non-shaded location in d. Let $d' = (L, C, R, R^*, S - \{x\}, \eta, A)$ be the diagram obtained by removing x from d. Then d logically entails d'.

Inference rule 4: Substitute spider Let $d = (L, C, R, R^*, S, \eta, A)$ be a concept diagram and let $x \in S$. Let y be a spider not in S. Let $d' = (L, C, R, R^*, (S - \{x\}) \cup \{y\}, (\eta - \{(x, \eta(x)\}) \cup \{(y, \eta(x)\}, A)$ be the diagram obtained by replacing x with y in d_1. Then d is logically equivalent d'.

Inference rule 5: Substitute anonymous curve Let $d = (L, C, R, R^*, S, \eta, A)$ be a concept diagram and let $c \in C$. Let c' be an anonymous curve not in C. Let d' be the diagram obtained from d by replacing all occurrences of c with c'. Then d is logically equivalent d'.

Lemma 2. *The inference rules are sound.*

We have sufficient inference rules to show that conjecture 2 is true.

Theorem 4. *Let $d = (L, C, R, R^*, S, \eta, A)$ and $d' = (L', C', R', R^{*\prime}, S', \eta', A')$ be concept diagrams such that $L = L'$. If d is strongly maximal with respect to d' and $d \vDash d'$ then $d \vdash d'$.*

Proof (Sketch). Assume d is strongly maximal with respect to d'. By the definition of strong maximality there is an injection, p, from the arrows of d' to the arrows of d. Therefore, we can apply rule 1, remove arrow, to d until its arrows match those of d', i.e. p becomes bijective. Similarly, we can apply rule 2, remove shading, repeatedly until the shading of d matches that of d'. Now, by the definition of strong maximality, each shaded region, r', in d', contains the same number of spiders as $h(r')$ in d. This means we can apply rule 3 to remove spiders until f is bijective. All that differs now are the spiders and the anonymous curves. Apply rules 4 and 5 to obtain d'.

We must now consider whether theorem 2 extends to concept diagrams:

Conjecture 3. Let $d = (L, C, R, R^*, S, \eta, A)$ and $d' = (L', C', R', R^{*\prime}, S', \eta', A')$ be concept diagrams such that $L = L'$. If $d \vDash d'$ then d is strongly maximal with respect to d'.

Figure 6 provides a counterexample to conjecture 3 (as well as conjecture 1). The problems arising from counterexamples like figure 6 may be easy to overcome (by defining inference rules that remove redundant anonymous curves, for instance). We will now demonstrate that problems also arise in more complex situations where arrows are involved.

Such a counterexample to conjecture 3 can be seen in figure 7. In d, the anonymous curves x and y are given labels for convenience. The arrow (f, A, B) tells us the image of f when its domain is restricted to A is B. One of the elements inside x is related to nothing under f, which we know by the arrow targeting the curve that represents the empty set (i.e. the curve containing shading but no spiders). At least one of the other elements inside A must therefore be related to the element inside B. In d', the arrows provide this information that we have just deduced from the arrows of d. The other information provided by d' 'agrees' with that provided by d, so $d \vDash d'$. However, d is not strongly maximal with respect to d': there is no appropriate injective mapping from arrows of d' to those of d.

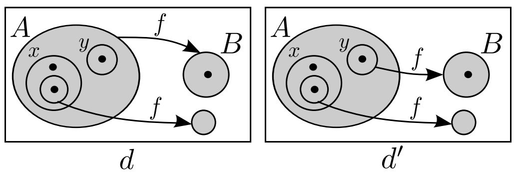

Fig. 7. Conjecture 3: not enough arrows.

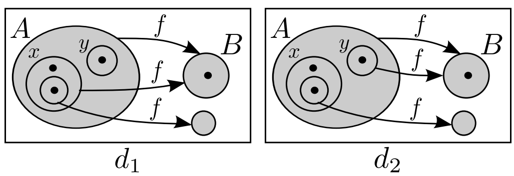

Fig. 8. Adding arrows.

As stated above, with regard to shading and spiders we can attempt to overcome the problems by devising inference rules for removing redundant anonymous curves, for example. With regard to arrows, the question arises as to whether we can add arrows to diagram d, figure 7, until there is an appropriate injection from the arrows of d' to those of d. This leads to the notion of *potential arrows*, those arrows which can be added to a diagram without changing its meaning:

Definition 8. *Let* $d = (L, C, R, R^*, S, \eta, A)$ *be a concept diagram and let* $a \notin A$ *be an arrow not in* d. *Let* $d' = (L, C, R, R^*, S, \eta, A \cup \{a\})$. *If* d *is semantically equivalent to* d' *then* a *is a* ***potential arrow*** *for* d.

In figure 7, there are two potential arrows for d. The arrow (f, A, B) tells us that at least one element of A is related under f to the element in B, and so we can add an arrow which represents this information explicitly. The arrows (f, x, B) in diagram d_1 and (f, y, B) in diagram d_2, figure 8, are potential arrows for d, since either arrow can be added to d without changing its meaning. After adding either arrow, however, the other arrow ceases to be a potential arrow. Neither d_1 nor d_2 have any potential arrows. Thus, an element of choice arises when adding potential arrows to concept diagrams, which again causes problems for completeness. Rather, adding potential arrows results in a set of obtainable diagrams each of which is semantically equivalent to the original diagram. In figure 8, $\{d_1, d_2\}$ is the set of such diagrams obtainable from d (figure 7).

If we are to extend the completeness proof strategy by adding arrows to the axiom, *all* of the diagrams obtained from d using this process must have an arrow set that can be injectively mapped to by the arrows of d' in the appropriate way; this is because the diagrams obtained are semantically equivalent to d and, therefore, semantically entail d'. We can see that we can remove syntax from d_2 to obtain d' since d_2 is strongly maximal with respect to d'. However , there is currently no sequence of rules that would, or general strategy that can be used to, transform d_1 into d', even though $d_1 \vDash d'$ (since d_1 is semantically equivalent to d and $d \vDash d'$); d_1 is not strongly maximal with respect to d'.

A possible approach to overcome this problem is to determine whether we can remove syntax from d' without changing its meaning until we have an appropriate injection from its arrows to those of, in this example, d_1. Unfortunately, no arrows can be removed from d' without weakening information, so such an approach is still insufficient.

Compared to the steps required to extend the definition of maximality, the non-uniqueness of the ways in which we can add arrows is the most serious blow so far to the aim of extending the completeness proof strategy. It is not at all clear how we need to change the syntax of an arbitrary axiom, d, to obtain d' in general. As we have demonstrated, we need to devise strategies for altering the spiders, shading and the arrows present in either the axiom and/or theorem until the axiom is strongly maximal with respect to the theorem. Even once this is solved, it will be challenging to extend the completeness proof strategies to larger fragments of the concept diagram logic.

4 Conclusion

We have identified commonality in the completeness proof strategies of various logics based on Euler diagrams and shown how, as expressiveness increases, the strategy readily extends in some cases. We have illustrated various ways in which this strategy breaks down for concept diagrams, which are syntactically richer and more expressive than earlier logics based on Euler diagrams. The problems identified with extending the completeness strategy to concept diagrams arise because we cannot simply delete syntax from the axiom to obtain the theorem, even for the very small fragment that we considered. Thus, we have established that the existing completeness proof strategies are limited. The non-unique ways of adding syntax to concept diagrams, which further complicates the issue, results from the syntactic richness of the notation and from their expressive power. We believe that the same phenomena will arise in equally expressive logics.

We examined ways in which parts of the completeness proof strategy might be 'patched up' but we conjecture that a new strategy needs to be developed. One (rather undesirable) route to obtaining completeness for fragments of concept diagrams is to derive inference rules for them inspired by complete symbolic logics[4]. This is not the route we want to pursue, strongly preferring a set of inference rules that makes use of *diagrammatic* reasoning. Even small fragments of the more expressive visual logics will require new completeness strategies.

We believe the need for expressive visual logics such as concept diagrams is clear, since they allow the techniques of diagrammatic reasoning to be applied in new domains, such as ontology specification. For these logics to be fully exploited, we need to develop sound inference rules with clearly understood metatheories, including establishing expressiveness and identifying complete fragments. Understanding the effect that increases in both syntactic richness and notational expressiveness have on completeness is essential for the informed design of new logics.

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[4] Symbolic logics have radically different (uncomparable) completeness proof strategies due to their vastly different syntax.